



**JANA SACKS**

# ELEMENTARY FINANCIAL DERIVATIVES

*A Guide to Trading and Valuation  
with Applications*

**WILEY**

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# **Elementary Financial Derivatives**

## **A Guide to Trading and Valuation with Applications**

By Jana Sacks Ph.D.

**WILEY**

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# DEDICATION

*To Jonathan*

## PREFACE

This book is written to be an easy-to-understand introductory textbook featuring hands-on application of presented theoretical concepts. The intention of the book is to introduce the subject of financial derivatives in an accessible and appealing way. The readers will be learning by doing. In the academic environment, the book is intended to be a textbook for an introductory course in financial derivatives.

The purpose of this book is essentially its focus on simplicity. Financial derivatives are often regarded as financial instruments that are just too complex to comprehend and individual investors tend to shy away. At the same time, market professionals state that financial derivatives trades now constitute about 40% of the overall alternative asset markets. This book offers explanations of complex concepts in the simplest possible form with hands-on applications through problems, case studies, and Microsoft® Office Excel®-based modules for valuation.

The book covers all the essential parts of the derivatives market. Aside from a section on options, it features sections on swaps and financial engineering concepts applied primarily, but not exclusively, to the futures markets.

The target audience is anyone who is interested in learning about financial derivatives in a relatively short time, such as individuals preparing for the CFA exam. I envision the readership anywhere from business school students—online and in-class—to individual investors interested in trading options using online trading accounts. After reading this book, investors will be able to better understand and assess their risk exposure and decide whether it is indeed one they are prepared to take.

The material in this book is organized into three parts: forwards and futures, swaps, and options. There is an Instructor's Manual available on the book's companion website hosted by John Wiley & Sons, which includes all solutions to end-of-chapter problems. Instructors may also obtain PowerPoint® slides from the website.

Jana Sacks

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I would also like to extend my special thanks to my wonderful family including my canine “writing companions”—Moses and Buddy.

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## ABOUT THE COMPANION WEBSITES

This book is accompanied by both Instructor and Student companion websites, which are available via the book's page on [www.wiley.com](http://www.wiley.com).

The Instructor website includes:

- Instructors' Solutions Manual
- PowerPoint® slides by chapter

The Student website includes:

- R and Microsoft® Office Excel® data sets

# 1

## DERIVATIVE INSTRUMENTS: CONCEPTS AND DEFINITIONS

### 1.1 Key Derivative Instruments Definitions

#### Forwards and Futures

A forward contract is a contract between two parties. It states that one of the two parties is to buy something from the other at a later date at a price agreed upon today.

A futures contract is also a contract between two parties. One party is bound to buy something from the other at a later date at a price agreed upon today, subject to a daily settlement of gains and losses and guaranteed against the risk that either party might default.

#### Swaps

A swap is a contract in which two parties agree to exchange a series of cash flows at predetermined dates over a period of time.

#### Options

Options are contracts made between two parties that give one party, the buyer, the right to buy or sell an asset from or to the other party, the seller, at a later date and price agreed upon today.

#### Positions

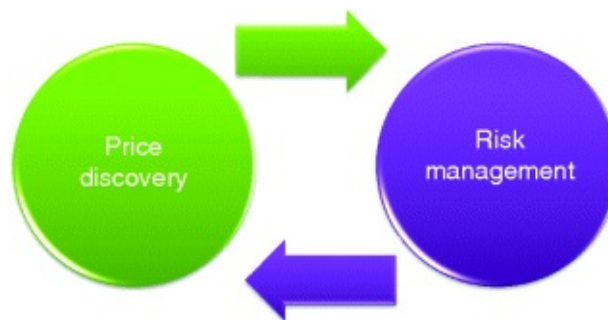
All derivative contracts have essentially two basic positions: long and short. Long position refers to buying, whereas short position refers to selling. The exact positions, rights, and obligations stemming from them differ for different types of financial derivatives.

### 1.2 The Role, Risks, and Benefits of Derivatives Markets



Derivative instruments are securities that derive their value from an underlying asset. They offer investors global diversification in financial instruments and currencies, and promise to generate returns that are superior to traditional investments. Investors in derivatives can profit from changes in interest rates and equity markets around the world, currency exchange rate shifts, and changes in global supply and demand for various types of commodities such as precious and industrial metals, oil, and grains.

There are two widely recognized benefits of derivative instruments: price discovery and risk management.



### Price Discovery

How do we determine prices? Prices depend on a continuous flow of information from around the world and require the highest possible degree of transparency. A broad range of various elements constantly have an impact on supply and demand for assets. Information flow concerning political situations, climatic and environmental conditions, debt situation, and societal behavioral patterns constantly impacts the price of a commodity, such as wheat, soybeans, and oil. This process is known as price discovery.

Futures markets in particular are a useful tool to help discover prices. Futures markets are considered a primary means for determining the spot price of an asset. The futures market is more active than the spot market; hence, information taken from it is often considered more reliable. Futures markets' underlying assets can be geographically quite dispersed and hence have more than one spot price in existence. The price of the contract with the shortest time to expiration often serves as a proxy for the underlying spot price of an asset.

Options are also relevant in price discovery, mostly in the way the market

participants view markets' volatility. If investors think that the markets will be volatile, the option premiums (i.e., their purchase prices) will spike higher.

## Risk Management

Companies and investors today use derivatives as tools in their strategies designed to enable them to manage risk exposures more effectively. Risk management is about both hedging and speculation. At times, investors want to increase their risk exposure to gain greater expected return. At other times, investors will want to protect themselves from undesirable risk exposure. Risk management has a very important purpose for the derivatives market. It is essentially the process of recognizing the desired level of risk, detecting the actual level of risk, and altering the actual to equal the desired.

Derivative securities represent additional risks to investors. Many of those risks are heightened by investors perhaps not fully understanding the proper use of said instruments. For example, options offer the potential for vast gains and losses. While the potential for gain is appealing, their complexity makes them suitable for only sophisticated investors with a high degree of risk tolerance.

Professional traders and money managers can use derivatives effectively. They are aware and trained to work with risk exposures stemming from (1) expiration time of the instrument, (2) market timing, (3) market direction, (4) market volatility, and (5) transaction costs.



Derivative instruments have expiration dates. As each day passes, the expiration date approaches and investors lose more and more “time value”. In the case of options, that alone makes an option's value decrease. In order to make money with most derivatives, investors need to accurately predict the direction in which the market will move. It is also very beneficial if they can predict the extent of the move within a specific amount of time.

### Operational Advantages

Derivative securities are associated with a number of operational advantages. They offer lower transaction costs and greater liquidity. They are generally easier to sell short and contribute to increasing market efficiency.

### Criticisms

It probably comes as no surprise that derivative instruments have been linked with periods of financial distress. Unlike financial markets, derivatives markets neither create nor destroy wealth—they merely provide means to transfer risk. Additionally, they enable speculation and as a result are often compared with gambling. They also enable greater leverage, which we'll see more clearly in the following chapters.

## 1.3 Arbitrage

Spot prices for the exact same commodities may vary around the world according to the place of trading. This represents an arbitrage opportunity. It exists whenever similar assets are sold at different price levels. This opportunity allows an investor to realize a profit with zero risk and, at times, limited funds, by selling the asset in the high-priced market and simultaneously buying it in the inexpensive market. Investors will continue buying and selling until the asset price reaches an equilibrium in both markets. This process of achieving equilibrium through buying and selling is referred to as the *Law of One Price*. In reality, fairly efficient markets have very little to no arbitrage opportunities. If they do, market participants quickly eliminate them.

### Case Study 1.3-1

Suppose that on NYMEX crude oil is trading at \$65/barrel and on another exchange (B) oil is trading at \$70/barrel.

If you buy crude oil on NYMEX and simultaneously sell it on another exchange, you can net a profit of \$5 without any risk or any outlay of cash. As people continue to buy on NYMEX, the price of crude will increase and all of the selling of oil on exchange B will force the price down. This will continue until equilibrium is reached. At equilibrium, the market is efficient. Thus, this is how arbitrage works to make the marketplace more efficient.

## Problems

**1.3-1.** How can market participants eliminate arbitrage opportunities? Discuss the effect of arbitrage on market efficiency.

## 1.4 Miscellaneous

### Short Selling

Short selling refers to a strategy where an investor is selling an asset that he/she doesn't own. The investor anticipates the asset price falling and wants to take an advantage of the price difference. In order to sell something we don't own, we have to temporarily borrow it. Thus, we

borrow the asset from a broker, with a promise to deliver the shares back to him at a later date. When we decide to close out the position, we buy back the underlying asset, hopefully for less than what we sold it for. That way we can unwind the debt to the broker and realize a gain.

## Repos

In repurchase agreements (repos), the seller agrees to sell an asset to a buyer and repurchase the asset sometime in the future for an agreed upon price (higher than the sell price). Repos are used as a way to borrow (usually short term) at a relatively low cost. The underlying asset serves as collateral.

## Discount Factor Calculations in Different Markets

### Notation

$B(t, T)$  Discount factor

$T$  Maturity time

$t$  Any time prior to maturity time

$r$  Interest rate (i.e., yield)

1. Bond equivalent yield (U.S. bond markets)

Equation 1.1 : Discount factor calculation in the U.S. bon

$$B(t, T) = \frac{1}{(1+r)^{(T-t)/360}}$$

2. Money market yield (money markets)

Equation 1.2 : Discount factor calculation in the money market add-on method

$$B(t, T) = \frac{1}{\left(1 + r \left(\frac{T-t}{360}\right)\right)}$$

3. Discount rates (commercial paper and treasury bill markets)

Equation 1.3 : Discount factor calculation for the T – bill discount method

$$B(t, T) = 1 - r \left(\frac{T-t}{360}\right)$$

### Eurodollar Time Deposits

Eurodollar time deposits are deposits held outside of the United States but denominated in U.S. dollars. They are time deposits denominated in USD held either with a foreign bank or with a subsidiary of a U.S. bank.

They have the “Euro” part in the name because they refer to deposits that were in the past held mostly by European banks, and thus became known as “Eurodollars.” Now they are part of the enormous global foreign exchange market. In 2006, China became the largest holder of foreign exchange reserves and most of those reserves are denominated in U.S. currency. Today, such deposits continue to be referred to as “Eurodollars,” regardless of the location.

Thus, Eurodollars are short-term obligations to pay dollars and they are obligations of banking offices located outside the United States.

Eurodollar time deposits are designed for corporate, commercial, institutional, and high-net-worth investors who want a short-term, high-

yield money market investment.

The interest paid for these dollar deposits is generally higher than that for funds deposited in U.S. banks because the foreign banks are riskier—they will not be supported by the U.S. government upon default. Furthermore, they may pay higher rates of interest because they are not regulated by the U.S. government. They are backed by the full faith and credit of the local domestic bank and are issued by its offshore branch.

### Libor

Libor, which stands for London Interbank Offered Rate, is the interest rate paid on interbank deposits in the international money markets (also called the Eurocurrency markets). Because Eurocurrency deposits priced at Libor are almost continually traded in highly liquid markets, Libor is commonly used as a benchmark for short-term interest rates in setting loan and deposit rates and as the floating rate on an interest rate swap. It's considered one of the most important barometers of the international cost of money. Libor is quoted on a one-month, three-month, six-month, or yearly basis.

### Euribor

Euribor (Euro Interbank Offered Rate) is similar to Libor, except it uses euros and euro deposits in the lending and borrowing between banks, instead of dollars. Euribor is the rate at which euro interbank term deposits are offered by one bank to another. It is compiled in Frankfurt, Germany, and published by the European Central Bank. Euribor is the benchmark rate of the large euro money market and is sponsored by the European Banking Federation, which represents the interests of 4500 banks in 24 Member States of the European Union and in Iceland, Norway, and Switzerland. The choice of banks quoting for Euribor is based on a number of market criteria, but all banks are selected to ensure that the diversity of the euro money market is adequately reflected, thereby making Euribor an efficient and representative benchmark.

### Treasury Bond Contracts

Treasury bond contract is a contract based on the delivery of a U.S. Treasury bond with any coupon and at least 15 years to maturity. There

are many different bonds that fit the above description. To give some type of standardization, the markets use a conversion factor to achieve a hypothetical bond with a 6% coupon. Because bond prices do not move in a linear fashion, there is a chance to use arbitrage to capitalize on the deviance of a bond when compared with the 6% standardized bond. To do this, traders look for the cheapest to deliver bond (CTD). This is the least expensive underlying product that can be delivered upon expiry to satisfy the requirements of a derivative contract. The CTD bond is always changing because prices and yields are always changing.

## Problems

**1.4-1.** Suppose  $r = 2.5\%$ , and there are 36 days left to maturity of an instrument. Please calculate the appropriate discount factor if this instrument is

- a. Treasury bond;
- b. money market instrument;
- c. T-bill.

**1.4-2.** Please discuss the difference between Libor and Euribor.

**1.4-3.** What are repos?

**1.4-4.** How can you make gains short-selling assets? Please discuss.



## 2

# FORWARDS AND FUTURES

## 2.1 Futures Fundamentals

### Futures Positions: Long and Short

A futures contract is a commitment between two or more counterparties who agree to engage in a transaction at a later date and a prespecified price. The contract gives one party the right to buy/sell the underlying asset for a specific price at a specific date in the future. We call this specific price a futures price.

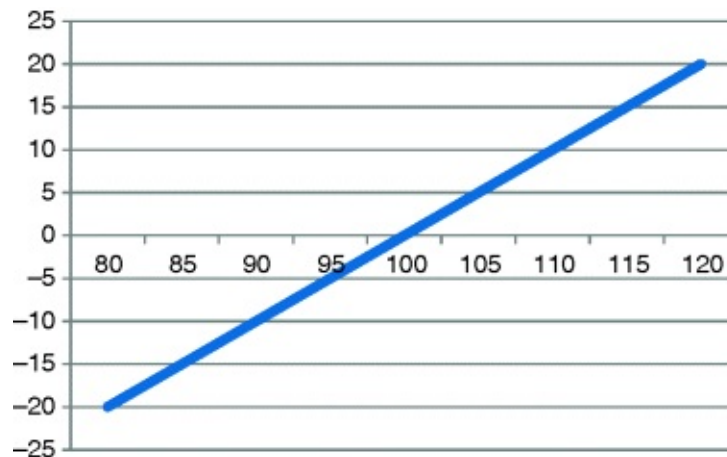
Depending on whether we acquire the right to buy or sell, we refer to our position as a long or a short position, respectively. The value of the contract at the outset is zero as both counterparties have the same probability of making gains at that point. No cash changes hands as a result.

Thus, the long position represents the right to buy the underlying asset at a specific futures price. The short position gives its holder the right to sell the underlying asset at a futures price sometime in the future. Futures (and forward) contracts can be written on a variety of underlying assets such as commodities, foreign exchange, short-term debt, and stock indices.

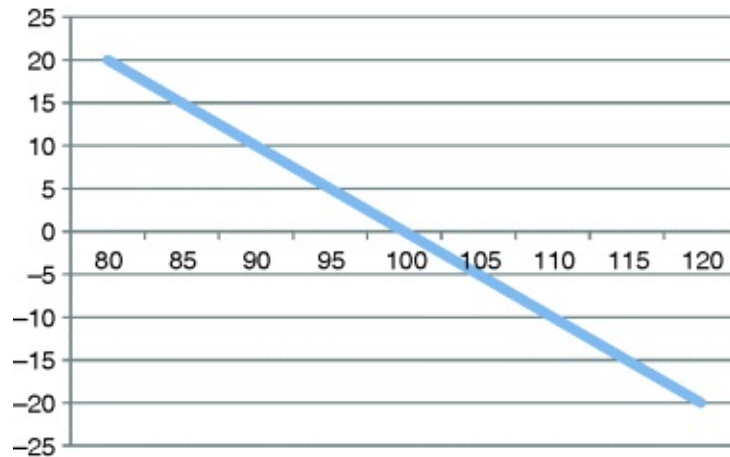


### Types of Futures Contracts

Figures 2.1 and 2.2 depict long and short positions of a futures contract with a futures price of \$100, respectively. The long position makes positive gains if the spot price ends up being above \$100 on the delivery date, whereas the short position gains if the spot price drops below \$100.



**FIGURE 2.1** Long Position with Futures Price of \$100



**FIGURE 2.2** Short Position with Futures Price of \$100

The exact payoff for each of these two positions can be calculated using the following payoff formulas.

*Notation*

$S_T$	Price of the underlying asset at maturity
$F_0$	Futures/forward price

Equation 2.1 : Long futures position payoff at maturity

$$Payoff_{LONG} = S_T - F_0$$

At maturity time of the futures contract, the payoff is the difference between the locked-in futures price and the price of the underlying asset at that time.

Equation 2.2 : Short futures position payoff at maturity

$$Payoff_{SHORT} = F_0 - S_T$$

At the time of the contract's maturity, there is a settlement that can be either physical or cash-settled. Settlement through actual physical delivery means that the buyer takes possession of the underlying asset. The vast majority of futures contracts, however, are settled with a single payment reflecting the market value of the derivative at the futures expiration time. An investor can offset his or her futures position by engaging in an opposite transaction before the stated maturity of the contract.

### Convergence Property

Futures price and spot price converge at maturity. This is called the convergence property. The reason for that is that a commodity available from two sources (spot and futures markets) must be priced identically or else investors will buy it cheaply and sell in the higher-priced market.

### Regulation of the Futures Markets

Regulation is nearly always at the federal level. In the United States, futures markets are regulated by the Commodities Futures Trading Commission (CFTC). It sets capital requirements for member firms of the futures exchanges, authorizes trading in new contracts, and oversees maintenance of daily trading records. In the United Kingdom, futures markets are regulated by Financial Services Authority. In Japan, futures markets are regulated by Financial Services Agency.

Arbitration of disputes is sometimes done through the federal government and the courts but often through self-regulatory organizations such as the National Futures Association in the United States. Forward markets are regulated only indirectly and, thus, are largely unregulated.

### Forward Contracts

A forward is just like a futures contract in many respects. It differs in that it is an over-the-counter (OTC) traded derivative instrument with customized features. A futures contract, on the other hand, is a standardized contract traded on futures exchanges. It specifies delivery places and dates, technical specifications such as commodity grades, and credit procedures. Most forwards do not pay dividends except for forwards that are total return forwards. These take into account payments and assume reinvestments of dividends.

In the forward market, transactions take place in large, private, and largely unregulated markets consisting of banks, investment banks, and government and corporations. Forwards tend to be held to maturity and have very little liquidity. There is a substantial exposure to counterparty default risk. In the case of forwards, this credit risk exposure is not necessarily addressed by imposing margin requirements.

Both futures and forwards are considered leveraged instruments because there is little or no cash requirement when entering into the contract. On the other hand, there exists a substantial counterparty credit risk as a result of this leverage. In the case of futures, this is addressed through exchange clearinghouses, margin requirements, and marking to market.

As the futures markets are able to eliminate all default risk by using clearinghouses as the middlemen guaranteeing contract performance, in the case of the forward markets the credit risk persists. If investors wish to terminate a forward contract prior to its expiration date, they may do so by entering into an offsetting position on the contract. In the absence of a clearinghouse, however, they will increase their credit risk exposure by adding yet another “uninsured” counterparty. The most optimal solution for an offset in the forward market would be to enter the reversing position with the same counterparty and under the same terms as the original agreement. That is, of course, very unlikely and obviously this makes it very difficult to get out of a forward contract prior to its expiration.

Suppose, for example, that you have a colleague, John, who is about to upgrade his boat to a newer larger model. He will be ready to make the purchase in six months from now. You would like to buy his existing boat at that point in time. You and John can enter into a forward agreement that states that you'll buy John's boat for \$20,000 in six months. At the end of the six-month period, you realize that the current market price for the used boat is about \$18,000. But you are still contractually bound to buy the boat for \$20,000. Thus, you'll end up losing \$2000 on this transaction and John will gain \$2000. The two of you can close the contract either by physical delivery or by a cash settlement.

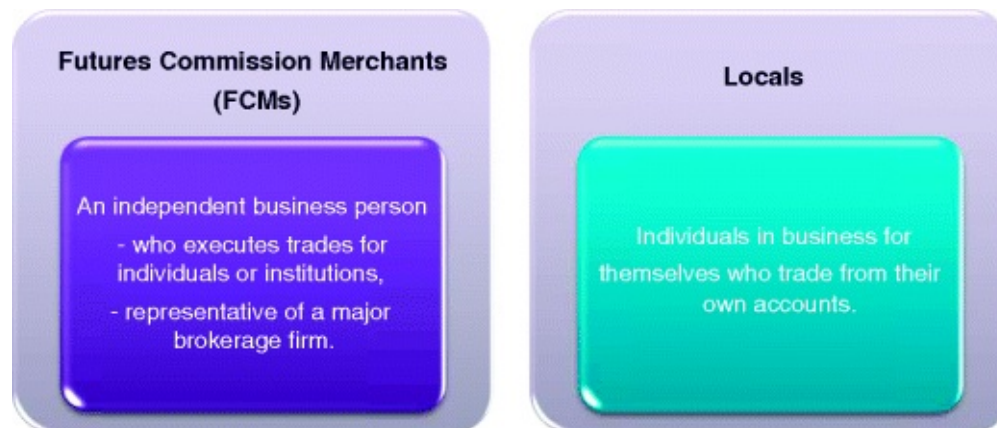
Importers and exporters, in order to hedge against currency exposures, often use forward agreements. For example, a U.S.-based importer wants to lock in a certain exchange rate so that he/she knows exactly how much they'll have to pay for a shipment in U.S. dollars. The same

goes for a foreign-based exporter—they will want to contractually lock in the amount of a foreign currency they will receive for delivery.

Forward contract dealers help to facilitate the transactions based on the counterparties' specific needs and requirements. A dealer differs from a broker in that it takes ownership of the asset. Firms, such as JP Morgan or Citigroup, can act as either a broker or a dealer in different transactions. This allows the dealer to earn additional revenue and reduce the end user's risk.

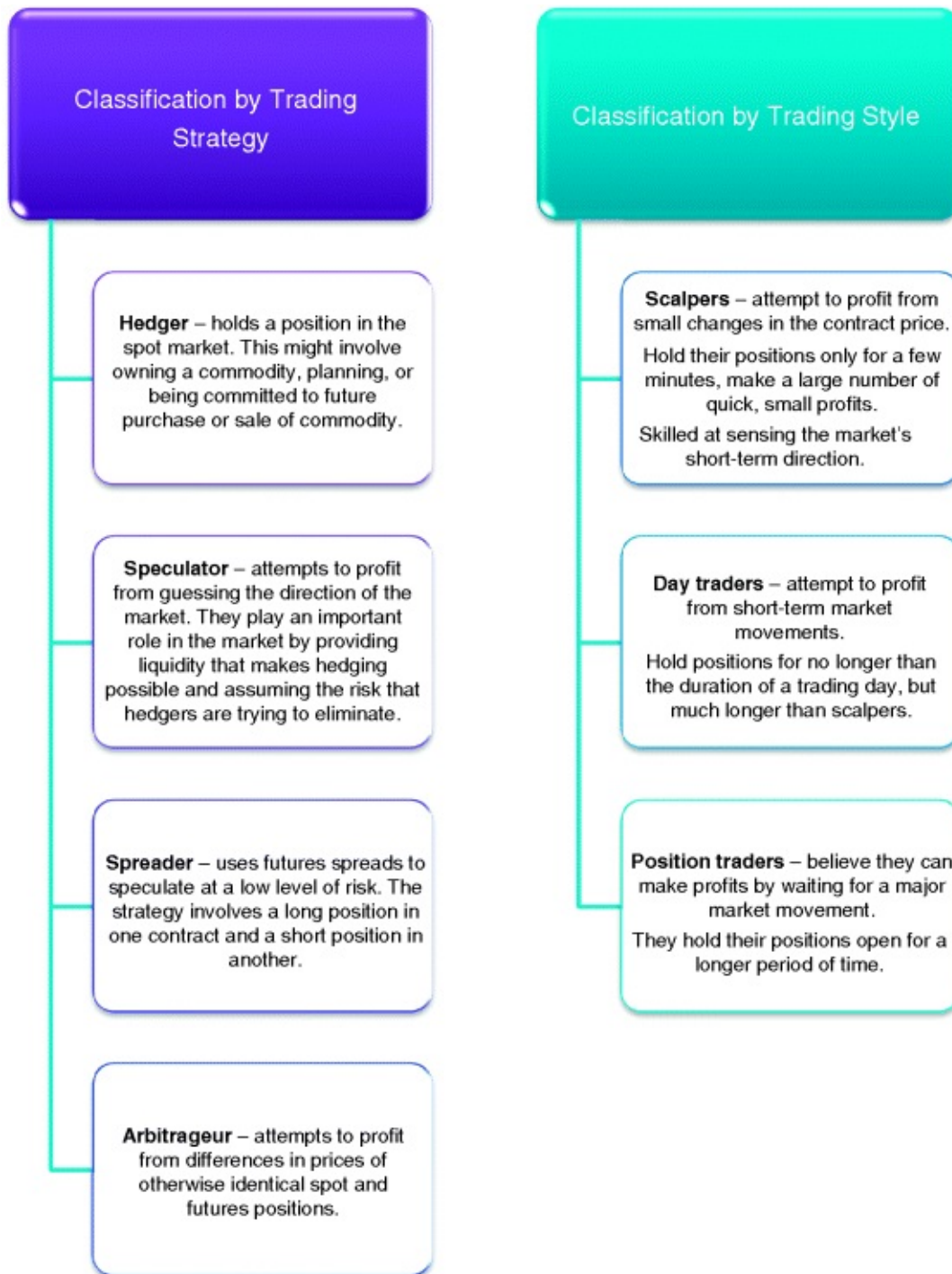
## Futures Traders

People who trade on the futures markets are referred to as futures traders. There are essentially two types of futures traders: an FCM (futures commission merchant) and a local.



Interestingly, a futures trader can be an FCM, a local, or both. As a result, a conflict may arise between trader's loyalty to themselves and the customers. This is known as *dual trading*. It has become controversial in recent years. For example, if a trader is to execute a large order for a client, he expects the price to move as a result. He will act on the anticipated price movement in his account before executing the client's order. Thus, he will capitalize on his “insider” information.

We can further classify futures traders by their trading style or trading strategy.



## Organized Futures Trading Process

There are fundamentally two ways to enact a futures trade.



The first one is called open outcry and it is the traditional form of floor trading. The second one is electronic trading. The basic steps are identical in either format. Investors submit orders that are filled by other traders who take equal but opposite positions.

The more traditional form of trading in the United States is actually the open outcry method. Brokers take orders by telephone or computer from traders. Those orders are then communicated orally to brokers in a trading pit. The pits are octagonal, multitiered areas on the floor of an exchange. The traders wear jackets that distinguish them from one another. To speed up the process, the jackets have different colors; they also wear badges that indicate who they work for and what type of traders they are.

In an open outcry, traders shout and use numerous hand signals to relay information and the price at which they are prepared to trade. Trades are executed when the traders agree on a price and the number of contracts. They do so either through verbal communication or simply by some sort of hand motion. The traders then turn their trade tickets over to their clerks who enter the transaction into the system. Investors are then notified of their trades and relevant information about each trade is sent to the clearinghouse and brokerages.

In electronic trading, investors have to be first pre-approved by a brokerage for electronic trading. When they are, they can send buy or sell orders directly from their computers to an electronic marketplace offered by an exchange. There are no physical brokers involved in the process. Traders see the various bids and offers on their computers and place orders.

A great advantage of electronic trading is that it offers much greater insight into pricing. It is so because the top five current bids and offers are posted on the trading screen for all market participants to see.



Computers process all trading and usually fill orders according to a first-in, first-out (FIFO) process. Another advantage of electronic trading is dissemination of information. It happens in milliseconds and results are instantly broadcasted to the public. Open outcry trading, however, takes from a few seconds to several minutes.

Now let us look at the different types of limit moves in the futures trading arena ([Fig. 2.3](#)).



**FIGURE 2.3** Limit Terminology

### Futures Contract Closing

Closing of a futures position is a matter of executing a reversing position—the so-called offset. For example, if you enter into a long position in the 30 December DJIA 500 contracts in July, you can offset your position, say in October, selling 500 contracts of the 30 December DJIA. By doing this, your position becomes flat because you are now both long and short the identical contract.

Closing out a futures contract will require one the following forms of closure.



## 2.1.1 Futures Margin Account and Marking to Market

### Margin Account in the Stock Market

In the stock market, investors can at times trade on margin. Margin is the percentage of the overall value of the investment that is owned by the investor. It is his account equity. The rest of the investment value is made up of margin loan. This is a loan to the investor from his broker. It helps the investor to reduce the amount of his own cash that he uses to purchase securities. This creates leverage for the investor, and that causes gains and losses to be amplified. The loan must always be paid back with interest.

### Equation 2.3 : Margin calculation

$$M = \frac{V-D}{V}$$

#### Notation

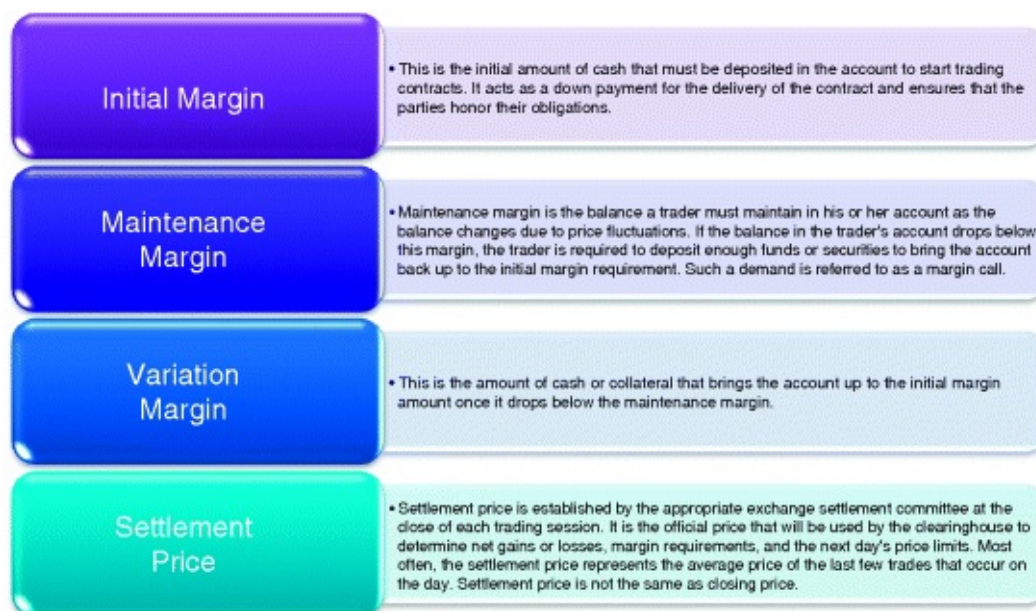
$M$	Margin percentage—the amount of own equity on the overall
-----	-----------------------------------------------------------

	investment
$V$	Market value of investment
$D$	Market value of the loan (i.e., debt)

Usually an initial margin loan in the United States is 40–60%. The market value of the securities minus the amount borrowed can often be less than that, but the investor must keep a balance of roughly 20–30% of the total market value of the securities in the margin account as a maintenance margin.

### Margin Account in the Futures Market

A margin in the futures market is the amount of cash an investor must put up to open an account to start trading (Fig. 2.4). This cash amount is the initial margin requirement and it is *not a loan*. It acts as a down payment on the underlying asset and helps ensure that both parties fulfill their obligations. Both buyers and sellers must put up payments. Maintenance margin exists to ensure that the margin account never becomes negative. If the balance in the margin account falls below the maintenance margin, the trader receives a margin call and is requested to put additional funds into the account equal to the initial margin level.



**FIGURE 2.4** Margin Terminology

One of the key roles of the exchange is to organize trading so that

contract defaults are minimized. The counterparties in the trade are anonymous to each other. The exchange chooses them randomly. Brokers require traders to deposit funds in a margin account. At the end of each trading day, the margin account is adjusted to reflect any gain or loss. The day's settlement price is established by the exchange at the close of each trading session using an exchange-specific methodology. This is known as marking to market (MTM).<sup>1</sup>

The chief task of a clearinghouse is to calculate the net position of each of its members on a daily basis. The clearinghouse has a number of members, all with offices close to the clearinghouse. Brokers who are not clearinghouse members themselves must channel their business through a member. Just as a trader is required to keep a margin account with a broker, a clearinghouse member is required to maintain a margin account with the clearinghouse. This account is known as the clearing margin. Periodic margin calls are commonplace.

### Marking to Market

The process of MTM is one that starts at the initiation of the trade. It is when the price is set and money is deposited in the account. At the end of the day, a settlement price is determined by the futures clearinghouse. The account is then adjusted accordingly, either in a positive or in a negative manner, with funds either being drawn from or being added to the account based on the difference in the initial price and the settlement price. The next day, the settlement price is used as the base price. As the market prices change through the next day, a new settlement price will be determined at the end of the day, and so on.

### Case Study 2.1.1-1

Consider a long and a short position of five futures contracts. Initial margin is \$5 per contract, and maintenance margin is \$3 per contract. Let us track the settlement price development over seven consecutive days. What is the effect of price changes on the balance of the margin account? Please use Excel spreadsheet to calculate results.

Assume the following settlement prices.

Day 0	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7

\$50.00	\$49.60	\$48.00	\$48.50	\$49.25	\$51.00	\$54.00	\$52.50
---------	---------	---------	---------	---------	---------	---------	---------

**Answer**

- Initial margin is \$5 per contract. That is 10% of the initial settlement price ( $5/50 = 0.1$ ).
- Maintenance margin is \$3 per contract, which is equal to 6% of the initial settlement price ( $3/50 = 0.06$ ).

Let us create a spreadsheet where the first column depicts time in days, the second column records settlement prices for all seven days, and the initial situation is depicted by day 0 entry.

Beginning balance	This is always zero at the outset. Each following day, it is equal to the ending balance of the previous day.
Funds deposited	At the outset, it is the number of contracts times the initial margin in dollars. After that you only deposit additional funds if the beginning balance falls below the maintenance margin times the number of contracts. In that case, we deposit additional funds to arrive at the level of the original funds deposited.
Futures price change	This is not applicable on day 0. After that, this is the difference between the current price and the previous settlement price.
Gain/loss	Price change times the number of contracts.
Ending balance	Beginning balance + funds deposited + gain/loss.

LONG FUTURES POSITION										
Day	Settlement Price	Beginning balance	Funds Deposited	Futures price change	Gain/Loss	Ending Balance	Initial Margin	Maintenance Margin	# of Contracts	
0	50	0	25				5	3	5	
1	49.6	25	0	-0.4	-2	23				
2	48	23	0	-1.6	-8	15				
3	48.5	15	0	0.5	2.5	17.5				
4	49.25	17.5	0	0.75	3.75	21.25				
5	51	21.25	0	1.75	8.75	30				
6	54	30	0	3	15	45				
7	52.5	45	0	-1.5	-7.5	37.5				

SHORT FUTURES POSITION										
Day	Settlement Price	Beginning balance	Funds Deposited	Futures price change	Gain/Loss	Ending Balance	Initial Margin	Maintenance Margin	# of Contracts	
0	50	0	25				5	3	5	
1	49.6	25	0	0.4	2	27				
2	48	27	0	1.6	8	35				
3	48.5	35	0	-0.5	-2.5	32.5				
4	49.25	32.5	0	-0.75	-3.75	28.75				
5	51	28.75	0	-1.75	-8.75	20				
6	54	20	0	-3	-15	5				
7	52.5	5	20	1.5	7.5	32.5				

Note that the long position holder did not receive a margin call during the time frame observed. The short position holder received a margin call on day 6 and had to increase his funds deposit.

## Problems

**2.1.1-1.** Suppose you are a futures trader and you enter into three long futures contracts written on frozen orange juice. Each contract is for delivery of 20,000 pounds. The current futures price is \$1.85 per pound, the initial margin is \$8500 per contract, and the maintenance margin is \$6500 per contract. How much would the price have to change to receive a margin call?

**2.1.1-2.** Suppose you enter into a short futures contract to sell silver for \$15 per troy oz. The size of the contract is 5000 ounces. The initial margin is \$15,000 and the maintenance margin is \$7500. At what price would you receive a margin call?

**2.1.1-3.** Consider a long and a short position of 10 futures contracts. Initial margin is \$8 per contract, and maintenance margin is \$5 per contract. Let us track the settlement price development over seven consecutive days. Will either the long position or the short position receive a margin call during these seven days?

Please use Excel spreadsheet to calculate results.

Assume the following settlement prices.

Day 0	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
\$80.00	\$78.23	\$81.16	\$82.34	\$86.15	\$83.24	\$81.14	\$79.20

### 2.1.2 Futures Hedging

Futures are often used for hedging. The hedge is frequently imperfect, but it offers higher degree of certainty than unhedged positions. If both spot price changes and futures prices changes had the same standard deviation and their coefficient of correlation would equal 1, we would use a 1:1 ratio for hedging with futures. In reality, these standard deviations often differ, as does the correlation coefficient resulting in the hedge ratio different from 1.

#### Notation

$\Delta S$	Change in price of the underlying asset
$\Delta F$	Change in the futures price
$h$	Hedge ratio
$V$	Variance of the hedge portfolio
$\sigma_S$	Standard deviation of the change in the price of the underlying asset over a period of time
$\sigma_F$	Standard deviation of the change in the futures price over a period of time
$\rho$	Coefficient of correlation between the change in futures price and the change in the price of the underlying
$h^{OPT}$	Optimal hedge ratio

A long position in the underlying asset can be hedged by a short position in the futures contract. The hedge ratio  $h$  gives us the appropriate amount of the futures contract to hedge one unit of the underlying asset.

Equation 2.4 : Hedge portfolio : short futures position

$$\Delta S - h \times \Delta F$$

A short position in the underlying is hedged by a long position in the futures contract.

Equation 2.5 : Hedge portfolio : long futures position

$$h \times \Delta F - \Delta S$$

Variance of the hedge portfolio is identical no matter if we hold the short or the long position.

$$V[\Delta S - h\Delta F] = V[h\Delta F - \Delta S]$$

Governed by  $V[aX + bY] = a^2V[X] + b^2V[Y] + 2ab \text{Cov}[X, Y]$ , the variance of both the short and the long position is equal to the following.

$$V[h\Delta F - \Delta S] = h^2V[\Delta F] + V[\Delta S] + 2h(-1)\text{Cov}[\Delta F, \Delta S]$$



Equation 2.6 : Variance of a hedge portfolio

$$V[h\Delta F - \Delta S] = h^2\sigma_F^2 + \sigma_S^2 - 2h\rho\sigma_F\sigma_S$$

Taking a partial derivative with respect to  $h$  yields

$$\frac{\partial V}{\partial h} = 2h\sigma_F^2 - 2\rho\sigma_S\sigma_F$$

We set this equal to zero in order to obtain the optimum hedge ratio.

$$2h\sigma_F^2 - 2\rho\sigma_S\sigma_F = 0$$

$$2h\sigma_F^2 = 2\rho\sigma_S\sigma_F$$

$$h^{OPT} = \frac{2\rho\sigma_S\sigma_F}{2\sigma_F^2}$$

## Equation 2.7 : Optimal hedge ratio

$$h^{OPT} = \rho \times \frac{\sigma_S}{\sigma_F}$$

### Case Study 2.1.2-1

Suppose a company is planning on buying two million gallons of jet fuel in six months. The standard deviation of the change in the price per gallon of jet fuel over the six months is estimated as 8%. The company chooses to hedge by buying jet fuel futures contracts. The standard deviation of the change in the futures price is 10% (over the next six months). The coefficient of correlation between the six-month change in the price of jet fuel and the six-month change in the futures price is 0.55. Assume that one jet fuel futures contract is on 50,000 gallons.

What is the optimal hedge ratio? How many futures contracts do we need to be hedged?

#### **Answer**

The optimal hedge ratio

$$h = 0.55 \times \frac{0.08}{0.1} = 0.44$$

$$0.44 \times \frac{2,000,000}{50,000} = 17.6$$

This company should therefore buy roughly 18 jet fuel futures contracts.

### Spot–Futures Parity Theorem

If a hedge is perfect, a portfolio consisting of an asset and a futures contract has no risk. In such a case, the hedged position must provide a rate of return equal to the rate of return on other risk-free investments. This insight is used to derive the theoretical relationship between a futures price and the price of its underlying asset.

#### Notation

$F_{t_0}$	Futures price at time $t_0$
$S_{t_0}$	Price of the underlying asset at time $t_0$
$r_{RF}$	Risk-free rate
$D$	Dividend expressed in \$
$d$	Dividend yield
$q$	Cost of carry

#### Equation 2.8 : Spot– futures parity

$$F_{t_0} = S_{t_0}(1 + r_{RF})$$

The spot–futures parity theorem gives the theoretically correct relationship between spot and futures prices. Any deviation from this parity would give rise to risk-free arbitrage opportunities.

If the underlying stock pays dividend, the relationship changes to the following.

$$F_{t_0} = S_{t_0}(1 + r_{RF}) - D$$

Suppose that dividends are measured in term of dividend yield, then

$$d = \frac{D}{S_{t_0}}$$

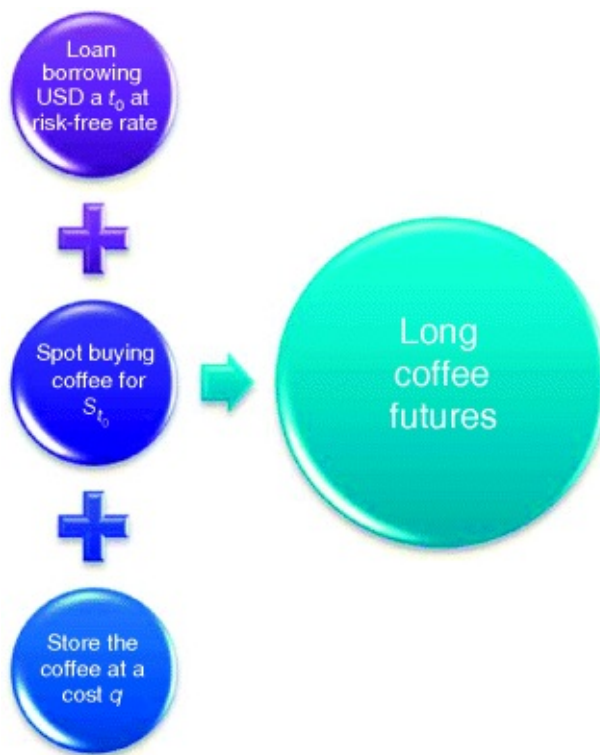
so that

Equation 2.9 : Spot– futures parity with dividend

$$F_{t_0} = S_{t_0}(1 + r_{RF} - d)$$

### Cost of Carry in the Futures Price

What are the components of the futures price if it is written on a commodity that has to be stored? Let's look at a contractual equation, which will be used in pricing throughout this text. Basically, this tells us that we can replace, in this case, a long futures contract for coffee bean by entering into a portfolio consisting of a loan, spot purchase of coffee, and storage rent ([Fig. 2.5](#)).



**FIGURE 2.5 Long Futures Replication**

If the cost of carry is expressed as a percentage of the price, at an annual rate, just like interest rates, the formula for futures price of a commodity becomes

Equation 2.10 : Spot– futures parity with cost of carry

$$F_{t_0} = S_{t_0}(1 + r_{t_0} + q_{t_0})$$

According to this, the more distant the expiration of the contract, the higher the price. This means that the term structures of futures will be normally upward sloping. We call this phenomenon contango. For commodities where storage is either impossible or prohibitive (i.e., crude oil), the slope of the term structure is negative. We refer to this as backwardation.

## Problems

**2.1.2-1.** Suppose Bake-a-Cake, Inc. is planning on purchasing one million bushels of corn in six months. The standard deviation of the change in the price per bushel of corn over the six months is estimated as 5%. The company chooses to hedge by buying corn futures contracts. The standard deviation of the change in the futures price is 4.5% (over the next six months). The coefficient of correlation between the six-month change in the spot price of corn and the six-month change in the futures price is 0.82. Assume that one corn futures contract is on 5000 bushels.

What is the optimal hedge ratio? How many futures contracts does Bake-a-Cake, Inc. need to be hedged?

**2.1.2-2.** Assume that the standard deviation of monthly changes in the spot price of pork bellies is 1.3, and the standard deviation of monthly changes in the pork bellies futures prices for the closest contract, expiring in the next month, is 1.2. Correlation between the futures price changes and the spot price changes has been estimated as 0.85.

A large meat producing company will need to purchase 100,000 pounds of pork. It wants to use pork bellies futures expiring next month to hedge its risk. Each contract is for delivery of 20,000 pounds of pork. How should this producer hedge its risk?

**2.1.2-3.** A cattle farmer expects to have 200,000 pounds of live cattle to sell in three months. The live cattle futures contract is for the delivery of 40,000 pounds of cattle. How can the farmer hedge his position? Show his situation on a payoff diagram.

**2.1.2-4.** An airline executive has argued: "There is no point in using jet fuel futures. There is just as much chance that the price of jet fuel in the future will be less than the futures price as there is that it will be greater than this price." Discuss this viewpoint.

**2.1.2-5.** Suppose  $S_{t_0} = 185.00$ ,  $r_{t_0} = 2.5\%$ ,  $d = 3\%$ , and  $q_{t_0} = 1\%$ .

Please calculate the appropriate futures price  $F_{t_0}$  given that this company pays

- a. no dividend, no cost of carry;
- b. cost of carry but no dividend;
- c. both dividend and the cost of carry.

## 2.2 Forward Rate Agreements

In order to explain forward rate agreements (FRAs), we will first need to introduce certain tools and notation. One of the initial tools is a cash flow diagram. Cash flow diagrams are graphical representations of cash flows expected to occur to each of the counterparties to the contract.

Second, we will use contractual equations that specify which contracts are to be used in order to synthetically replicate a more complex derivative contract. The cash flow diagrams and contractual equations should be used in tandem for achieving the best explanatory value.

### Cash Flow Diagrams

Cash flow diagrams show cash flows occurring at a certain time as a result of a financial contract/security. For example, a short bond or a loan is a situation when one party issues a bond or receives a loan. Thus, there is a cash inflow at present with the obligation to repay the loan in the future. In a time period, let's say one year from now, the counterparty holding the short bond has to repay the original loan amount with some interest. In Case Study 2.2-1, we will look at cash flows reflecting short and long bonds, forward loans, forward deposits, and so on.

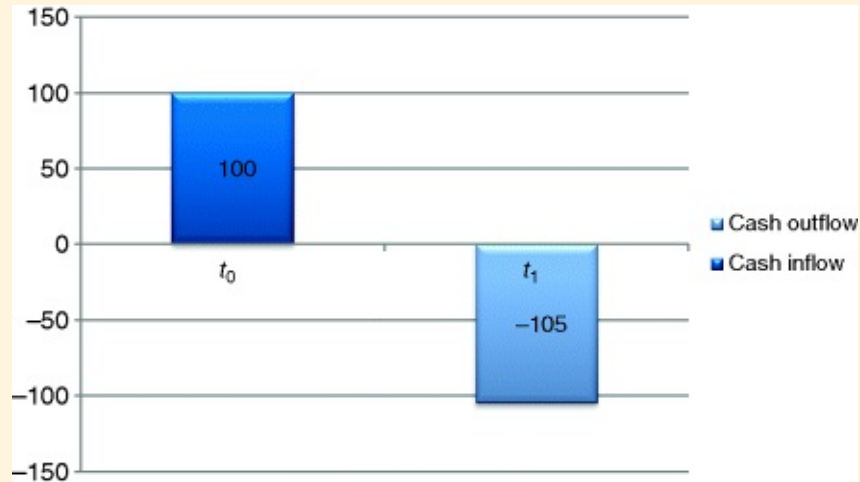
### Case Study 2.2-1

In the case below, the original loan is for \$100, and one-period interest rate is 5%. At  $t_1$ , this party will repay \$105 to its counterparty. The counterparty in this situation is the lender, or also referred to as the holder of a long position in a bond. The lender will have a cash outflow at present of \$100. However, one period from now, he will receive \$105.

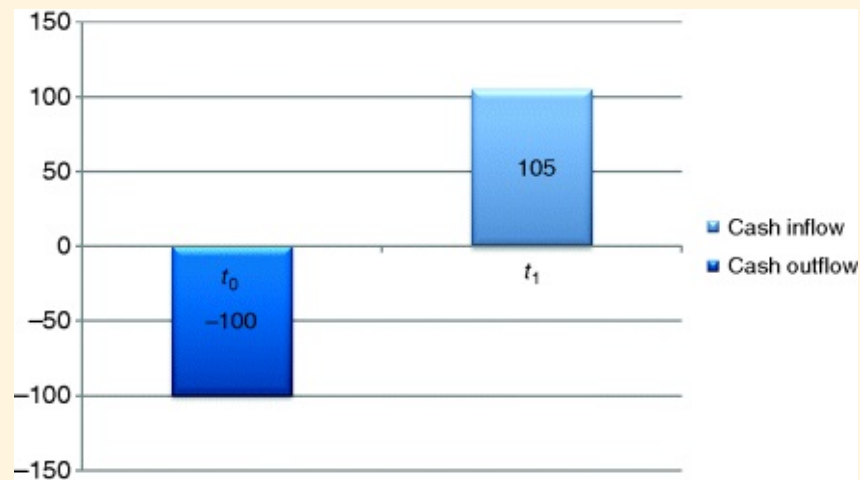
Please sketch the corresponding cash flow diagrams for a short bond from the point of view of the lender and the borrower.

### Answer

The cash flow diagrams show cash outflows in the bottom portion of the graph and inflows in the top portion (Figs. 2.6 and 2.7).



**FIGURE 2.6** Short Bond Cash Flows: Borrower



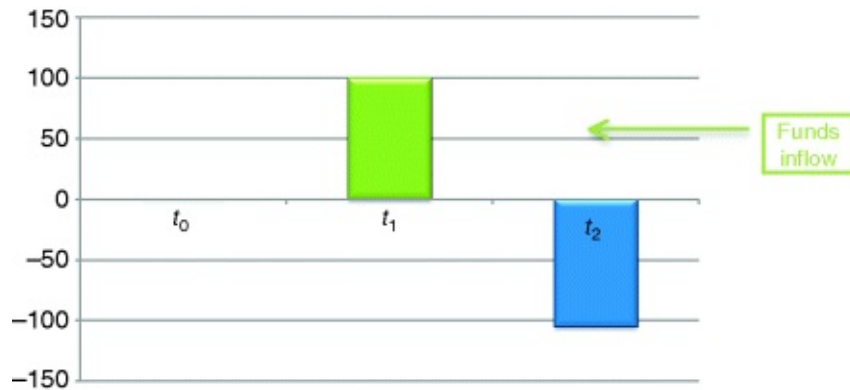
**FIGURE 2.7** Long Bond Cash Flows: Lender

The lender's point of view is identical to holding a long position in a bond.

### Forward Loans

Now let us look at a slightly more complicated security. It is called a forward loan; alternatively, you can think about it as a short bond starting at some future time from now (Fig. 2.8).





**FIGURE 2.8** Cash Flow Diagram for Forward Loan

*Notation*

$F(t_0, t_1, t_2)$	Forward loan interest rate fixed at time $t_0$ , applicable for time period between $t_1$ and $t_2$
$F_{t_0}$	Forward rate fixed at $t_0$ . Any $F(t_0, t_i, t_{i+n})$
$L_{t_1}$	Libor interest rate observable at $t_1$
$(t_2 - t_1)$	Time period of the loan
$\delta = \frac{t_2 - t_1}{360}$	Time period of the loan as a fraction of a year

Given that Libor rate for period 1 ( $L_{t_1}$ ) is unknown at present,  $F(t_0, t_1, t_2)$  will be used as the best guess. It eliminates the uncertainty of  $L_{t_1}$  at time  $t_0$ .

In Case Study 2.2-2, we will learn how to create a synthetic forward loan. Essentially, in a synthetic we use a combination of simpler well-understood and liquid contracts to replicate a more complex one. A forward loan can be replicated by combining a short bond and a deposit.

### Case Study 2.2-2

Assume that there is a cash flow at time  $t_1$  of \$100. Please calculate the cash flows for times  $t_0$  and  $t_2$ , given that we know the appropriate forward rates.

$$F(t_0, t_0, t_1) = 4.5\%$$

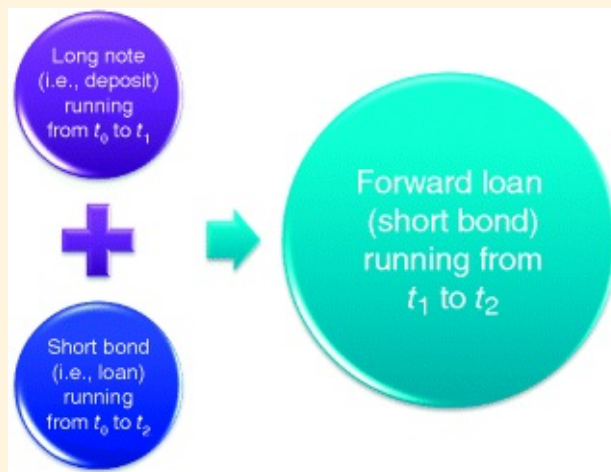
$$F(t_0, t_1, t_2) = 5\%$$

$$\delta = 1$$

Replicate a forward loan using cash flow diagrams.

**Answer**

[Figure 2.9](#) shows a contractual equation. A contractual equation is a visual tool for understanding the synthetic we are creating. This equation tells us that if we add cash flows from a deposit running from time  $t_0$  to time  $t_1$  and a loan running from  $t_0$  to  $t_2$ , we will have replicated a forward loan running from  $t_1$  to  $t_2$ .



**FIGURE 2.9** Replication of a Forward Loan

Cash flow at time  $t_2$  is a function of the cash flow at  $t_1$  and the appropriate forward rate.

$$CF_1(1 + F(t_0, t_1, t_2)\delta) = CF_2$$

$$100(1 + 0.05 \times 1) = 105$$

Cash flow at present is a function of the cash flow of  $t_1$  discounted by the current one-period interest rate.

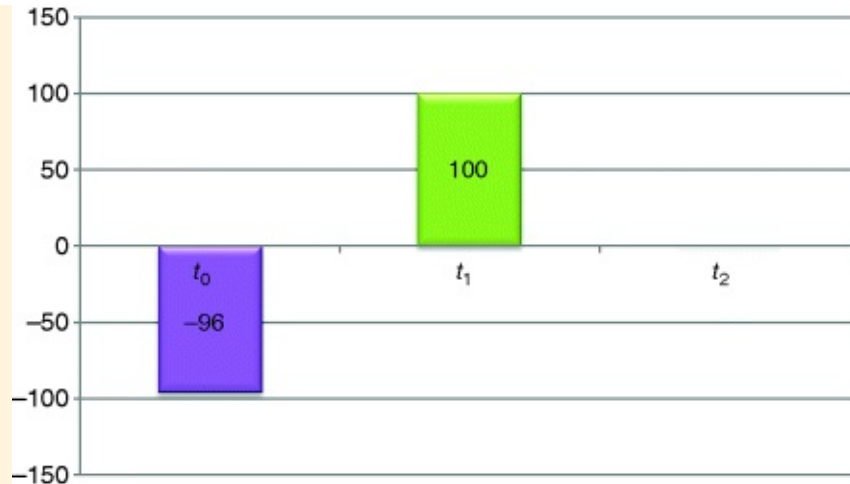
Equation 2.11 : Present value of future cash flow

$$\frac{CF_1}{(1 + F(t_0, t_0, t_1)\delta)} = CF_0$$

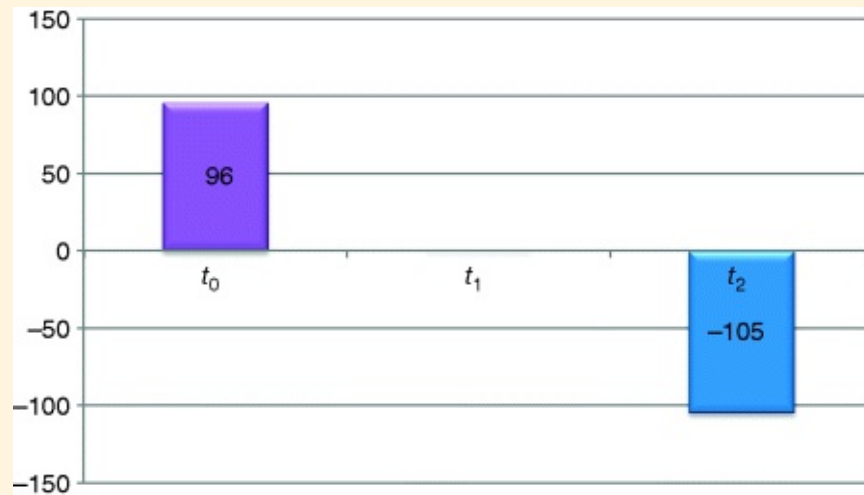
$$\frac{100}{(1 + 0.045 \times 1)} = 96$$

Note that  $F(t_0, t_0, t_1) = L_{t_0}$ .

Now we'll put these results into cash flow diagrams. [Figure 2.10](#) shows a one-period deposit, and [Fig. 2.11](#) shows cash flows stemming from a two-period loan.

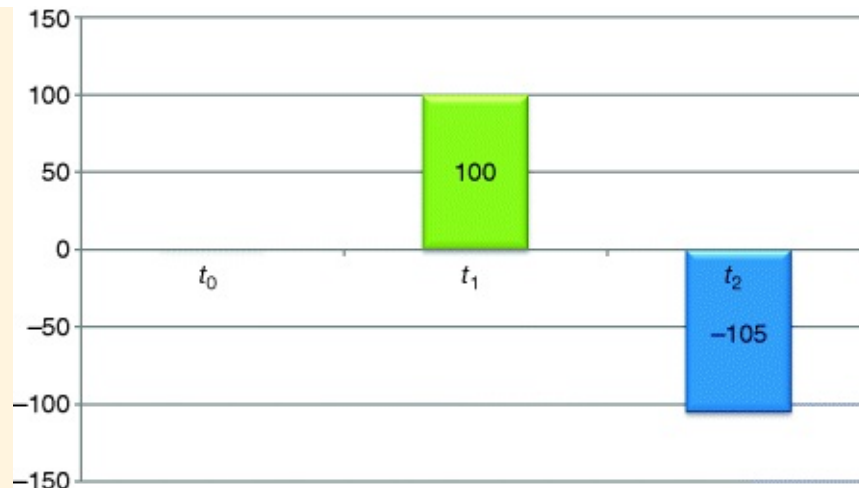


**FIGURE 2.10** Deposit



**FIGURE 2.11** Short Two-Period Bond

Adding cash flow streams from [Figs. 2.10](#) and [2.11](#) vertically together gives us an exact replica of the forward loan cash flows. The present-time cash flows cancel each other out and we are left with the \$100 inflow at time  $t_1$  and the \$105 outflow at  $t_2$  ([Fig. 2.12](#)).



**FIGURE 2.12** Forward Loan

A forward loan is just like any other loan, the difference being that it originates at some time in the future.

### Pricing Implications of Forward Loans

A one-period forward rate going from time  $t_0$  to  $t_1$  is identical to the one-period Libor rate observable at present.

Equation 2.12 : Current Libor rate is identical to a forward r

$$F(t_0, t_0, t_1) = L_{t_0}^1$$

Note that the two-period forward rate going from time  $t_0$  to  $t_2$  is just as well identical to the two-period Libor rate observable at present.

$$F(t_0, t_0, t_2) = L_{t_0}^2$$

## Notation

$F(t_0, t_0, t_2)$	Two-period forward rate for a loan starting today and going on for two periods
$L_{t_0}^2$	Two-period Libor rate known at present
$F(t_0, t_0, t_1)$	One-period forward rate for a loan starting today and going on for one period
$L_{t_0}^1$	One-period Libor rate known at present

Assume  $\delta = 1$ . Let us borrow data from Case Study 2.2-2 to show pricing implications of forward loans.

$$105 = 96 \times (1 + L_{t_0}^2 \times 2\delta)$$

$$100 \times (1 + F(t_0, t_1, t_2)\delta) = 96 \times (1 + L_{t_0}^2 \times 2\delta)$$

$$100 \times (1 + F(t_0, t_1, t_2)\delta) = \left( \frac{100}{(1 + L_{t_0}^1 \times \delta)} \right) (1 + L_{t_0}^2 \times 2\delta)$$

Simplifying this, we get a result that is commonly referred to as the *pure expectations hypothesis* (PEH). It is a powerful result as it enables us to figure out forward rates, and thus make educated guesses about the future, using information from the current yield curve.

$$(1 + F(t_0, t_1, t_2)\delta) = \frac{(1 + L_{t_0}^2 \times 2\delta)}{(1 + L_{t_0}^1 \times \delta)}$$

Equation 2.13 : Forward rate as a function of current yield curve

$$F(t_0, t_1, t_2) = \frac{\left( \frac{(1 + L_{t_0}^2 \times 2\delta)}{(1 + L_{t_0}^1 \times \delta)} - 1 \right)}{\delta}$$

We can also determine the forward rate using data from the bond market. Given a yield curve of Libor rates at time  $t_0$  for several maturities, we can calculate the appropriate discount factors  $B(t_0, t_1)$ ,  $B(t_0, t_2)$ , and so on, which reflect zero-coupon bond pricing.

*Notation*

$B(t_0, t_1)$	Discount factor, the present value of \$1 discounted over one period
$B(t_0, t_2)$	Discount factor, the present value of \$1 discounted over two periods

Note that  $B(t_0, t_0)$  is always equal to 1.

Equation 2.14 : Bond market forward rate

$$1 + F(t_0, t_1, t_2)\delta = \frac{B(t_0, t_1)}{B(t_0, t_2)}$$

Taking a ratio of two consecutive zero-coupon bonds' prices (i.e., the discount factors) enables us to figure out forward rates for the specific accrual period. We do that by subtracting one and dividing the result by the accrual period.

## Case Study 2.2-3

Assume that  $\delta = 180/360$ .

Suppose that the Libor yield curve shows the following.

Libor Rate	
$L_{t_0}^{180}$	1.1%
$L_{t_0}^{360}$	1.45%
$L_{t_0}^{540}$	1.75%
$L_{t_0}^{720}$	2.2%

Please use an Excel spreadsheet to calculate the appropriate discount factors and forward rates for the given maturity times.

### Answer

Time	Libor Yield Curve	Future Time	Time Fraction of a Year	Discount Factor	1+Forward Rates*Delta	Forward Rates* Delta	Forward Rates
180-day	1%	180	0.5	0.9945	1.0055	0.0055	1.10%
360-day	1.45%	360	1	0.9857	1.0090	0.0090	1.79%
540-day	1.75%	540	1.5	0.9744	1.0116	0.0116	2.32%
720-day	2.20%	720	2	0.9579	1.0173	0.0173	3.46%

Time	Libor Yield Curve	Future Time	Time Fraction of a Year	Discount Factor	1+Forward Rates*Delta	Forward Rates* Delta	Forward Rates
180-day	0.011	180	=C2/360	=1/(1+B2*D2)	=1/E2	=F2-1	=G2*2
360-day	0.0145	360	=C3/360	=1/(1+B3*D3)	=E2/E3	=F3-1	=G3*2
540-day	0.0175	540	=C4/360	=1/(1+B4*D4)	=E3/E4	=F4-1	=G4*2
720-day	0.022	720	=C5/360	=1/(1+B5*D5)	=E4/E5	=F5-1	=G5*2

The forward rates for a one-period loan starting sometime in the future are



$$F(t_0, t_0, t_1) = 1.1\%$$

$$F(t_0, t_1, t_2) = 1.79\%$$

$$F(t_0, t_2, t_3) = 2.32\%$$

$$F(t_0, t_3, t_4) = 3.46\%$$

## FRA

In an FRA, we end up exchanging interest payments at some future date. Let us start with an example where we exchange a fixed payment against a floating one.

An FRA is a contract where the underlying asset is merely an interest payment. Thus, a forward rate agreement is a forward contract on a specific short-term interest rate, such as Libor. At maturity, counterparties exchange payments based on the difference between the contractually determined forward rate and the actual prevailing benchmark rate (i.e., Libor) on that date. The actual dollar amount is calculated using a notional principal specified in the contract.

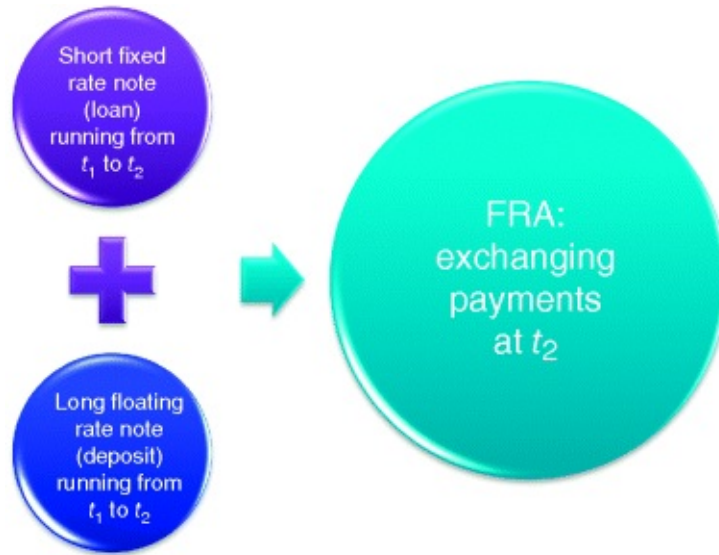
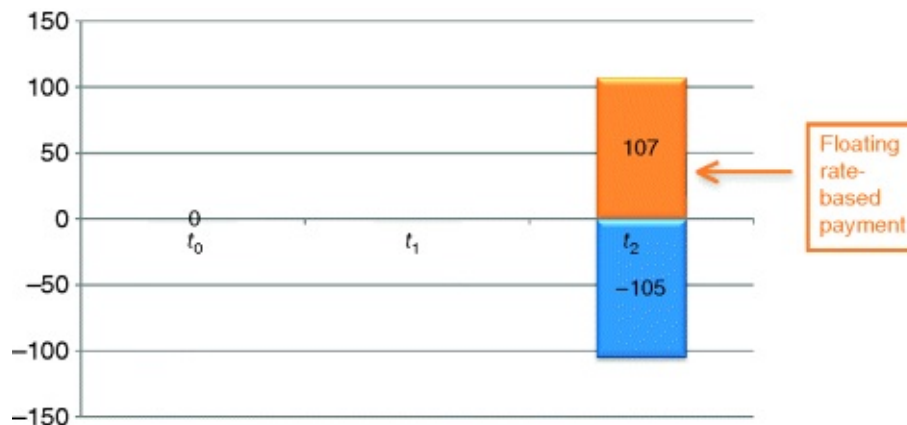
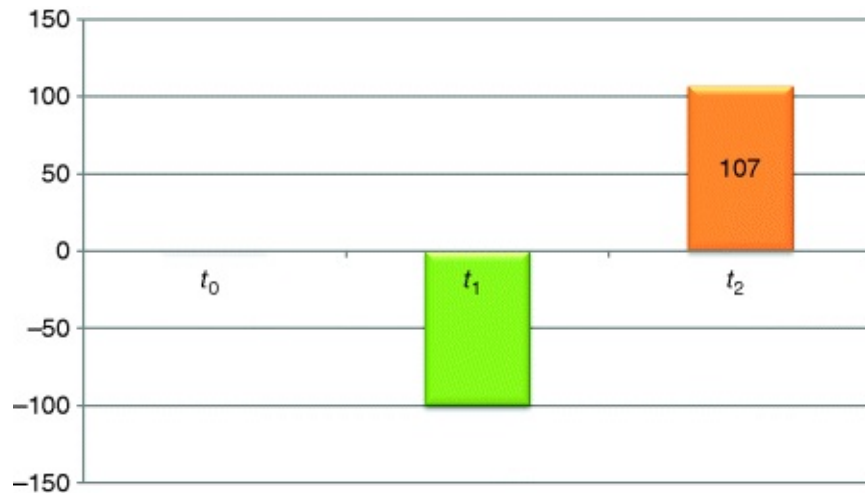


Figure 2.13 shows the cash flows that will occur between counterparties of the FRA in the future time  $t_2$ . In this case, we are looking at the cash flows of the counterparty that is a payer of fixed rate and a receiver of the floating rate.

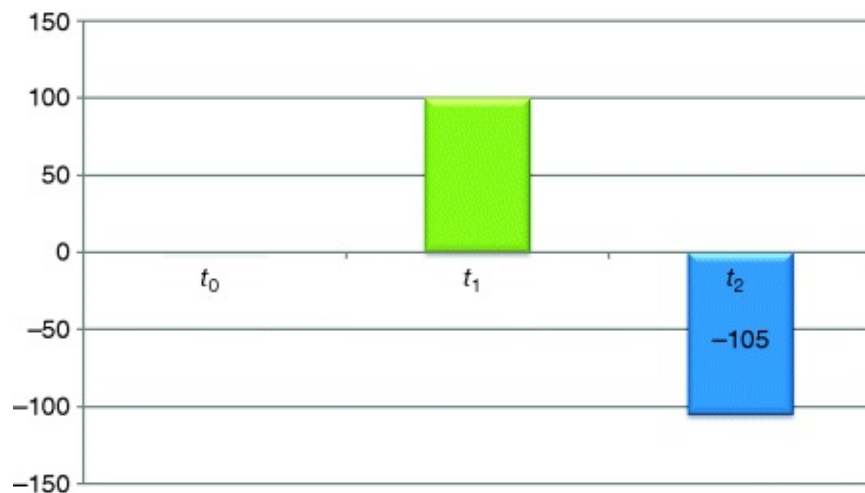


**FIGURE 2.13** FRA Float Against Fix Cash Flows

Let us synthesize this position. We will do so by decomposing the previous cash flows into two separate securities. The first one is a floating rate deposit, and the second one is a fixed loan (Figs. 2.14 and 2.15). Both contracts are running from  $t_1$  to  $t_2$ .



**FIGURE 2.14** Floating Rate Deposit



**FIGURE 2.15** Fixed Rate Loan

Both the floating rate deposit and the fixed rate loan have the same size  $t_1$  cash flows. Adding them together, these cash flows exactly offset each other—creating the resulting FRA flows as seen in [Fig. 2.13](#).

The long FRA position benefits if the actual rate ends up being higher than the contractual forward rate, and that happens to be the case in this example.

In an FRA, we need to have a system that determines the future dates of importance. Suppose that we see a quote for a  $6 \times 12$  FRA. This means that the contract expires in six months based on the underlying rate of a 180-day Libor in six months from now.



The comparison rates are usually based on exact day counts such as 60-day Libor or 90-day Libor. If a client wants to customize an FRA in any unusual way, it can be done but it will be considered an off-the-run contract, and will be illiquid and as a consequence relatively more expensive.

The FRA structure is the same for all currencies.

Equation 2.15 : FRA discounted payoff : long position

$$Payoff_{FRA}^{LONG} = \frac{\left( L_{t_1}^{(t_2-t_1)} - F_{t_1}^{(t_2-t_1)} \right) (\delta)}{\left( 1 + L_{t_1}^{(t_2-t_1)} (\delta) \right)} \times NP$$

Equation 2.16 : FRA discounted payoff : short position

$$Payoff_{FRA}^{SHORT} = \frac{\left( F_{t_1}^{(t_2-t_1)} - L_{t_1}^{(t_2-t_1)} \right) (\delta)}{\left( 1 + L_{t_1}^{(t_2-t_1)} (\delta) \right)} \times NP$$

## Case Study 2.2-4

Let's assume that a dealer quotes a rate of 4.5% on this instrument and end user (i.e., long position) agrees. He is hoping that rates will

increase. Expiration date is 90 days from now. The notional amount is \$50 million. The underlying interest rate is the 180-day Libor. It is a  $3 \times 9$ , so in 90 days we are curious about the current 180-day Libor. Suppose it will be 5%. That 5% interest will be paid 180 days later. What is the long position's payoff?

**Answer**

*Notation*

$L_{3m}^{180}$	Underlying rate for a 180-day loan starting 90 days from now. This is the rate known and applicable at the FRA expiration time, that is, three months from now. This rate is unknown as we enter into the FRA.
$F_{3m}^{180} = F(t_0, 90, 270)$	Forward contract rate that the two parties agree will be paid. This rate is fixed at time zero.
$\delta$	Delta is the fraction of a year that is specified as a number of days to maturity on the underlying instrument. The number of days to maturity is the difference between the settlement date and the initial future date.

$$\delta = \frac{(t_2 - t_1)}{360} = \frac{270 - 90}{360} = \frac{180}{360}$$

$$Payoff^{LONG} = \frac{(0.05 - 0.045)(180/360)}{(1 + 0.05(180/360))} \times 50,000,000 = 1.1$$

The above formula reflects a discounted payoff. This is the value of the payoff at the end of the initial 90-day period and at the beginning of the 180-day period. We are discounting the payment at the Libor rate at that time.

The long position holder's payoff will be positive at \$121,951.22. It is a gain for the long positions because rates had increased. The short position loses \$121,951.22.

## Problems

**2.2-1.** What is the appropriate two-period forward rate (known today) given that  $L_{t_0}^1 = 4.5\%$  and  $F(t_0, t_1, t_2) = 5\%$ ? Assume that  $\delta = 1$ .

**2.2-2.** (Cash flow diagrams) What is the price of a 1-year zero-coupon bond if  $L_{t_0}^1 = 3.75\%$  and its par value is \$1000? Please sketch cash flow diagrams for both the

- short position and
- long position in this bond.

**2.2-3.** (Cash flow diagrams) What is the appropriate  $L_{t_0}^{6m}$  for a 180-day note with the par value of \$100 and present value of \$97? Assume  $\delta = 1$ . Please sketch cash flow diagrams for both the short and the long position in this bond.

**2.2-4.** (Forward loan) Current yield curve shows the following Libor rates:

$$L_{t_0}^1 = 3.75\%$$

$$L_{t_0}^2 = 4\%$$

- Using the PEH result, please calculate the following forward rate:  $F(t_0, t_1, t_2)$ . Assume that  $\delta = 360/360$ .
- Suppose that  $CF_{t_0} = 100$ . Using the information obtained above, please calculate  $CF_{t_1}$  and  $CF_{t_2}$ .
- Using cash flow diagrams, sketch this forward loan along with its synthetic.
- How would your diagram change if this were a forward deposit?

**2.2-5.** Consider a  $3 \times 6$  FRA. 90-day Libor is  $L_{90}^{90} = 2\%$ . The locked-in forward rate is  $F(0, 90, 180) = 2.25\%$  and  $\delta = (180 - 90)/360$ . The notional principal is \$10,000,000.

- What is long and short position payoff in this FRA? *Hint:* The long position pays  $F(0, 90, 180)$  and receives  $L_{90}^{90}$ .
- Sketch the corresponding cash flow diagram for the long position in the FRA.
- Sketch the cash flow diagrams for the FRA synthetic consisting of a forward loan and a forward deposit.

**2.2-6.** Suppose a dealer quotes a forward rate of 3% on an FRA. Both the long and the short position agree. One is hoping that interest rates will increase and the other one that they will decrease relative to the forward rates. This is a  $6 \times 12$  FRA. The notional amount is \$100 million. The underlying interest rate is the 180-day Libor ( $L_{180}^{180}$ ). Suppose in six months from now the six-month Libor is 2.75%. This will be paid 180 days later. What is the long and short position payoff six months from now?

**2.2-7.** Suppose a dealer quotes a forward rate of 2.5% on an FRA. This is a  $6 \times 9$  FRA. The notional amount is \$20 million. The underlying interest rate is the 90-day Libor ( $L_{180}^{90}$ ). Suppose in six months from now the three-month Libor is 2.6%. This will be paid 90 days later. What are the values of long and short positions six months from now?

## 2.3 Currency Forwards

Currency forwards are instruments mostly used by importers and exporters looking to mitigate uncertainty about the direction of the exchange rate. For example, an exporter might elect to lock in the rate at which he will sell EUR and buy USD in three months from now. By doing this, he effectively hedges his EUR receivables by entering into a currency forward. This contract leaves him fully hedged should the currency depreciate below the contract level. On the other hand, if the currency appreciates in the meantime, he will have lost the opportunity to capitalize on the appreciation. For most companies, this opportunity cost is an acceptable one.

Like other futures contracts, foreign currency futures have standardized contract form specifying sizes, time periods, settlement procedures, and so on. They are traded on organized exchanges, and cleared through futures clearinghouses. Currency forward contracts may have different and unusual contract sizes, settlement procedures, and so on. These are trades that are done OTC directly between counterparties.

In currency forwards, we deal with currencies. We must therefore pay attention to the way currencies are quoted. Suppose we see a quote such as EUR/USD =  $x$ . The way we read this is that  $x$  is a certain number of USD per one EUR. Analogously, USD/EUR gives us the number of EUR per each USD.

### Case Study 2.3-1

In the following currency forward, the future exchange rate  $F_{t_0}$  is quoted as EUR/USD. Suppose we have been quoted the following spot and forward rates.

	Spot Exchange Rate	Forward Exchange Rate
USD/EUR	→ 0.77	
EUR/USD		→ 1.2923

The Libor rates for a single time period are quoted as  $L_{t_0}^{1,USD} = 3.5\%$  and  $L_{t_0}^{1,EUR} = 4\%$ . Let us assume that  $\delta = 360/360$ .



Please sketch a currency forward cash flow diagram. How can we engineer a synthetic? How much are the exact cash flows in the synthetic?

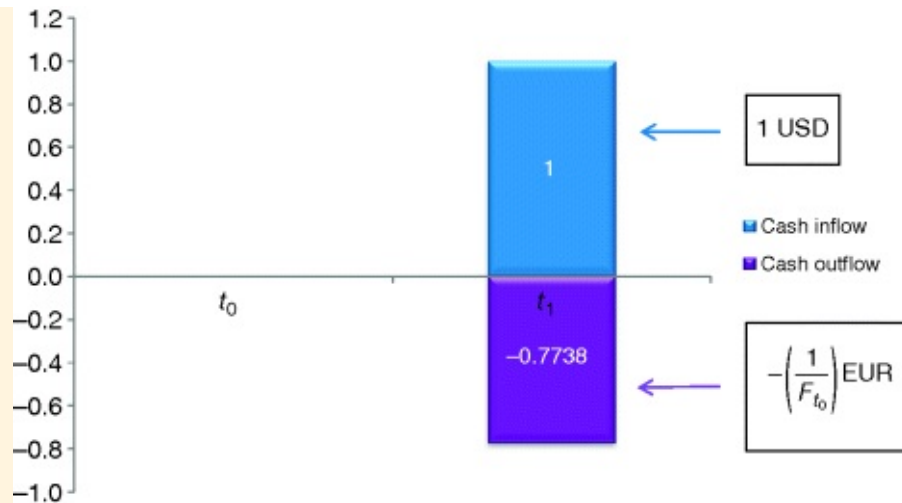
**Answer**

*Notation*

$F_{t_0}$	Forward exchange rate, can be quoted either as USD/EUR or as EUR/USD
$e_{t_0}$	Spot exchange rate, can be quoted either as USD/EUR or as EUR/USD
$L_{t_0}^{1,USD}$	One-period USD Libor rate
$L_{t_0}^{1,EUR}$	One-period EUR Libor rate
$B(t_0, t_1)^{EUR}$	Present value of a EUR-denominated zero-coupon bond; EUR-denominated discount factor
$B(t_0, t_1)$	Present value of a USD-denominated zero-coupon bond; USD-denominated discount factor

In this currency forward, we are receiving \$1 and paying  $\text{€}1/F_{t_0}$  one year from now. The reason for this ratio is the forward exchange rate being quoted as EUR/USD ([Fig. 2.16](#)).

$$\frac{1}{F_{t_0}} = 1/1.2923 = 0.7738$$



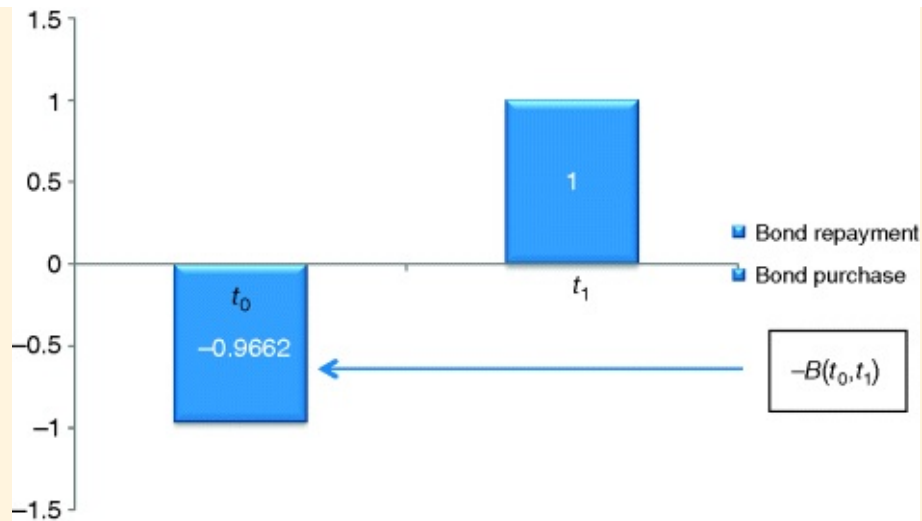
**FIGURE 2.16** EUR/USD Currency Forward

Thus, in this contract we will receive \$1 and pay €0.7738.

In order to understand currency forwards better, we will now replicate the currency forward using other, more common, financial instruments. Let us begin with separating the currency forward's cash flows into two separate cash flow events. Next, we need to add accompanying cash flows.

The first instrument is going to be a long bond denominated in USD (Fig. 2.17). This is a zero-coupon bond that has a par value of \$1. In order to calculate its present value, we need the appropriate USD Libor rate.

$$B(t_0, t_1) = \frac{1}{\left(1 + 0.035 \times \frac{360}{360}\right)} = 0.9662$$

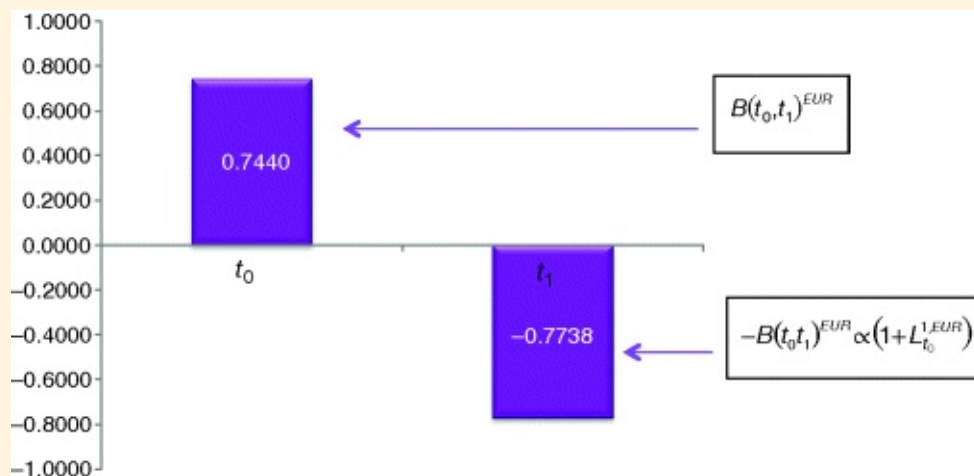


**FIGURE 2.17** Long USD Bond

The second security is a short zero-coupon bond denominated in EUR (Fig. 2.18). Recall that the present value of the EUR-denominated bond is equal to  $B(t_0, t_1)^{EUR}$ . It is the present value of the USD-denominated bond multiplied by the spot exchange rate, where  $e_{t_0}$  is quoted as USD/EUR.

Equation 2.17 : PV of EUR – denominated bond

$$B(t_0, t_1)^{EUR} = B(t_0, t_1) \times e_{t_0}$$

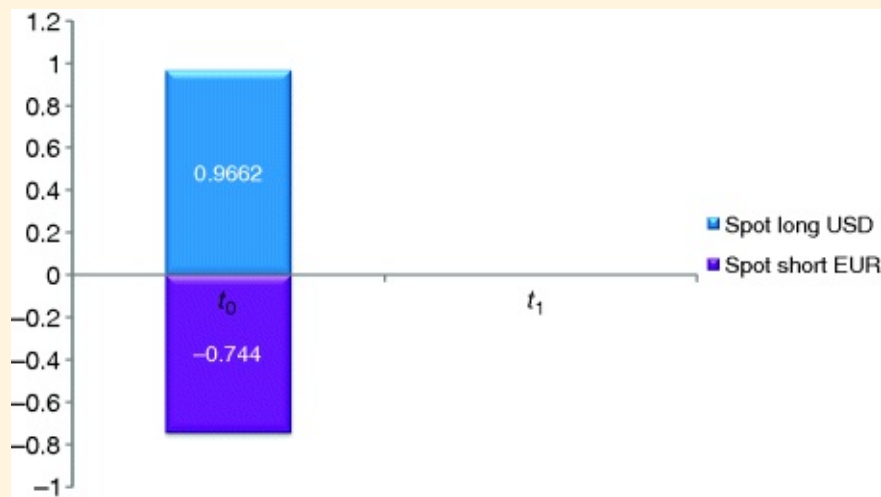


### FIGURE 2.18 Short EUR Bond

The future value of the EUR-denominated bond can be calculated by compounding its present value over one period with EUR Libor or by translating the future value of the USD-denominated bond into EUR using  $F_{t_0}$ . Note that in this instance  $F_{t_0}$  is quoted as EUR/USD.

$$B(t_0, t_1)^{EUR} = 0.9662 \times 0.77 = 0.7440$$

The last cash flow that will complete the replication is a spot FX transaction (Fig. 2.19). This transaction will cancel out the newly added initial cash flows that appear in the diagrams for the short and long bonds.



**FIGURE 2.19** Spot Transaction of Purchasing USD Against EUR

Putting the above figures together yields a contractual equation for a currency forward. In the replication process, we create a currency forward synthetically. The synthetic can be used to derive a well-known result in FX trading—the interest rate parity.



The present values of both EUR-and USD-denominated payments are a function of their respective Libor rates and time elapsed between payment dates.

Equation 2.18 : PV of USD zero – coupon bond, par = \$1

$$B(t_0, t_1) = \frac{1}{(1 + L_{t_0}^{1,USD} \times \delta)}$$

Equation 2.19 : PV of EUR zero – coupon bond

$$B(t_0, t_1)^{EUR} = \frac{1/F_{t_0}}{(1 + L_{t_0}^{1,EUR} \times \delta)}$$

At the outset, the contract value has to be equal to zero. [Equation 2.17](#) has to hold in order to ensure that.

### Pricing Implications

Let us now use this result to derive the interest rate parity. First, however, we need to translate both sides of the equation into a single currency. Let us assume that USD is the *base currency* and recall that  $e_{t_0}$  is quoted as USD/EUR.

Recall from [Equation 2.17](#) that

$$B(t_0, t_1) \times e_{t_0} = B(t_0, t_1)^{EUR}$$

$$\frac{1}{(1 + L_{t_0}^{USD} \times \delta)} \times e_{t_0} = \frac{1/F_{t_0}}{(1 + L_{t_0}^{EUR} \times \delta)}$$

Thus far,  $F_{t_0} = 1.29$  has been quoted as EUR/USD. We will change that so that  $F_{t_0} = 0.7738$  is quoted as USD/EUR. That changes the above equation in the following way:

$$\frac{1}{(1 + L_{t_0}^{USD} \times \delta)} \times e_{t_0} = \frac{1}{(1 + L_{t_0}^{EUR} \times \delta)} \times F_{t_0}$$

$$\frac{e_{t_0}}{F_{t_0}} = \frac{\frac{1}{(1 + L_{t_0}^{EUR} \times \delta)}}{\frac{1}{(1 + L_{t_0}^{USD} \times \delta)}}$$

$$\frac{e_{t_0}}{F_{t_0}} = \frac{(1 + L_{t_0}^{USD} \times \delta)}{(1 + L_{t_0}^{EUR} \times \delta)}$$

$$\frac{F_{t_0}}{e_{t_0}} = \frac{(1 + L_{t_0}^{EUR} \times \delta)}{(1 + L_{t_0}^{USD} \times \delta)}$$

This yields the *interest rate parity*. Here the base currency is USD. The USD Libor is the discount rate used in the parity.

Equation 2.20 : Interest rate parity

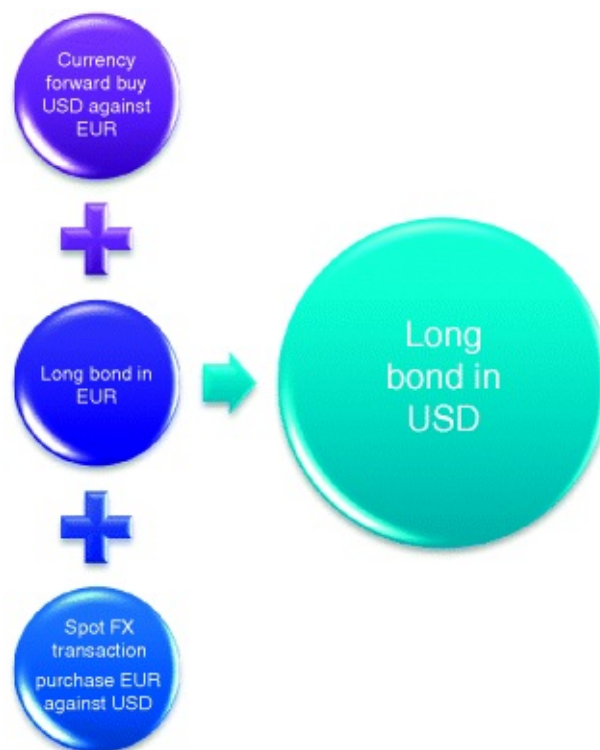
$$F_{t_0} = e_{t_0} \times \frac{(1 + L_{t_0}^{EUR} \times \delta)}{(1 + L_{t_0}^{USD} \times \delta)}$$

Creating Other Synthetics

We use currency forwards for an array of different reasons. Some of the more recurrent reasons are tax withholding and capital controls.

## Taxes

Oftentimes, investors are less than enthusiastic about paying taxes. But, of course, taxes can be scarcely avoided. Employing derivative securities in financial engineering strategies, however, sometimes offers a solution. For example, instead of investing in a regular zero-coupon bond we can create a synthetic bond using securities that are not subject to taxation.



The most liquid forward contracts are denominated in USD. Contracts that are liquid typically have much lower transaction costs. At times, clients that are not U.S.-based will want a forward contract denominated in USD. It will have to be created synthetically as they are foreign-based.

Creating the position synthetically, in order to satisfy a EUR-based client, requires three steps. First, we buy spot EUR. Second, we buy a EUR-denominated bond, and finally we sell a EUR-denominated forward at the futures price.



## Case Study 2.3-2

Let  $e_{t_0} = 0.77$  EUR per one USD. Buying spot EUR has the following associated cash flow. Suppose that  $B(t_0, t_1) = 0.96$ .  $NP = 100$ , and suppose that  $F_{t_0} = 0.78$  EUR per one USD. Using cash flow diagrams, please show a EUR-based synthetic to a long USD-denominated zero-coupon bond.

### Answer

#### Notation

$B(t_0, t_1)$	Discount factor for USD-denominated currency starting at present and maturing at $t_1$
$B(t_0, t_1)^{EUR}$	Discount factor for EUR-denominated currency going from $t_0$ to $t_1$
$b(t_0, t_1)$	USD-denominated discount bond, starting at present and maturing at $t_1$
$b(t_0, t_1)^{EUR}$	EUR-denominated discount bond going from $t_0$ to $t_1$ . Both of these bonds have zero default risk
$e_{t_0}$	Spot exchange rate
$F_{t_0}$	Forward exchange rate

$$b(t_0, t_1) = B(t_0, t_1) \times NP$$

Recall that

$$b(t_0, t_1)^{EUR} = b(t_0, t_1) \times e_{t_0}$$

and

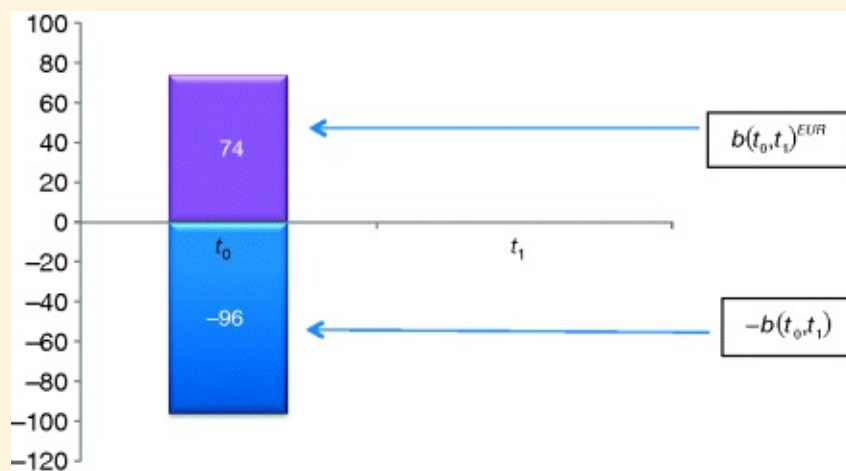
$$b(t_1, t_1)^{EUR} = b(t_0, t_1)^{EUR} \times (1 + L_{t_0}^{EUR} \delta)$$

The present value of a one-year zero-coupon bond denominated in USD with \$100 par value will be the starting point.

$$b(t_0, t_1) = 0.96 \times 100 = 96$$

Now we need to exchange the dollar amount into euros. We'll purchase euros on the spot market ([Fig. 2.20](#)).

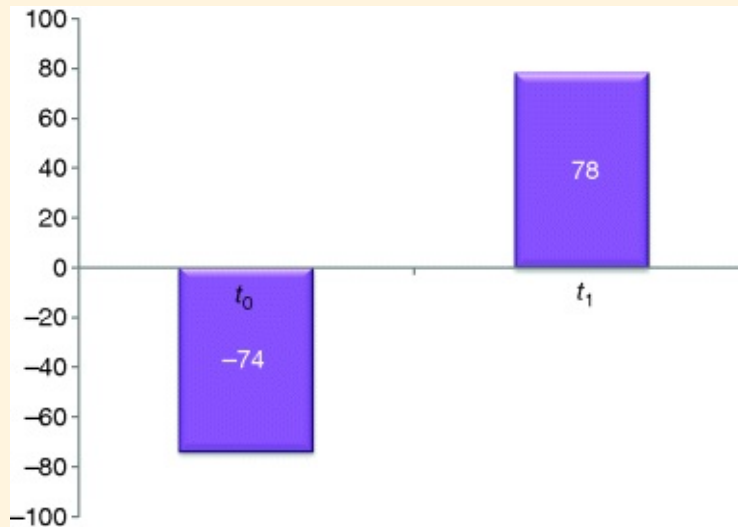
$$b(t_0, t_1)^{EUR} = 96 \times 0.77 = 74$$



**FIGURE 2.20** Spot Purchase of EUR

In the next step, we'll enter a long position in a EUR-denominated

bond ([Fig. 2.21](#)).



**FIGURE 2.21** Long EUR-Denominated Bond

The EUR-denominated bond grows at 6% annually.

$$b(t_1, t_1)^{EUR} = 74 \times (1 + 0.06 \times 1) = 78$$

The value in one year from now will be 78 euros.

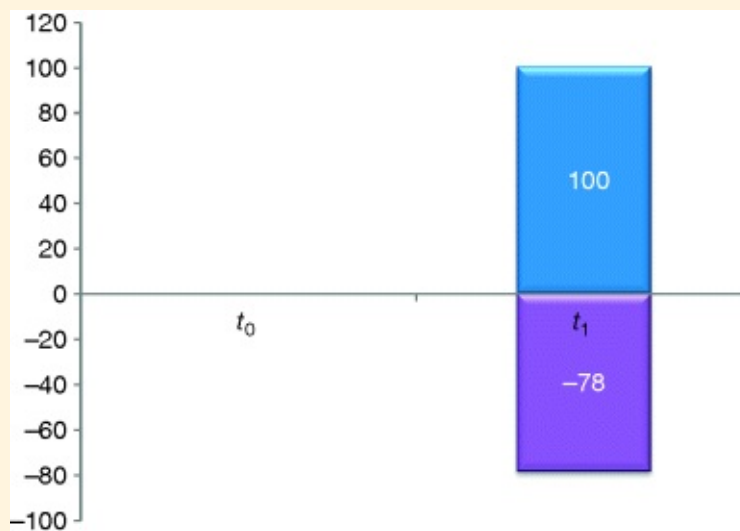
If we want to express the forward exchange rate as the number of USD per one EUR, we invert the exchange rate.

$$F_{t_0} = \frac{1}{0.78} = 1.3$$

Finally, we need to exchange the EUR holdings back to USD ([Fig. 2.22](#)).

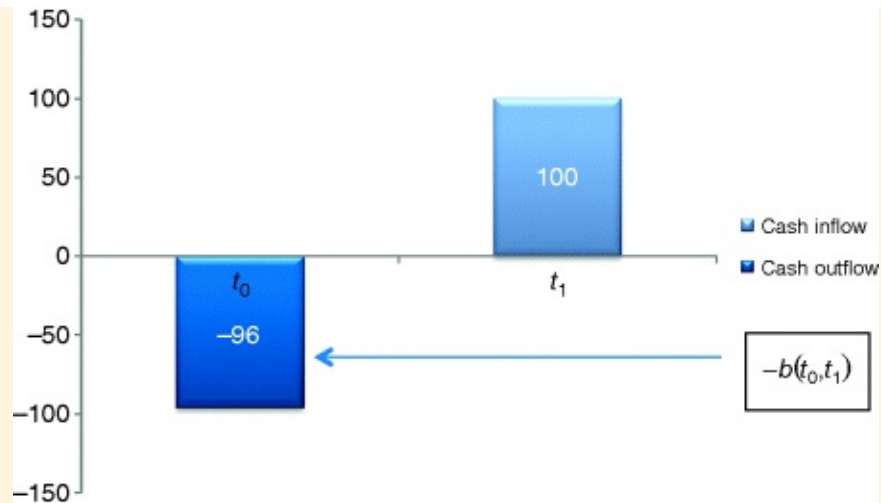
$$b(t_1, t_1) = b(t_1, t_1)^{EUR} \times F_{t_0}$$

$$b(t_1, t_1) = 78 \times 1.3 = 100$$



**FIGURE 2.22** Forward Contract: Selling EUR Against USD

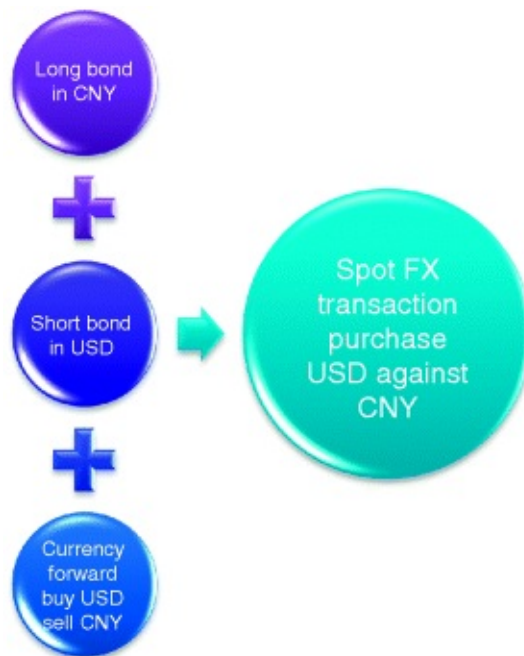
Adding the cash flows in [Figs. 2.20–2.22](#) vertically replicates a USD-denominated zero-coupon bond ([Fig. 2.23](#)).



**FIGURE 2.23** Long USD-Denominated Zero-Coupon Bond

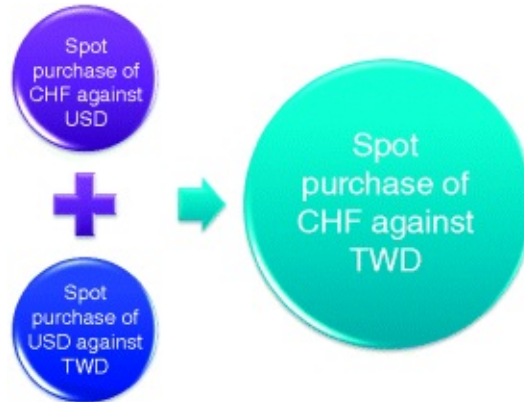
### Capital Controls

Another example where we can use the concept of currency forward and its synthetic is in a situation of imposed capital controls. If capital controls are imposed, spot transactions are not permitted. Thus, we can use the known contractual equation for currency forward and manipulate it so that we create the needed spot transaction synthetically.

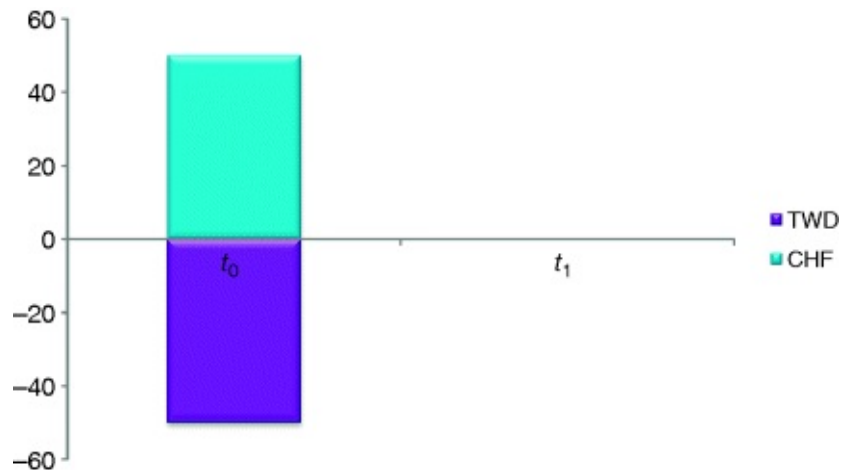


## Cross Currencies

Cross-currency operations are generally spot market transactions. We'd like to buy Swiss franc against Taiwanese dollar. Suppose that this transaction is currently prohibited due to capital controls. We can, however, get around this by creating a synthetic. It will consist of a spot purchase of easily available currency such as USD or EUR against TWD plus a spot purchase of CHF using USD (EUR).

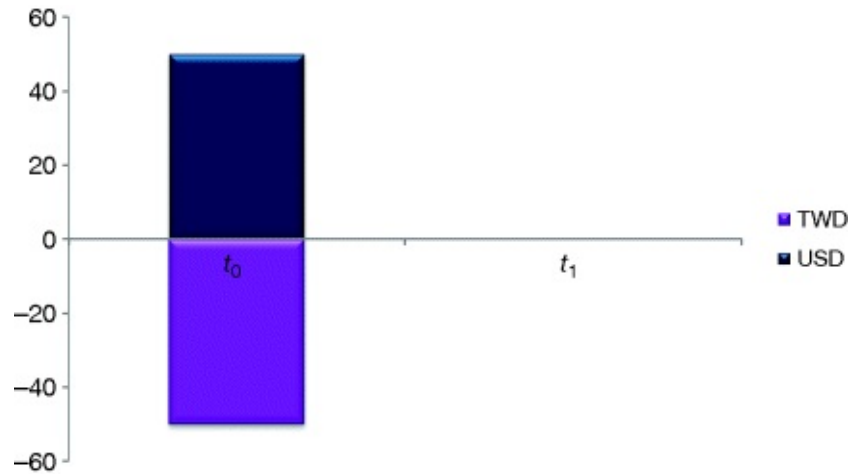


Let us show the appropriate positions in a set of cash flow diagrams. Creating the position synthetically requires a combination of two spot transactions. [Figure 2.24](#) shows the trade that the counterparties would like to do but are unable to due to capital controls.

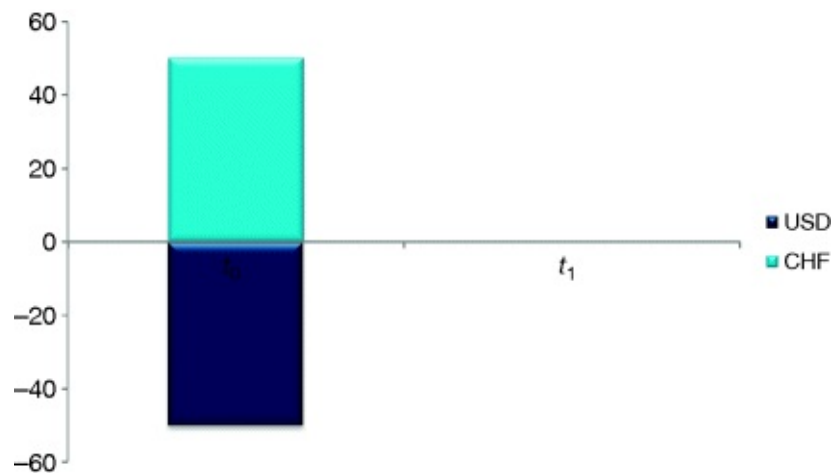


**FIGURE 2.24** Spot Purchase of CHF Against TWD

[Figure 2.25](#) shows a spot transaction of buying USD against TWD, and [Fig. 2.26](#) shows a spot transaction of buying CHF against USD.



**FIGURE 2.25** Spot Purchase of USD Against TWD



**FIGURE 2.26** Spot Purchase of CHF Against USD

Adding [Figs. 2.25](#) and [2.26](#) vertically results in the desired position of exchanging TWD for CHF.

## Problems

**2.3-1.** Suppose we have been quoted the following spot and forward rates.

	Spot Exchange Rate	Forward Exchange Rate
USD/GBP	0.55	
GBP/USD		1.8094

The Libor rates for a single time period are quoted as  $L_{t_0}^{1,USD} = 3\%$  and  $L_{t_0}^{1,GBP} = 3.5\%$ . Let us assume that  $\delta = 360/360$ . Please calculate the spot exchange rate for GBP/USD and the forward exchange rate for USD/GBP.

**2.3-2.** Please sketch cash flow diagrams reflecting a synthetic long bond denominated in USD. How would the cash flow diagrams change if we wanted to synthesize a short USD-denominated bond?

**2.3-3.** Please sketch cash flow diagrams reflecting a synthetic long bond denominated in GBP. Please use USD as the companion currency in both the currency forward and the spot FX transaction.

Suppose

	Spot Exchange Rate	Forward Exchange Rate
USD/GBP	0.64	0.65
GBP/USD	1.56	1.54

The Libor rates for a single time period are quoted as  $L_{t_0}^{1,USD} = 1.75\%$  and  $L_{t_0}^{1,GBP} = 2.5\%$ . Let us assume that  $\delta = 360/360$ .

**2.3-4.** Please sketch cash flow diagrams for a synthetic spot FX transaction—buying USD against CNY.

Adding these diagrams together vertically produces a spot transaction of selling CNY to purchase USD.

**2.3-5.** (Cross currencies) Please sketch cash flow diagrams reflecting a synthetic spot FX transaction of purchasing CHF against CNY. Assume that this transaction is currently prohibited due to capital controls.

**2.3-6.** (Currency forward) The spot exchange rate USD/EUR is 0.7 EUR per one USD. What is the EUR/USD rate? Suppose the one-year Libor rate on USD-denominated securities is 2.25%, and the one-year Libor on EUR securities is 3.50%. Using the interest rate parity, determine the forward exchange rates in EUR/USD and USD/EUR forms. Suppose we are purchasing a USD-denominated zero-coupon bond with par value of \$100. What is this bond's present



value? Using the above information, determine the future value of a short EUR-denominated bond. Also please calculate the present value of this EUR bond.

## Note

1. The daily cash flows associated with margining can skew futures prices, causing them to deviate from corresponding forward prices.

# 3

## SWAPS

### 3.1 Swaps Fundamentals

Swaps were first introduced to the public in 1981 when IBM and the World Bank entered into a swap. Today, swaps are the most heavily traded financial contracts in the world. The swap markets are enormous in size on the world scale. According to the Bank for International Settlements (BIS), the total amount of interest rates and currency swaps outstanding was around \$348 trillion in 2010. The whole derivatives market was over \$600 trillion in 2013.

In that it may be surprising that swaps as of now are still nonstandardized contracts and therefore are traded over the counter (OTC). Given the high frequency of contract structuring, in order to facilitate trading, market participants have developed the ISDA Master Agreement, which covers the legal terms of a swap contract, such as representations and warranties, and events of termination and default. Swaps are subject to credit risk. Other than that counterparties in a swap only need to negotiate the notional principal, interest rate or price, expiration, and collateral, if necessary. Interestingly, swap transactions are usually not disclosed on corporate balance sheets. Most recently proposed legislation changes may alter that.

Swaps are contract agreements in which one party agrees to make a series of regularly scheduled payments to another party and receive payments from the other party. In a swap, parties exchange assets, liabilities, currencies, securities, and commodities. There is a spectrum of complexity that swap structures may exhibit. An example of a rather simple one is a floating-for-fixed-rate swap. The most complex ones include multiple currencies, interest rates, commodities, and options. As is the case with OTC derivatives, swaps are flexible in terms of specifications such as timing or contractual horizons, settlement procedures, and so on.

Generally, swaps are used for risk management by relatively large institutions such as banks, brokers, dealers, and corporations. Individuals

are not excluded but have to be qualified in order to be permitted to participate in swap transactions. At the outset of a swap, the swap has value of zero. No cash changes hands between counterparties. The reason for the zero value is that the future anticipated flows of each of the two counterparties' cash flows exactly equal one another in their present value. Swap payments are settled on regularly scheduled settlement dates. Exchanging the difference between swap payments only is called netting the payment. The final payment coincides with the termination date of the swap contract (Fig. 3.1).



**FIGURE 3.1** List of Ways to Terminate a Swap

### 3.1.1 Interest Rate Swap

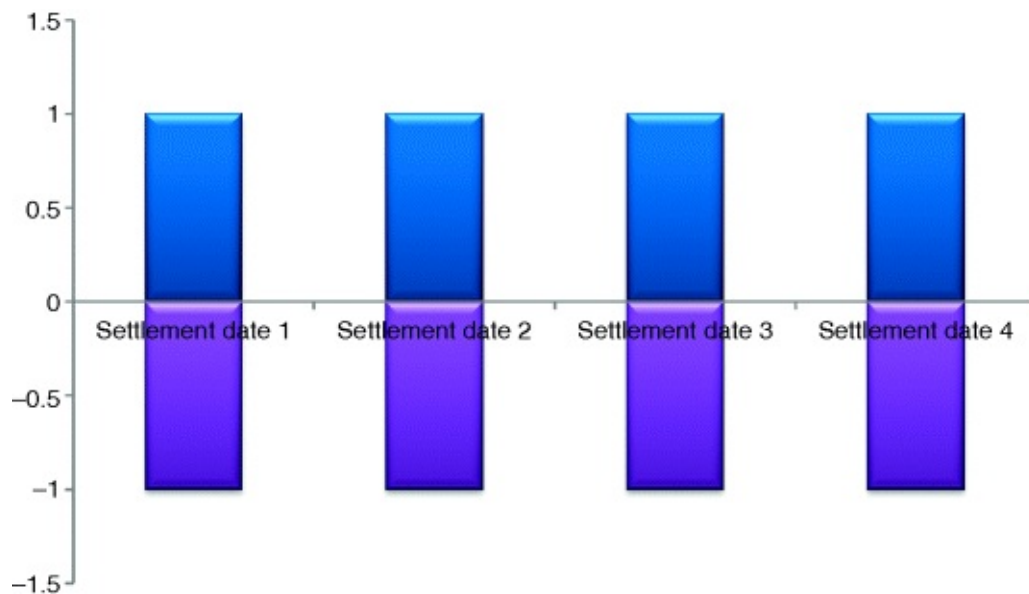
In an interest rate swap (IRS), investors enter into an agreement to swap interest rate payments. A plain vanilla swap typically refers to a generic interest rate swap in which one party pays a fixed rate and the other party pays a floating rate such as Libor. There are at least two counterparties to each swap. One of them pays a fixed swap rate in exchange for a floating rate. The other counterparty pays the floating rate and receives the fixed swap rate.

Swaps are valued on the basis of the present value of the cash flows both parties expect to receive and the present value of the payments they expect to make. At the origination of the contract, the value for both

parties is usually identical because both the payer leg and the receiver leg are of the same value. As the net value is zero at that time, no cash exchanges hands. Over the life of the contract, it becomes a zero-sum game. As interest rates fluctuate, the value of the swap creates a profit for one of the counterparties and a proportionate loss to the other. The amount, either a loss or a gain, is exchanged on *settlement dates*.

In this chapter, you'll learn how to value an IRS and to come up with the appropriate fixed swap rate. The appropriate swap rate will be such that equalizes the present values of both counterparties' streams of cash flows. Swap payments are made on settlement dates reflecting the most recent *settlement period*.

[Figure 3.2](#) shows a cash flow diagram for one of the counterparties to the swap. In this case, we have cash inflows, which are depicted above the x-axis, exactly equal to cash outflows, in the negative area below the x-axis.



**FIGURE 3.2** Interest Rate Swap Cash Inflows and Outflows

In reality, the two cash flows would not be exactly the same. Whereas the fixed payment would be fixed throughout the life of the swap, the floating payment would fluctuate according to market conditions.

The actual swap payments are netted at each settlement date. The payments are calculated as the difference between the fixed and floating rates, subsequently multiplied by the swap notional principal and its

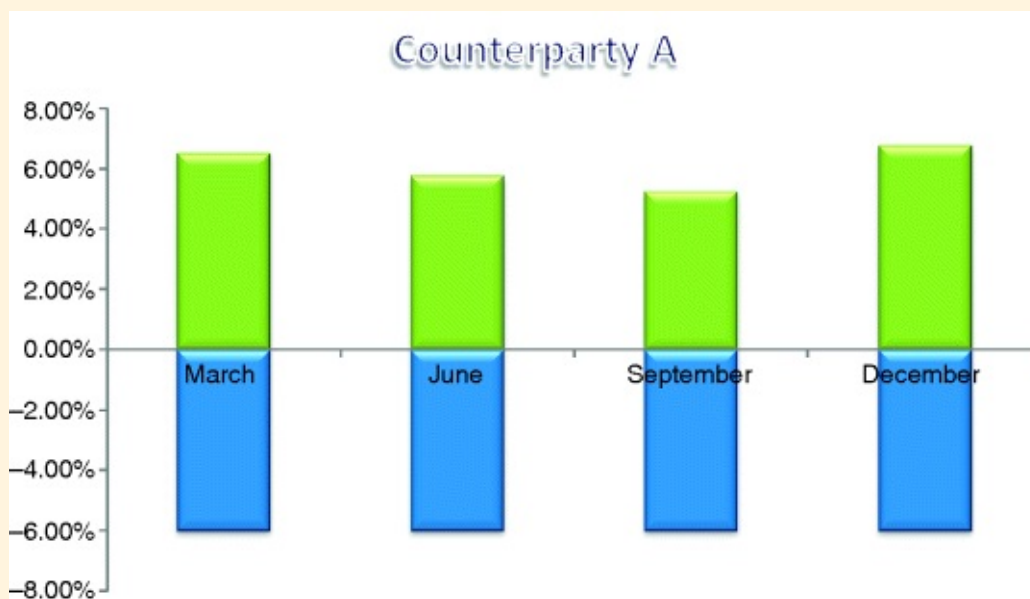
accrual period ( $\delta$ ).  $NP$  denotes notional principal.

Equation 3.1: Net swap payment at settlement dates

$$NetSwapPmt = NP \times (Float - Fix) \times \delta$$

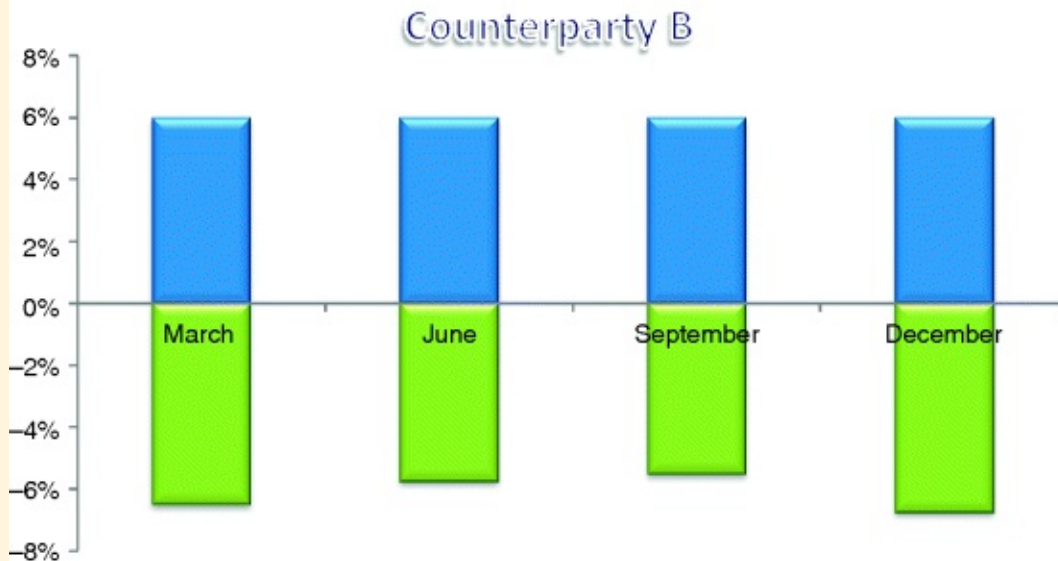
### Case Study 3.1.1-1

Suppose that counterparty A is a receiver of a floating rate, based on Libor, and in return it pays a fixed swap rate determined at the outset of the swap contract. In [Fig. 3.3](#), we depict this situation using a one-year swap with four three-month settlement periods. The fixed swap rate had been calculated as 6%. It stays constant throughout the life of the swap.



**FIGURE 3.3** IRS: Fixed Payer at 6% Against 10-Year Libor—Counterparty A

Counterparty B to the swap has a reversed position. Counterparty B receives fixed payments and pays floating rates ([Fig. 3.4](#)).



**FIGURE 3.4** IRS: 10-Year Libor Payer Against 6% Fixed—Counterparty B

Suppose that we have the following information in March:  
 $\delta = 90/360 = 0.25$ ;  $NP = \$100,000,000$ . And 3-month Libor at each settlement date reads as follows:

Libor Rate	
$L_{Dec}^{3m}$	6.5%
$L_{Mar}^{3m}$	5.5%
$L_{Jun}^{3m}$	5.75%
$L_{Sep}^{3m}$	6.25%

How large is the March net swap payment? Which counterparty, A or B, would receive this payment?

**Answer**

$$NetSwapPmt = 100,000,000 \times (0.065 - 0.06) \times (0.25) = 125$$

On the first settlement date in March, the floating rate was greater than the fixed rate benefiting counterparty A. Counterparty A receives the net swap payment of \$125,000. In the next two periods, it will be counterparty B that will be benefiting from the relatively lower interest rate environment and receiving a positive net swap payment.

## Swap Rate Determination

In order to value a swap, all we need to do is to find the appropriate swap rate for the fixed payments. There are essentially two ways to do this. One is the current term structure of interest rates and the other is the term structure of forward rates.

### Notation

$B(t_0, t_1)$	One-period discount factor
$L_{t_0}^1$	One-period Libor, observable at present
$\delta$	Accrual period
$F(t_0, t_0, t_1)$	One-period futures interest rate starting today
$F(t_0, t_1, t_2)$	One-period futures interest rate starting next period
$L_{t_0}^1 = F(t_0, t_0, t_1)$	One-period futures interest rate starting today is exactly equal to the currently observable one-period Libor rate
$L_{t_1}^1 \approx F(t_0, t_1, t_2)$	One-period futures interest rate known today is a good approximation for next period's Libor rate
$s_{t_0}$	Fixed swap rate

## Term Structure of Forward Rates

If we elect to work with the term structure of forward rates, we may need to determine them first. We can determine the forward rates by using discount factors for different time periods—all known at present.

### Equation 3.2: Discount factors

$$\begin{aligned} B(t_0, t_1) &= \frac{1}{1 + L_{t_0}^{t_1} \times \frac{t_1}{360}} \\ &\vdots \\ B(t_0, t_n) &= \frac{1}{1 + L_{t_0}^{t_n} \times \frac{t_n}{360}} \end{aligned}$$

### Equation 3.3: Relationship between futures rates and discount factors

$$\begin{aligned} 1 + F(t_0, t_0, t_1)\delta &= \frac{B(t_0, t_0)}{B(t_0, t_1)} \\ 1 + F(t_0, t_1, t_2)\delta &= \frac{B(t_0, t_1)}{B(t_0, t_2)} \\ 1 + F(t_0, t_2, t_3)\delta &= \frac{B(t_0, t_2)}{B(t_0, t_3)} \\ &\vdots \\ 1 + F(t_0, t_{n-1}, t_n)\delta &= \frac{B(t_0, t_{n-1})}{B(t_0, t_n)} \end{aligned}$$

In order to gain the present value of the fixed leg of the swap, we need to discount all future swap payments to present. The completely accurate result will appear if we discount with the actual future Libor rates for the appropriate time period.



$$\begin{aligned}
PV_{FIX} = & \frac{s_{t_0} \delta \times NP}{(1+L_{t_0}^{180} \delta)} + \frac{s_{t_0} \delta \times NP}{(1+L_{t_0}^{180} \delta) (1+L_{t_1}^{180} \delta)} + \frac{s_{t_0} \delta \times NP}{(1+L_{t_0}^{180} \delta) (1+L_{t_1}^{180} \delta)} \\
& + \frac{s_{t_0} \delta \times NP + NP}{(1+L_{t_0}^{180} \delta) (1+L_{t_1}^{180} \delta) (1+L_{t_2}^{180} \delta) \cdots (1+L_{t_{n-1}}^{180} \delta)}
\end{aligned}$$

Unfortunately, these Libor rates are not yet known at the present time. Thus, we need to use the second best tool, which in this case are the futures interest rates. Thus, the resulting present value is essentially an approximation obtained using futures data.

### Equation 3.4: Present value calculation of the fixed payment

$$\begin{aligned}
PV_{FIX} = & \frac{s_{t_0} \delta \times NP}{(1+F(t_0, t_0, t_1) \delta)} + \frac{s_{t_0} \delta \times NP}{(1+F(t_0, t_0, t_1) \delta) (1+F(t_0, t_1, t_2) \delta)} \\
& + \frac{s_{t_0} \delta \times NP}{(1+F(t_0, t_0, t_1) \delta) (1+F(t_0, t_1, t_2) \delta) (1+F(t_0, t_2, t_3) \delta)} + \cdots \\
& + \frac{s_{t_0} \delta \times NP + NP}{(1+F(t_0, t_0, t_1) \delta) (1+F(t_0, t_1, t_2) \delta) (1+F(t_0, t_2, t_3) \delta) \cdots (1+F(t_0, t_{n-1}, t_n) \delta)}
\end{aligned}$$

$$\begin{aligned}
PV_{FLOAT} = & \frac{L_{t_0}^{180} \delta \times NP}{(1+L_{t_0}^{180} \delta)} + \frac{L_{t_1}^{180} \delta \times NP}{(1+L_{t_0}^{180} \delta) (1+L_{t_1}^{180} \delta)} + \frac{L_{t_2}^{180} \delta \times NP}{(1+L_{t_0}^{180} \delta) (1+L_{t_1}^{180} \delta)} \\
& + \frac{L_{t_{n-1}}^{180} \delta \times NP + NP}{(1+L_{t_0}^{180} \delta) (1+L_{t_1}^{180} \delta) (1+L_{t_2}^{180} \delta) \cdots (1+L_{t_{n-1}}^{180} \delta)}
\end{aligned}$$

Simplify

$$PV_{FLOAT} = NP$$

However, once again, at present ( $t_0$ ) none of the future one-period Libor rates are actually known. We need to approximate them with our best guesses—the forward rates.

Equation 3.5: Present value of the floating payment leg of the swap

In order to have both counterparties entering the contract with equal chances of making positive gains, the present values of both the floating and the fixed leg of the swap have to be equal.

Let us first simplify both the present value of the fixed leg and the present value of the floating leg of the swap.

$$PV_{FIX} = (s_{t_0} \delta \times NP) B(t_0, t_1) + (s_{t_0} \delta \times NP) B(t_0, t_2) + (s_{t_0} \delta \times NP) B(t_0, t_n) + NP \times B(t_0, t_n)$$

Simplify

Equation 3.6: Present value of the fixed leg

$$PV_{FIX} = (s_{t_0} \delta \times NP) \times \sum_{i=1}^n B(t_0, t_i) + NP \times B(t_0, t_n)$$

$$PV_{FLOAT} = (F(t_0, t_0, t_1)\delta \times NP)B(t_0, t_1) + (F(t_0, t_1, t_2)\delta \times NP)B(t_0, t_2) + \dots + (F(t_0, t_{n-1}, t_n)\delta \times NP)B(t_0, t_n) + NP \times B(t_0, t_n)$$

Simplify

Equation 3.7: Present value of the floating leg

$$PV_{FLOAT} = (\delta \times NP) \times \sum_{i=1}^n F(t_0, t_{i-1}, t_i) B(t_0, t_i) + NP \times B(t_0, t_n)$$

Set  $PV_{FIX} = PV_{FLOAT}$ .

Solve for the fixed swap rate ( $s_{t_0}$ ).

$$\underbrace{(s_{t_0} \delta \times NP) \times \sum_{i=1}^n B(t_0, t_i) + NP \times B(t_0, t_n)}_{FIX} = \underbrace{(\delta \times NP) \times \sum_{i=1}^n F(t_0, t_{i-1}, t_i) B(t_0, t_i) + NP \times B(t_0, t_n)}_{FLOAT}$$

Subtract the discounted notional principal from both sides of the equation.

$$(s_{t_0} \delta \times NP) \times \sum_{i=1}^n B(t_0, t_i) = (\delta \times NP) \times \sum_{i=1}^n F(t_0, t_{i-1}, t_i) B(t_0, t_i)$$

Divide both sides by the product of delta and the swap notional principal.

$$s_{t_0} \times \sum_{i=1}^n B(t_0, t_i) = \sum_{i=1}^n F(t_0, t_{i-1}, t_i) B(t_0, t_i)$$

Solve for the fixed swap rate.

Equation 3.8: Swap rate calculation using futures rates

Term Structure of Current Interest Rates

$$PV_{FLOAT} = \frac{NP \times (1 + L_{t_0}^{180} \delta)}{(1 + L_{t_0}^{180} \delta)}$$

Equation 3.9 : Present value of the floating leg

$$PV_{FLOAT} = NP$$

Equation 3.10 : Present value of the fixed leg

$$PV_{FIX} = (s_{t_0} \delta \times NP) \times \sum_{i=1}^n B(t_0, t_i) + NP \times B(t_0, t_n)$$

$$PV_{FLOAT} = PV_{FIX}$$

Solve for  $s_{t_0}$ .

$$NP = (s_{t_0} \delta \times NP) \times \sum_{i=1}^n B(t_0, t_i) + NP \times B(t_0, t_n)$$

$$NP = NP \times \left( s_{t_0} \delta \times \sum_{i=1}^n B(t_0, t_i) + B(t_0, t_n) \right)$$

$$1 = 1 \times \left( s_{t_0} \delta \times \sum_{i=1}^n B(t_0, t_i) + B(t_0, t_n) \right)$$

$$1 - B(t_0, t_n) = s_{t_0} \delta \times \sum_{i=1}^n B(t_0, t_i)$$

Equation 3.11 : Swap rate based on discount factors

$$s_{t_0} = \frac{1 - B(t_0, t_n)}{\delta \times \sum_{i=1}^n B(t_0, t_i)}$$

Equation 3.12 : Current swap value

$$V_{SWAP} = PV_{FLOAT} - PV_{FIX}$$

### Case study 3.1.1-2

Suppose we would like to price a plain vanilla interest rate swap for a firm that is going to pay fixed and receive floating payments from its swap counterparty. This swap will run for two years. The notional principal on the swap is \$10,000,000. The payments will be exchanged every six months so that the accrual period will be equal to  $\delta = 180/360$ . We will assume 30-day month in our calculations. Calculate the fixed swap rate. What is the value of the swap at present?

Let the current yield curve read the following.

Libor Rate	
$L_{t_0}^{180}$	1%
$L_{t_0}^{360}$	1.5%
$L_{t_0}^{540}$	2%
$L_{t_0}^{720}$	2.2%

### Answer

This yield curve produces the following discount factors reflecting today's value of a dollar received in the future.

$$B(t_0, t_1) = \frac{1}{1 + L_{t_0}^{180} \times \left(\frac{t_1}{360}\right)} = \frac{1}{1 + 0.01 \times \frac{180}{360}} = 0.9950$$

$$B(t_0, t_2) = \frac{1}{1 + L_{t_0}^{360} \times \left(\frac{t_2}{360}\right)} = \frac{1}{1 + 0.015 \times \frac{360}{360}} = 0.9852$$

$$B(t_0, t_3) = \frac{1}{1 + L_{t_0}^{540} \times \left(\frac{t_3}{360}\right)} = \frac{1}{1 + 0.02 \times \frac{540}{360}} = 0.9709$$

$$B(t_0, t_4) = \frac{1}{1 + L_{t_0}^{720} \times \left(\frac{t_4}{360}\right)} = \frac{1}{1 + 0.022 \times \frac{720}{360}} = 0.9579$$

Fixed swap rate for this swap will be equal to

$$s_{t_0} = \frac{1 - B(t_0, t_n)}{\delta \times \sum_{i=1}^n B(t_0, t_i)} = \frac{1 - 0.9579}{0.5 \times 3.909} = 0.0216$$

The appropriate swap rate for this two-year swap is 2.16%.

The value of this swap at present is zero. At the beginning of a swap's life, the value is always zero by definition. You can, however, verify this result by calculating the value of the fixed and floating legs of the swap and comparing them. If the values are exactly equal, then the swap value is nil.

$$PV_{FLOAT} = NP = 1 \times 10,000,000 = 10,000,000$$

$$\begin{aligned} PV_{FIX} &= (s_{t_0} \delta \times NP) \times \sum_{i=1}^n B(t_0, t_i) + NP \times B(t_0, t_n) \\ &= 0.0216 \times 0.5 \times 10,000,000 \times 3.909 + 10,000,000 \\ &= 1 \times 10,000,000 = 10,000,000 \end{aligned}$$



$$V_{SWAP} = 10,000,000 - 10,000,000 = 0$$

### *x* Days into the Life of the Swap

Now we have verified that the value of a swap at its outset is indeed zero. What about other times? What is the value of an IRS two weeks into the life of it? Or two months into the life of the swap? Well, the mechanics of the value calculations are essentially the same as we have just used to calculate the value at the outset. However, once we move forward in time, we have to be careful about using the relevant information for our valuation calculations.

We need to recalculate the values of the appropriate discount factors. We will be using information from the relevant yield curve at the time of the valuation. Assuming that the yield curve changed, we would use the new interest rate numbers and also shorten their tenure as we are now closer to the end of the swap life.

Equation 3.13 : Discount factors *x* days into the swap life

$$\begin{array}{l} B(t_0 + x, t_1 - x) = \frac{1}{1 + L_{(t_0+x)}^{(t_1-x)} \times \left(\frac{t_1-x}{360}\right)} \\ \vdots \\ B(t_0 + x, t_n - x) = \frac{1}{1 + L_{(t_0+x)}^{(t_n-x)} \times \left(\frac{t_n-x}{360}\right)} \end{array}$$

### Case Study 3.1.1-3

Assume that our initial information is identical to Case Study 3.1.1-2.

What is the value of the swap three months into the life of the swap?  
 Let the new yield curve at 90 days into the life of the swap read

Libor Rate at 90 Days into Life	
$L_{90}^{90}$	0.9%
$L_{90}^{270}$	1.6%
$L_{90}^{450}$	1.9%
$L_{90}^{630}$	2.1%

### Answer

present											
Libor	time 0	Future Time	Time Fraction of a Year	Discount Factor - B(t0,ti)	Accrual Period - Delta	Notional Principal	Fixed Swap Rate	Fixed Swap Payment	Value of Fixed Pmts	Value of Floating Pmts	Value of the Swap (Pay Fix Receive Float)
180-day	1.0%	180	0.5	0.9950							
360-day	1.5%	360	1	0.9852							
540-day	2.0%	540	1.5	0.9709							
720-day	2.2%	720	2	0.9579							
				Sum							
				3.9090	0.5	\$ 10,000,000.00	2.16%	\$ 107,817.51	\$ 10,000,000.00	\$ 10,000,000.00	\$ 0.000000000
90 days into the life of the swap											
Libor	time 90	Future Time	Time Fraction of a Year	Discount Factor - B(90,ti)							
90-day	0.9%	90	0.25	0.9978							
270-day	1.6%	270	0.75	0.9881							
450-day	1.9%	450	1.25	0.9768							
630-day	2.1%	630	1.75	0.9646							
				Sum							
				3.9273					1.006895333	1.002743826	-0.0041515063
									\$ 10,068,953.33	\$ 10,027,438.26	\$ (41,515.06)

The value of this interest rate swap at 90 days into the life of the instrument is \$(41,515.06) for the counterparty that pays fixed and receives floating.

present											
Libor	time 0	Future Time	Time Fraction of a Year	Discount Factor - B(t0,ti)	Accrual Period - Delta	Notional Principal	Fixed Swap Rate	Fixed Swap Payment	Value of Fixed Pmts	Value of Floating Pmts	Value of the Swap (Pay Fix Receive Float)
180-day	0.01	180	=C3/360	=1/(1+B3^D3)							
360-day	0.015	360	=C4/360	=1/(1+B4^D4)							
540-day	0.02	540	=C5/360	=1/(1+B5^D5)							
720-day	0.022	720	=C6/360	=1/(1+B6^D6)							
			Sum	=SUM(F3:F6)	=180/360	1000000			=B3*B4*SUM(F3:F6)+F6	=1/(1+B3^D3)*(1+B3^D3)	=(O7-N7)
									=N7^S58	=O7^S58	=(O8-N8)
90 days into the life											
Libor	time 90	Future Time	Time Fraction of a Year	Discount Factor - B(90,ti)							
90-day	0.009	=C3-90	=C12/360	=1/(1+B12^D12)							
270-day	0.016	=C4-90	=C13/360	=1/(1+B13^D13)							
450-day	0.019	=C5-90	=C14/360	=1/(1+B14^D14)							
630-day	0.021	=C6-90	=C15/360	=1/(1+B15^D15)							
			Sum	=SUM(I12:I15)					=B12*B13*SUM(I12:I15)+I15	=1/(1+B12^D12)*(1+B12^D12)	=(O16-N16)
									=N16^S58	=O16^S58	=(O17-N17)

## Problems

3.1.1-1. Please refer back to Case Study 3.1.1-1. Please calculate the

net swap payments for the remaining three settlement periods.

**3.1.1-2.** Two parties enter a three-year, plain vanilla interest rate swap agreement to exchange the Libor rate for a 10% fixed rate on \$10 million. Libor is 11% now, 12% at the end of the first year, and 9% at the end of the second year.

What are the net cash flows to be received by the fixed-rate receiver?

**3.1.1-3.** Notional amount = \$1 million, and payments are made semiannually. The corporation will pay a floating rate of three-month Libor, which is 3%, and will receive a fixed payment of 3.5%. What is the correct floating rate payment in \$? What is the size of the fixed rate payment in \$? How much is the net payment on the first payment date? Is it a net gain or loss for this corporation?

**3.1.1-4.** Consider your Excel model for Case Study 3.1.1-3. How would you have to change the yield curve inputs should you want to arrive at a positive value for this swap 90 days into its life? Please consider the value of the swap for the counterparty that pays fix and receives floating.

## 3.2 Equity, Currency, And Fx Swaps

### 3.2.1 Equity Swaps

An equity swap is an arrangement between two counterparties to exchange a set of payments. The payments are determined usually by a stock index return but can be based on an individual stock return as well. The counterparty makes payments, most commonly, using the proceeds of a fixed or floating rate debt instrument, but payments can also be based on the return on another stock or index. Equity swaps are used to substitute for a direct transaction in stock. Just like the case with interest rate swaps, the difference in payments is netted. A total return swap includes both capital gains and current income on an underlying stock or index. No principal is exchanged.

At the outset of the contract, the value of the swap is zero. This holds true no matter if we are dealing with a swap exchanging equity return against a fixed payment, or an equity return-based payment against a

floating payment.

An equity or index return can be calculated as a ratio of two consecutive prices less 1.

Equation 3.14 : Rate of return on equity

$$r_{t_1} = \left( \frac{S_{t_1}}{S_{t_0}} \right) - 1$$

The value of the equity leg in an equity swap is obtained by adding 1 to the rate of return on equity.

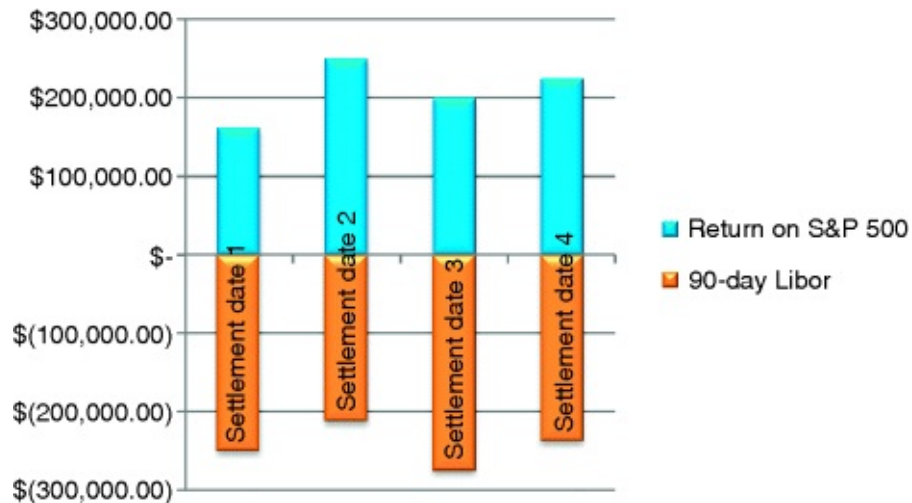
Equation 3.15 : Value of the equity leg of an equity swap

$$V_{EQUITY} = \frac{S_t}{S_{t_0}}$$

Suppose we'd like to pay the 90-day Libor rate and receive the return on the S&P 500 index. In the case of the equity return against a floating payment, we end up essentially exchanging two floating payments because neither of them is set and known at the beginning of the life of the swap ([Fig. 3.5](#)).

Equation 3.16 : Value of a floating leg of a swap

$$V_{FLOAT} = \left( 1 + L_{t_0}^{t_1} \times \delta \right) \times B(t_0, t_1)$$



**FIGURE 3.5** Equity Swap: Floating Against Floating

In the event of a swap between fixed swap rate payments and equity return, the value of the fixed swap leg is the following.

Equation 3.17 : Value of a fixed leg of a swap

$$V_{FIX} = s_{t_0} \times \delta \times \sum_{i=1}^n B(t_0, t_i) + B(t_0, t_n)$$

Equation 3.18 : Floating payment calculation on any settlement date

$$Pmt_{FLOAT}^{SDi} = NP \times \delta \times L_{t_{i-1}}$$

Equation 3.19 : Equity payment calculation on any settlement date

$$Pmt_{EQUITY}^{SDi} = NP \times (r_{t_i} + d_{t_i})$$

where  $d_{t_i}$  is the dividend on the stock or index at time  $i$ . Assume  $\delta = 90/360$ .

In equity swap exchanging fixed payments for the equity return, we'd have to first determine the appropriate fixed swap rate. The fixed payment is then a multiple of the notional principal, accrual period, and the fixed swap rate.

Equation 3.20 : Fixed payment calculation at any settlement

Equation 3.21 : Net swap payment : equity receiver, fix payer

$$NetSwapPmt^{SDi} = Pmt_{EQUITY} - Pmt_{FIX}$$

Equation 3.22 : Equity swap value : fix payer, equity receiver

$$V_{SWAP} = (V_{EQUITY} - V_{FIX}) \times NP$$

### Case Study 3.2.1-1

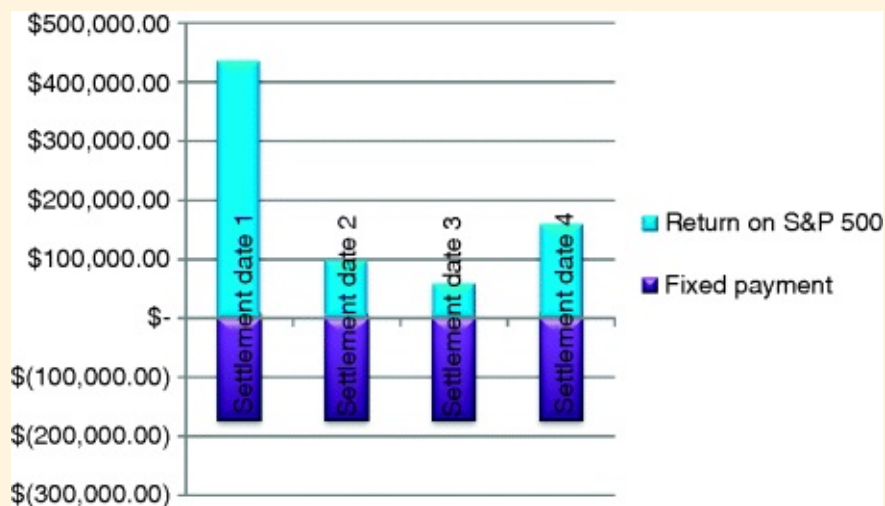
Let us consider an example where counterparty A is paying a fixed swap rate and receiving a return on the S&P 500 index. The accrual period is three months so that  $\delta = 90/360$ . The fixed swap rate offered by the swap dealer is 3.45%. Assume that the notional

principal is \$20,000,000.

The swap will go on for one year and have four settlement dates. Please calculate the net swap payment on the first settlement date. Assume that the S&P index was at 1100 at the outset of the first settlement period, and 1124 at the end of the first period.

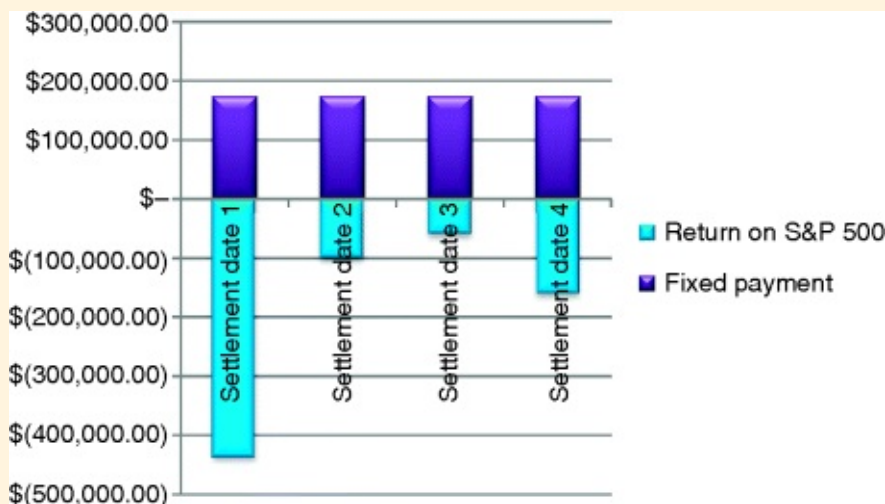
**Answer**

Suppose these are the cash flows observed for this swap. In this example, we will concentrate on the first settlement date only. [Figure 3.6](#) shows the cash flows for counterparty A.



**FIGURE 3.6 Counterparty A: Cash Flows**

[Figure 3.7](#) depicts the cash flows of counterparty B in this swap. Counterparty B is receiving payments based on a fixed swap rate and paying equity return-based payments.



### FIGURE 3.7 Counterparty B: Cash Flows

The index return can be calculated as follows:

$$r_{t_1} = \left( \frac{1124}{1100} \right) - 1 = 0.0218$$

In order to obtain the swap floating payments, we multiply the notional principal by the equity return. Here the equity return is covering the time of the first settlement period, so we don't have to adjust the calculation by multiplying it by the accrual period  $\delta$ .

$$Pmt_{EQUITY}^{SD1} = 20,000,000 \times 0.0218 = 436,363.64$$

$$Pmt_{FIX}^{SD1} = 20,000,000 \times 0.25 \times 0.0345 = 172,500.00$$

The value of the swap is the difference between the two payments. If we are looking at the problem from the point of view of counterparty A—fixed payer and floating equity receiver—we experience the following net payment at settlement date 1.

$$NetSwapPmt^{SD1} = 436,363.64 - 172,500.00 = 263,863.64$$



On the first settlement date of this swap, the payoff is positive for counterparty A at \$263,863.64. Reflecting back on the initial cash flow diagrams, we see that subsequent settlement dates favor counterparty B. Discuss why that is so.

## Pricing an Equity Swap

### Case Study 3.2.1-2

Suppose BuMos is a company that is entering into a swap receiving fixed payments and paying a short-term bond index-based return. The swap will go on for one year, with four settlement dates. The accrual period is 90/360, and the notional principal is \$20,000,000.00. Is the value of the swap at the beginning of its life equal to zero? Why is that? Show numerically.

The short end of the Libor yield curve shows the following information at the beginning of this swap.

Libor	
$L_{t_0}^{90}$	1.6%
$L_{t_0}^{180}$	1.7%
$L_{t_0}^{270}$	1.9%
$L_{t_0}^{360}$	2.3%

#### **Answer**

Suppose that instead of the fixed swap rate we are given only information from the current yield curve. Thus, we have to calculate the appropriate swap rate first. The formula is identical to the one used in calculating swap rates for interest rate swaps.

We will use the yield curve numbers to calculate the appropriate discount factors.

$$B(t_0, t_1) = \frac{1}{1 + L_{t_0}^{90} \times \left(\frac{t_1}{360}\right)} = \frac{1}{1 + 0.016 \times 0.25} = 0.9960$$

$$B(t_0, t_2) = \frac{1}{1 + L_{t_0}^{180} \times \left(\frac{t_2}{360}\right)} = \frac{1}{1 + 0.017 \times 0.5} = 0.9916$$

$$B(t_0, t_3) = \frac{1}{1 + L_{t_0}^{270} \times \left(\frac{t_3}{360}\right)} = \frac{1}{1 + 0.019 \times 0.75} = 0.9860$$

$$B(t_0, t_4) = \frac{1}{1 + L_{t_0}^{360} \times \left(\frac{t_4}{360}\right)} = \frac{1}{1 + 0.023 \times 1} = 0.9775$$

Calculate the fixed swap rate.

$$s_{t_0} = \frac{1 - B(t_0, t_4)}{\delta \times \sum_{i=1}^4 B(t_0, t_i)} = \left( \frac{1 - 0.9775}{0.25 \times (0.996 + 0.9916 + 0.986 + 0.9775)} \right)$$

The fixed swap rate for this equity swap is 2.28%.

$$V_{FIX}^{SD0} = (0.0228 \times 0.25 \times (0.996 + 0.9916 + 0.986 + 0.9775))$$

$$V_{EQUITY}^{SD0} = \left( \frac{1100}{1100} \right) = 1$$

$$V_{SWAP} = (1 - 1) \times 20,000,000 = 0$$

## Valuing an Equity Swap During Its Life

### Case Study 3.2.1-3

Let us now examine the way that the swap value changes during the life of the swap. We'll use an example where we assume that we are now 60 days into the life of the swap. That means that we still have 30 days left until reaching the first settlement date of the swap. We will build on the data obtained in Case Study 3.2.1-2.

The value of the equity leg of the swap at 60 days into its life is  $S_{60} = 1124$ . Recall that at the outset of the swap it was  $S_{t_0} = 1100$ . The current Libor curve shows the following information.

Libor	
$L_{60}^{30}$	1.7%
$L_{60}^{120}$	1.8%
$L_{60}^{210}$	2.2%
$L_{60}^{300}$	2.4%

**Answer**

The discount factors change accordingly.

$$B(60,30) = \frac{1}{1 + L_{60}^{30} \times \left(\frac{90-60}{360}\right)} = \frac{1}{1 + 0.017 \times 0.0833} = 0.99$$

$$B(60,120) = \frac{1}{1 + L_{60}^{120} \times \left(\frac{180-60}{360}\right)} = \frac{1}{1 + 0.018 \times 0.333} = 0.99$$

$$B(60,210) = \frac{1}{1 + L_{60}^{210} \times \left(\frac{270-60}{360}\right)} = \frac{1}{1 + 0.022 \times 0.5833} = 0.99$$

$$B(60,300) = \frac{1}{1 + L_{60}^{300} \times \left(\frac{360-60}{360}\right)} = \frac{1}{1 + 0.024 \times 0.833} = 0.9772$$

The value of the fixed leg of the swap at 60 days into the life of the swap is

$$V_{FIX} = 0.0228 \times 0.25 \times (0.9986 + 0.994 + 0.9873 + 0.9804) = 0.2112$$

The value of the equity leg of the swap at 60 days into its life is

$$V_{EQUITY} = \frac{1124}{1100} = 1.02182$$

The value of the swap 60 days into the life of the swap turns out to be negative for BuMos. Its counterparty in the swap would be making the same amount that BuMos would have lost.

$$V_{SWAP} = (1.00293 - 1.02182) \times 20,000,000 = -377,805.63$$

Case Study 3.2.1-4

What is the value of this swap 60 days into its existence if the swap is constructed as a floating interest rate receiver and equity return payer? Everything else is the same as in the previous cases.

**Answer**

The value of the floating leg of the swap at 60 days into the life of the swap is

$$V_{FLOAT} = (1 + 0.016 \times 0.25) \times 0.9986 = 1.00258$$

$$V_{SWAP} = (V_{FLOAT} - V_{EQUITY}) \times NP$$

$$V_{SWAP} = (1.00258 - 1.02182) \times 20,000,000 = -384,770.06$$

### Case Study 3.2.1-5

Given that in most cases equity swaps are constructed in such a way that counterparties are swapping actual stock price index, such as S&P 500, return with a fixed or floating rate based on Libor rates. Libor rates are generally on average lower than the stock market return. Therefore, we include a fixed percentage adjustment to the swap usually added to the fixed/floating payment.

Thus, what would be the value of a fixed to equity swap 60 days into its life if we structure it as receiving the fixed payment plus 200 basis points (bps) against paying equity index return? How would this

change if the swap we are considering is receiving floating rate plus 200 bps against paying the equity index return? Find your answer using Excel spreadsheet.

### Answer

Present										Value of Fixed Pmts	Value of Floating Pmts	Stock Index Return	Value of the Equity Swap (receive fx pay eq)	Value of Equity Swap (receive fl pay eq)
Libor	time 0	Future Time	Time Fraction of a Year	delta	Discount Factor	Stock Index Values	Notional Principal	Fixed Swap Rate						
90-day	0.016	90	0.25		0.996016									
180-day	0.017	180	0.5		0.991572									
270-day	0.019	270	0.75		0.98595									
360-day	0.023	360	1		0.977517									
				0.25		1100	\$ 20,000,000.00	2.28%	1	1	1	0	0	
60 days into the life of the swap										Value of Fixed Pmts	Value of Floating Pmts	Stock Index Return	Value of the Equity Swap (receive fx pay eq)	Value of Equity Swap (receive fl pay eq)
Libor	time 60													
30-day	0.017	30	0.083333333		0.998585									
120-day	0.018	120	0.333333333		0.994036									
210-day	0.022	210	0.583333333		0.987329									
300-day	0.024	300	0.833333333		0.980392									
				0.25		1124			1.002928	1.002579679	1.021818182	\$ (377,805.63)	\$ (384,770.06)	
									Plus 200 Bps			\$ 22,194.37	\$ 15,229.94	
									1.002928	1.002579679	1.021818182	\$ 22,194.37	\$ 15,229.94	

Present										Value of Fixed Pmts	Value of Floating Pmts	Stock Index Return	Value of the Equity Swap (receive fx pay eq)	Value of Equity Swap (receive fl pay eq)
Libor	time 0	Future Time	Time Fraction of a Year	delta	Discount Factor	Stock Index Values	Notional Principal	Fixed Swap Rate						
90-day	0.016	90	=C3/\$C\$6		=1/(1+B3*D3)									
180-day	0.017	180	=C4/\$C\$6		=1/(1+B4*D4)									
270-day	0.019	270	=C5/\$C\$6		=1/(1+B5*D5)									
360-day	0.023	360	=C6/\$C\$6		=1/(1+B6*D6)									
				=D5/3		1100	20000000	=1/E7*(1-F6)/(SUM(F3:F6))	=S15*\$E57*SUM(F3:F6)*(1-S1853*\$D53)	=G7/G7		=(L7-J7)*\$H57	=(K7-L7)*\$H57	
60 days into the life of the swap										Value of Fixed Pmts	Value of Floating Pmts	Stock Index Return	Value of the Equity Swap (receive fx pay eq)	Value of Equity Swap (receive fl pay eq)
Libor	time 60													
30-day	0.017	30	=C10/\$C\$6		=1/(1+B10*D10)									
120-day	0.018	120	=C11/\$C\$6		=1/(1+B11*D11)									
210-day	0.022	210	=C12/\$C\$6		=1/(1+B12*D12)									
300-day	0.024	300	=C13/\$C\$6		=1/(1+B13*D13)									
				=D5/3		1124			=(S15*\$E57*SUM(F1:F10)*(1-S1853*\$D53)	=(G14/G7)		=(L14-L10)*\$H57	=(K14-L10)*\$H57	
									Plus 200 Bps			=(L16-L10)*\$H57	=(K16-L10)*\$H57	
									=J14+0.02	=K14+0.02	1.021818181818	=(L16-L10)*\$H57	=(K16-L10)*\$H57	

Once we add 200 bps, the value of this equity swap that is paying equity and receiving a fixed rate turns positive for the party receiving fixed. The gain is \$22,194.37.

Considering an equity swap that pays equity return and receives the Libor floating rate; this swap will also benefit. The gain here is \$15,229.94.

## Problems

**3.2.1-1.** A fund manager will pay the broker/dealer the return of the S&P 500 and will receive an interest payment of 5% every six months.

Notional principal amount = \$1 million. Payments are made semiannually (i.e.,  $\delta = 180/360$ ). Index is at 10,500 at the start of

the swap. Six months from now, the index is at 10,700.

What is the net payment the fund receives at the end of the first six months?

**3.2.1-2.** Suppose counterparty to a swap is paying a fixed swap rate and receiving a return on the S&P 500 index. The accrual period is three months so that  $\delta = 90/360$ . The swap will go on for one year.

The fixed swap rate offered by the swap dealer is 3.05%. Assume that the notional principal is \$50,000,000. The S&P index was at 1210 at the outset of the first settlement period, and 1240 at the end of the first period.

Please calculate the net swap payment on the first settlement date.

**3.2.1-3.** Suppose counterparty to a swap is paying a fixed swap rate and receiving a return on an index. The accrual period is six months so that  $\delta = 180/360$ . The swap will go on for one year.

The fixed swap rate offered by the swap dealer is 13%. Assume that the notional principal is \$100,000,000. The S&P index was at 1210 at the outset of the first settlement period, and 1300 at the end of the first period.

Please calculate the net swap payment on the first settlement date.

**3.2.1-4.** Suppose that SSI Investments, LLC is entering into a swap receiving a short-term bond index-based return and paying fixed swap rate. The swap will go on for one year, with four settlement dates. The accrual period is 90/360, and the notional principal is \$10,000,000.00.

The short end of the Libor yield curve currently shows the following information.

Libor	
$L_{t_0}^{90}$	0.6%
$L_{t_0}^{180}$	0.7%
$L_{t_0}^{270}$	1.3%
$L_{t_0}^{360}$	1.5%



Please show numerically that the swap value is zero at the outset of the contract. You may use Excel spreadsheet for your calculations.

**3.2.1-5.** What is the value of the following equity swap at time 0 and 50 days into the life of the swap? Note that at 50 days into the life of the swap we still have 40 days left until reaching the first settlement date.

The value of the equity leg of the swap at 50 days into its life is  $S_{50} = 1103$ . At the outset of the swap, the index value was  $S_{t_0} = 1100$ . The current Libor curve shows the following information.

Libor at $t_{50}$	
$L_{50}^{40}$	1.8%
$L_{50}^{130}$	1.85%
$L_{50}^{220}$	1.9%
$L_{50}^{310}$	2%

The short end of the Libor yield curve shows the following information at the beginning of this swap.

Libor at $t_0$	
$L_{t_0}^{90}$	1.6%
$L_{t_0}^{180}$	1.7%
$L_{t_0}^{270}$	1.8%
$L_{t_0}^{360}$	1.9%

- Please calculate the value of an equity receiver and fixed payer swap at time 0 and 50 days into the life of the swap.
- How would your answer change if the swap was equity receiver and float payer?

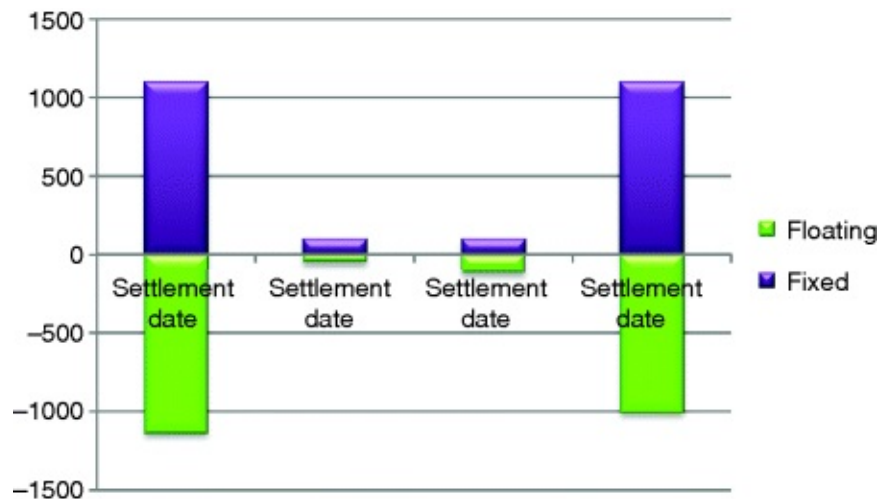
### 3.2.2 Currency/FX Swap

## Currency Swap

In a currency swap, we typically exchange two fixed payments in different currencies. Currency swap transactions are usually not disclosed on corporate balance sheets. Most recent proposed legislation changes may alter that.

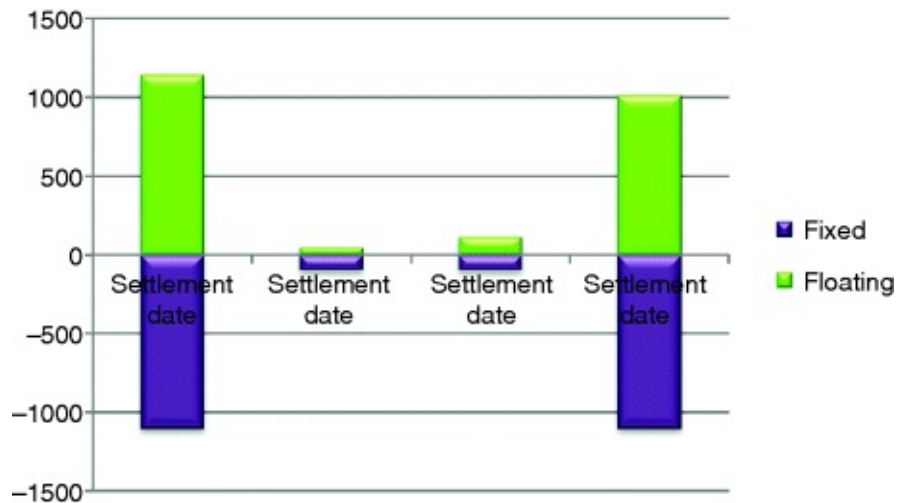
Most commonly, parties in a currency swap contract have some comparative advantage and are willing to swap it in order to satisfy their current needs. One of the counterparties may have an advantage over the other with respect to fixed or floating rates for a particular currency.

Suppose there are two firms, A and B, that wish to enter into a swap. Firm A can get a loan with a fixed rate of 12% in USD and firm B can obtain 10% fixed rate in EUR. In this case, firm B has the comparative advantage in the fixed rates but desires a floating rate. That leaves firm A to have a comparative advantage in the floating rate, USD-denominated, arena. Firm A will be a receiver of the fixed rate and a payer of the floating rate ([Fig. 3.8](#)).



**FIGURE 3.8** Firm A's Cash Flows

Firm B will have the reversed position paying fixed EUR rate and receiving floating rates equal to USD Libor ([Fig. 3.9](#)).



**FIGURE 3.9** Firm B's Cash Flows

At the origination of the swap, the counterparties exchange notional principals in the two currencies. During the life of the swap, each party pays to the other the interest in the currency of the notional principal received. At maturity, each party makes a final exchange of the initial principal amounts at the initial spot exchange rate.

Recall that cash flows from an interest rate swap occur on concurrent dates and are netted against one another. With a currency swap, the cash flows are in different currencies, so they can't net.

Given that the typical setup is about swapping two fixed rates, we need to calculate the appropriate fixed swap rate for both currencies. Just like previously, we will use the current yield curves for that. Note that each currency has its own yield curve.

Equation 3.23 : Fixed swap rate : foreign currency

$$s_{t_0}^{EUR} = \frac{1}{\delta} \left( \frac{1 - B^{EUR}(t_0, t_n)}{\sum_{i=1}^n B^{EUR}(t_0, t_i)} \right)$$

### Equation 3.24 : Fixed swap rate : domestic currency

$$s_{t_0}^{USD} = \frac{1}{\delta} \left( \frac{1 - B(t_0, t_n)}{\sum_{i=1}^n B(t_0, t_i)} \right)$$

Additionally, we need to figure out the appropriate notional principals in the two currencies. We use the spot exchange rate to do that. The notional principals have to be of equal value at the outset of the swap.

### Equation 3.25 : Notional principal translation

$$NP^{USD} = NP^{EUR} \times e_{t_0}$$

#### Notation

$s_{t_0}^{EUR}$	Fixed swap rate in EUR
$s_{t_0}^{USD}$	Fixed swap rate in USD
$B^{EUR}(t_0, t_i)$	Discount factor derived from EUR yield curve
$B(t_0, t_i)$	Discount factor derived from USD yield curve
$NP^{USD}$	Notional principal denominated in USD
$NP^{EUR}$	Notional principal denominated in EUR
$e_{t_0}$	Spot exchange rate quoted as a number of USD per one EUR
$\delta$	Accrual period

### Case Study 3.2.2-1

Let us consider the following situation. Suppose there are two

companies, A and B, that want to enter into a currency swap exchanging fixed payments based on their respective yield curves. Company A is a European firm, and company B is a U.S.-based firm. They want to set up a two-year swap with payments made semiannually. Each EUR costs \$1.35. The swap notional principal is 10,000,000 EUR.

USD yield curve:

Libor USD: $t_0$	
$L_{t_0}^{180}$	3%
$L_{t_0}^{360}$	3.5%
$L_{t_0}^{540}$	3.75%
$L_{t_0}^{720}$	4%

EUR yield curve:

Libor EUR: $t_0$	
$L_{t_0}^{180}$	3.5%
$L_{t_0}^{360}$	3.8%
$L_{t_0}^{540}$	4.1%
$L_{t_0}^{720}$	4.4%

What is the amount of notional principal in USD? What are the respective fixed swap rates? What is the value of this swap at the outset?

**Answer**

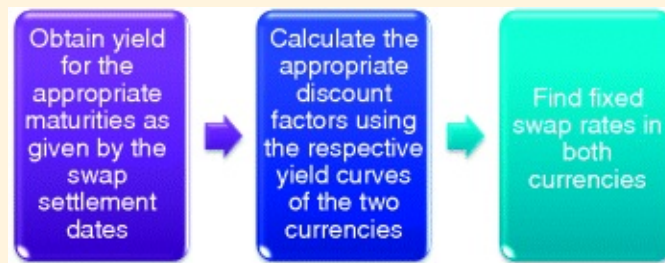
$$\delta = \frac{180}{360}$$

$$e_{t_0} = EUR/USD = 1.35$$

$$NP^{EUR} = 10,000,000$$

The value of the notional principal in USD is the notional principal in EUR multiplied by the spot exchange rate.

$$NP^{USD} = 10,000,000 \times 1.35 = 13,500,000$$



USD discount factors:

$$B(t_0, 180) = \frac{1}{1 + L_{t_0}^{180} \left( \frac{180}{360} \right)} = \frac{1}{1 + 0.03 \times 0.5} = 0.9852$$

$$B(t_0, 360) = \frac{1}{1 + L_{t_0}^{360} \times \left(\frac{360}{360}\right)} = \frac{1}{1 + 0.035 \times 1} = 0.9662$$

$$B(t_0, 540) = \frac{1}{1 + L_{t_0}^{540} \times \left(\frac{540}{360}\right)} = \frac{1}{1 + 0.0375 \times 1.5} = 0.9467$$

$$B(t_0, 720) = \frac{1}{1 + L_{t_0}^{720} \times \left(\frac{720}{360}\right)} = \frac{1}{1 + 0.04 \times 2} = 0.9259$$

EUR discount factors:

$$B^{EUR}(t_0, 180) = \frac{1}{1 + L_{t_0}^{EUR,180} \left(\frac{180}{360}\right)} = \frac{1}{1 + 0.035 \times 0.5} = 0.9825$$

$$B^{EUR}(t_0, 360) = \frac{1}{1 + L_{t_0}^{EUR,360} \times \left(\frac{360}{360}\right)} = \frac{1}{1 + 0.038 \times 1} = 0.9633$$

$$B^{EUR}(t_0, 540) = \frac{1}{1 + L_{t_0}^{EUR,540} \times \left(\frac{540}{360}\right)} = \frac{1}{1 + 0.041 \times 1.5} =$$

$$B^{EUR}(t_0, 720) = \frac{1}{1 + L_{t_0}^{EUR,720} \times \left(\frac{720}{360}\right)} = \frac{1}{1 + 0.044 \times 2} = 0$$

Fixed swap rate in USD:

$$s_{t_0}^{USD} = \frac{1}{0.5} \left( \frac{1 - 0.9259}{0.9852 + 0.9662 + 0.9467 + 0.9259} \right) = 0.0387$$

Fixed swap rate in EUR:

$$s_{t_0}^{EUR} = \frac{1}{0.5} \left( \frac{1 - 0.9191}{0.9828 + 0.9634 + 0.9421 + 0.9191} \right) = 0.0425$$

Recall that the value of the swap at the outset of the contract is equal to zero. That means that both the legs of the swap must equal in value. Let us confirm that this is true.



$$V_{FIX}^{USD} = s_{t_0}^{USD} \times \delta \times \sum_{i=180}^n B(t_0, t_i) + B(t_0, t_n) = 1$$

$$V_{FIX}^{EUR} = s_{t_0}^{EUR} \times \delta \times \sum_{i=180}^n B^{EUR}(t_0, t_i) + B^{EUR}(t_0, t_n) = 1$$

The value of this swap denominated in USD is zero.

$$V_{SWAP}^{USD} = (1 - 1) \times 13,500,000 = 0$$

The value of the swap denominated in EUR is also zero.

$$V_{SWAP}^{EUR} = (1 - 1) \times 10,000,000 = 0$$

## Case Study 3.2.2-2

In the following example, we will value the swap described in Case Study 3.2.2-1, 90 days into its life. This is the value of the swap in-between settlement dates. What is the value of the swap 90 days into its life? Give the result in USD and in EUR.

USD yield curve 90 days into the life of the swap:

Libor USD: $t_{90}$	
$L_{90}^{90}$	2.9%
$L_{90}^{270}$	3.4%
$L_{90}^{450}$	3.6%
$L_{90}^{630}$	3.9%

EUR yield curve 90 days into the life of the swap:

Libor EUR: $t_{90}$	
$L_{90}^{90}$	3.3%
$L_{90}^{270}$	3.7%
$L_{90}^{450}$	4%
$L_{90}^{630}$	4.4%

**Answer**



$$V_{FIX}^{USD} = s_{t_0}^{USD} \times \delta \times \sum_{i=(180-90)}^{(720-90)} B(90, t_{i-90}) + B(90, t_{n-90})$$

$$V_{FIX}^{USD} = 0.0387 \times 0.5 \times (0.9928 + 0.9751 + 0.9569 + 0.9361)$$

$$V_{FIX}^{EUR} = s_{t_0}^{EUR} \times \delta \times \sum_{i=(180-90)}^{(720-90)} B^{EUR}(90, t_{i-90}) + B^{EUR}(90, t_r)$$

$$V_{FIX}^{EUR} = 0.0425 \times 0.5 \times (0.9918 + 0.973 + 0.9524 + 0.9285) -$$

The value of this swap 90 days into its life, looking at it from the point of view of USD receiver and EUR payer, is the following denominated in USD.

$$V_{SWAP}^{USD} = (1.0109 \times 13,500,000) - (1.0102 \times 10,000,000 \times 1.35)$$

In EUR, the swap value at 90 days into its life would be nearly 7000 EUR.

$$V_{SWAP}^{EUR} = \left( 1.0109 \times 13,500,000 \times \frac{1}{1.35} \right) - (1.0102 \times 10,000,000)$$

### Foreign Exchange (FX) Swap

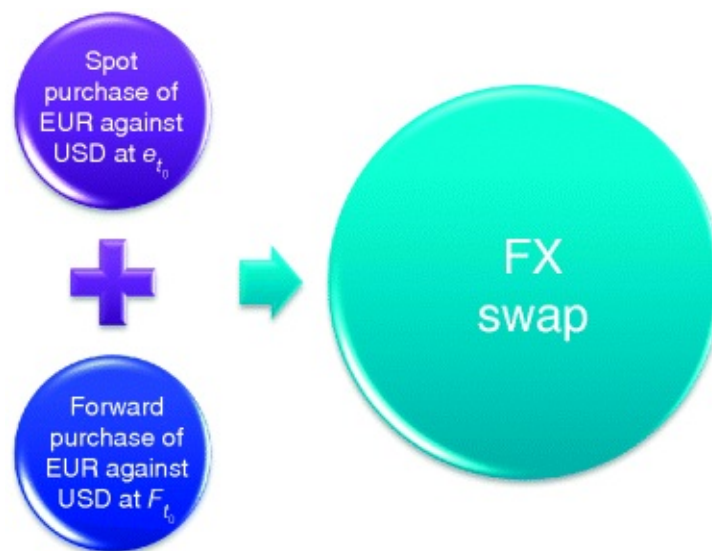
FX swaps are also instruments designed to facilitate exchange of cash

flows. An FX swap can be constructed using a short and a long bond denominated in USD and another currency, respectively.

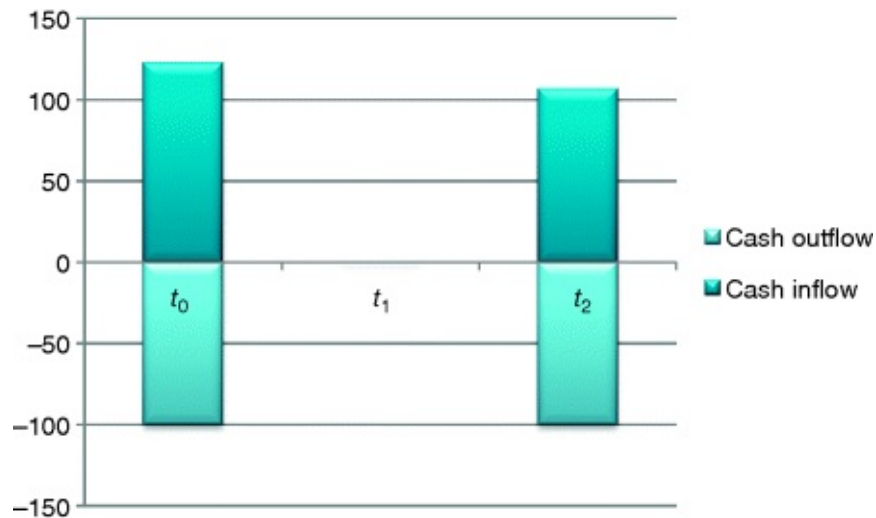
We can synthesize an existing FX swap using a spot purchase of USD against a currency, let's use once again the EUR, at the spot exchange rate. Along with that we need a forward purchase of USD against EUR at the forward exchange rate.

*Notation*

$F_{t_0}$	Forward exchange rate
$e_{t_0}$	Spot exchange rate



In [Fig. 3.10](#), we are swapping \$100 USD for EUR. The current exchange rate is EUR/USD = 1.23. In this example, the forward expectation-based exchange rate  $F_{t_0}$  is reflecting strengthening USD.  $F_{t_0}$  is EUR/USD = 1.07.



**FIGURE 3.10** FX Swap: Purchase EUR, Pay USD

### Difference Between FX Swaps and Currency Swaps

There is a difference between FX swaps and currency swaps. Currency swaps exchange interest payments during the life of the swap as well as notional principals at the beginning and at the end of the life of the swap at the spot exchange rate valid at the outset of the swap.

FX swaps exchange notional principals at the beginning and at the end of the swap with no interim payments during the life of the swap. Notional principals are exchanged at the spot rate at the beginning of the swap and at the forward exchange rate at the end of swap. The forward exchange rate is determined using the interest rate parity.

## Problems

**3.2.2-1.** Consider Case Studies 3.2.2-1 and 3.2.2-2. Given this information, set up a currency pricing module in Excel. Once done, please confirm that your resulting swap values equal to the results above.

**3.2.2-2.** Using your model from Problem 3.2.2-1, please do the following:

- Insert different values of the spot exchange rate and recalculate swap values.
- Assume different shapes of the yield curves and observe the effect

on the swap values.

- c. Plug in information from a different currency—its yield curve and exchange rate—and calculate the value of such a swap.
- d. Discuss a situation where the original swap would go on for three years. How would the swap rates and swap values change?

**3.2.2-3.** Using cash flow diagrams, please show the difference between a currency swap and a FX swap.

## 3.3 Other Yield Curve-Dependent Swaps

### 3.3.1 Basis Swap

In a basis swap, counterparties exchange benchmark rates such as the T-bill rate and the Libor. The two interest rates are seldom priced identically. In fact, the only time they would show identical values would be when the credit spread between Treasuries and Libor is zero. Thus, inevitably we arrive at a basis swap spread that in turn becomes a part of the basis swap pricing.

Usually, the basis swap creation process involves two individual interest rate swaps. In these IRSs, counterparties are swapping a fixed rate with a floating one.

While calculating the appropriate fixed swap rates, we also easily obtain the basis swap spread as the difference between the two fixed rates ([Fig. 3.11](#)).

Equation 3.26 : Fixed swap rate– Libor yield curve

$$s_{t_0}^L = \frac{1}{\delta} \left( \frac{1 - B(t_0, t_n)}{\sum_{i=1}^n B(t_0, t_i)} \right)$$

Equation 3.27 : Fixed swap rate– TB yield curve

$$s_{t_0}^T = \frac{1}{\delta} \left( \frac{1 - B^T(t_0, t_n)}{\sum_{i=1}^n B^T(t_0, t_i)} \right)$$

Equation 3.28 : Basis swap spread

$$BSpread = s_{t_0}^L - s_{t_0}^T$$

In the basis swap, the party paying floating T-bill rate would, in fact, be paying T-bill rate + basis swap spread in order to receive the floating Libor rate. Alternatively, we can have the party paying Libor receive Libor – basis spread and pay the T-bill rate. At the outset of the swap, its value is once again equal to zero.

Equation 3.29 : Value of a basis swap at its outset

$$V_{SWAP}^{t_0} = \left( V_{FLOAT}^{L, t_0} - V_{FLOAT}^{T, t_0} \right) \times NP$$

### Case Study 3.3.1-1

Let us consider a one-year basis swap that makes payments semiannually. Assume that the notional principal is \$50,000,000. The accrual period is  $\delta = 180/360$ . What is its basis spread given the information below?

The Libor yield curve at the beginning of the swap shows the following.

Libor	
$L_{t_0}^{180}$	3%
$L_{t_0}^{360}$	3.2%

The T-bill yield curve at the outset shows the following.

T-Bill	
$T_{t_0}^{180}$	1.05%
$T_{t_0}^{360}$	1.95%

### **Answer**

First, we calculate the Libor-based discount factors. Calculations are performed based on the add-on method.

$$B(t_0, 180) = \frac{1}{1 + L_{t_0}^{180} \left( \frac{180}{360} \right)} = \frac{1}{1 + 0.03 \times 0.5} = 0.9852$$

$$B(t_0, 360) = \frac{1}{1 + L_{t_0}^{360} \times \left( \frac{360}{360} \right)} = \frac{1}{1 + 0.032 \times 1} = 0.9690$$

Second, the T-bill-based discount factors need to be determined. Note that this time, as is the convention with T-bill pricing, our calculations follow the discount method.



$$B^T(t_0, 180) = 1 - 0.0105 \times \left( \frac{180}{360} \right) = 0.9948$$

$$B^T(t_0, 360) = 1 - 0.0195 \times \left( \frac{360}{360} \right) = 0.9805$$

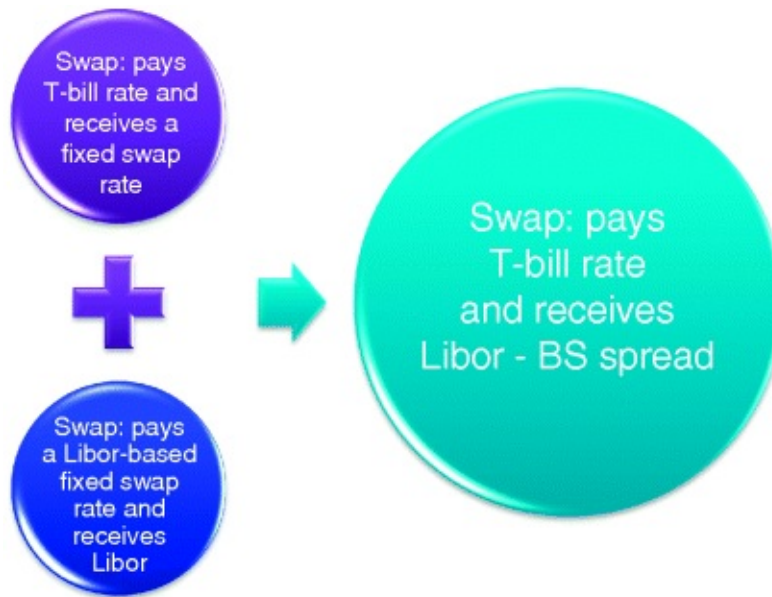
Determining discount factors for both yield curves enables us to calculate the respective fixed swap rates for both Libor and T-bill rates.

Libor-based fixed swap rate:

$$s_{t_0}^L = \frac{1}{0.5} \left( \frac{1 - 0.969}{0.9852 + 0.969} \right) = 0.0317 = 3.17\%$$

T-bill-based fixed swap rate:

$$s_{t_0}^T = \frac{1}{0.5} \left( \frac{1 - 0.9805}{0.9948 + 0.9805} \right) = 0.0197 = 1.97\%$$



**FIGURE 3.11** Financial Engineering of Basis Swap

### Basis Swap Spread

$$BSpread = 0.0317 - 0.0197 = 0.012 = 1.2\%$$

### Valuing the Swap During Its Life

Once we move  $x$  days into the life of the swap, we value the swap based on the present value of the floating payments, both in Libor and in the T-bill rate.

Equation 3.30 : Value of a Libor – based floating leg  $x$  days i

$$V_{FLOAT}^{L,x} = \left( 1 + \left( L_{t_0}^{t_1} - BSpread \right) \times \frac{t_1}{360} \right) \times B(x, t_1)$$

Equation 3.31 : Value of a TB – based floating leg  $x$  days into

$$V_{FLOAT}^{T,x} = \left( 1 + T_{t_0}^{t_1} \times \frac{t_1}{360} \right) \times B^T(x, t_1)$$

Equation 3.32 : Value of the basis swap  $x$  days into its life

$$V_{SWAP}^x = \left( V_{FLOAT}^{L,x} - V_{FLOAT}^{T,x} \right) \times NP$$

### Case Study 3.3.1-2

Assume that information from Case Study 3.3.1-1 holds. What is the value of this basis swap 90 days into the life of the swap?

Libor 90 Days In	
$L_{90}^{180}$	3.1%
$L_{90}^{360}$	3.35%

T-Bill 90 Days In	
$T_{90}^{180}$	1.65%
$T_{90}^{360}$	1.75%

#### **Answer**

First we calculate the new discount factors.

$$B(90,180) = \frac{1}{1 + L_{90}^{180} \left( \frac{180-90}{360} \right)} = \frac{1}{1 + 0.031 \times 0.25} = 0.992$$

$$B(90,360) = \frac{1}{1 + L_{90}^{360} \times \left( \frac{360-90}{360} \right)} = \frac{1}{1 + 0.0335 \times 0.75} = 0.9869$$

$$B^T(90,180) = 1 - 0.0165 \times \left( \frac{180 - 90}{360} \right) = 0.9959$$

$$B^T(90,360) = 1 - 0.0175 \times \left( \frac{360 - 90}{360} \right) = 0.9869$$

We then use our results to obtain the value of the floating legs for both the Libor-based and the T-bill rate-based legs 90 days into the life of the swap.

$$V_{FLOAT}^{L,90} = \left( 1 + (0.03 - 0.012) \times \frac{180}{360} \right) \times (0.9923) = 1.001245$$

$$V_{FLOAT}^{T,90} = \left( 1 + 0.0105 \times \frac{180}{360} \right) \times (0.9959) = 1.001103$$

The value of the swap at day 90 into the life of the swap is the following. We look at the swap value from the point of view of the counterparty that is paying T-bill rate and is receiving Libor – spread.

$$V_{SWAP} = (1.001245 - 1.001103) \times 50,000,000 = 7102.50$$

present								
	time 0	Future Time	Time Fraction of a Year	Discount Factor - B(t0,ti)	Accrual Period - Delta	Notional Principal	Fixed Swap Rate	
<b>Libor</b>								
180-day	3.00%	180	0.5	0.9852				
360-day	3.20%	360	1	0.9690	0.5	\$ 50,000,000.00	3.17%	
<b>T-Bill</b>								
180-day	1.05%	180	0.5	0.9948				
360-day	1.95%	360	1	0.9805	0.5	\$ 50,000,000.00	1.97%	
							<b>Basis Spread</b>	
							1.20%	
90 days into the life of the swap								
	time 90	Future Time	Time Fraction of a Year	Discount Factor - B(90,ti)			Value of Floating Pmts 90 Days Into Life	Swap Value 90 Days Into Life
<b>Libor</b>								
90-day	3.10%	90	0.25	0.9923			1.00124539	
270-day	3.35%	270	0.75	0.9755			\$ 50,062,269.68	
<b>T-Bill</b>								
90-day	1.65%	90	0.25	0.9959			1.00110334	
270-day	1.75%	270	0.75	0.9869			\$ 50,055,167.19	\$ 7,102.50

present								
Libor	time 0	Future Time	Time Fraction of a Year	Discount Factor - B(t0,ti)	Accrual Period - Delta	Notional Principal	Fixed Swap Rate	
180-day	0.03	180	=C3/360	=1/(1+B3*D3)				
360-day	0.032	360	=C4/360	=1/(1+B4*D4)	=180/360	50000000	=(1-E4)/SUM(E3:E4)*(1/F4)	
<b>T-Bill</b>								
180-day	0.0105	180	=C7/360	=1-B7*D7				
360-day	0.0195	360	=C8/360	=1-B8*D8	=180/360	50000000	=(1-E8)/SUM(E7:E8)*(1/F8)	
							Basis Spread	
							=H4-H8	
90 days into the life								
Libor	time 90	Future Time	Time Fraction of a Year	Discount Factor - B(90,ti)			Value of Floating Pmts 90 Days Into Life	Swap Value 90 Days Into Life
90-day	0.031	=C3-90	=C14/360	=1/(1+B14*D14)			=E14*(1+(B3-\$H\$10)*\$F\$4)	
270-day	0.0335	=C4-90	=C15/360	=1/(1+B15*D15)			=H14*\$G\$4	
<b>T-Bill</b>								
90-day	0.0165	=C7-90	=C18/360	=1-B18*D18			=E18*(1+\$B\$7*\$D\$7)	
270-day	0.0175	=C8-90	=C19/360	=1-B19*D19			=H18*\$G\$4	=H15-H19

## Problems

**3.3.1-1.** Let us consider an 18-month basis swap that makes payments semiannually. Assume the notional principal is \$100,000,000. The accrual period is  $\delta = 180/360$ .

- What is its basis spread given the information below?
- What is the swap value at the outset of the swap?

The Libor yield curve at the beginning of the swap shows the following.

Libor	
$L_{t_0}^{180}$	2.7%
$L_{t_0}^{360}$	3%
$L_{t_0}^{540}$	3.1%

The T-bill yield curve at the outset shows the following.

T-Bill	
$T_{t_0}^{180}$	1%
$T_{t_0}^{360}$	1.5%

$T_{t_0}^{540}$	1.6%
-----------------	------

**3.3.1-2.** Using assumptions from Problem 3.3.1-1.

- Please look up current T-bill and Libor yield curve data. Recalculate the basis spread given these current numbers.
- How would your answer change if this was a 3-year swap instead of an 18-month one?

**3.3.1-3.** Assume that the information from Problem 3.3.1-1 holds.

- What is the value of this basis swap 30 days into its life?
- How would you go about valuing the swap at 60 and 90 days into its life? Discuss.

Libor 30 Days In	
$L_{30}^{180}$	2.75%
$L_{30}^{360}$	2.9%
$L_{30}^{540}$	3.15%

T-Bill 30 Days In	
$T_{30}^{180}$	1.05%
$T_{30}^{360}$	1.15%
$T_{30}^{540}$	1.55%

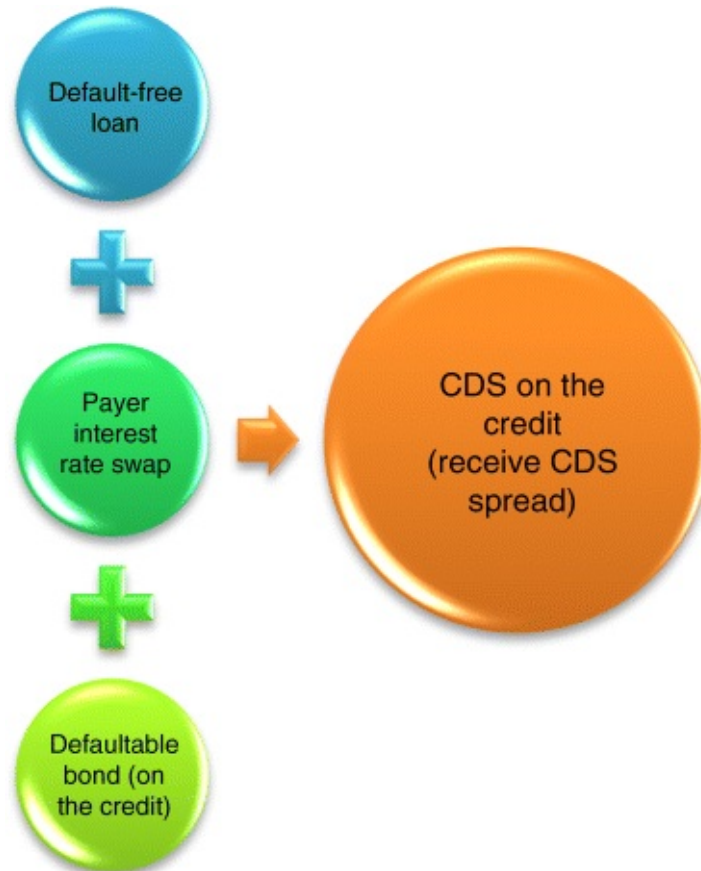
### 3.3.2 Credit Default Swap

A credit default swap (CDS) is similar in concept to an insurance contract. In a credit default swap, a fee, called the CDS spread, is periodically paid until default occurs. If there is no default, the protection ends at contract expiration with no other cash exchanging hands. If there is default, the CDS spread payments cease and the protection seller makes a payment to the protection buyer based on the CDS spread and the notional principal of the swap— $s_{t_0} \times \delta \times NP$ .

The size of the CDS spread is dependent on a particular bond issuer's probability of default.

In order to create a CDS, we take a bond that has default risk and decompose the cash flows from this bond into simpler, liquid constituents. CDS will be one of these instruments.

Let us look at a CDS from the point of view of a CDS seller, collecting CDS spread ([Fig. 3.12](#)).



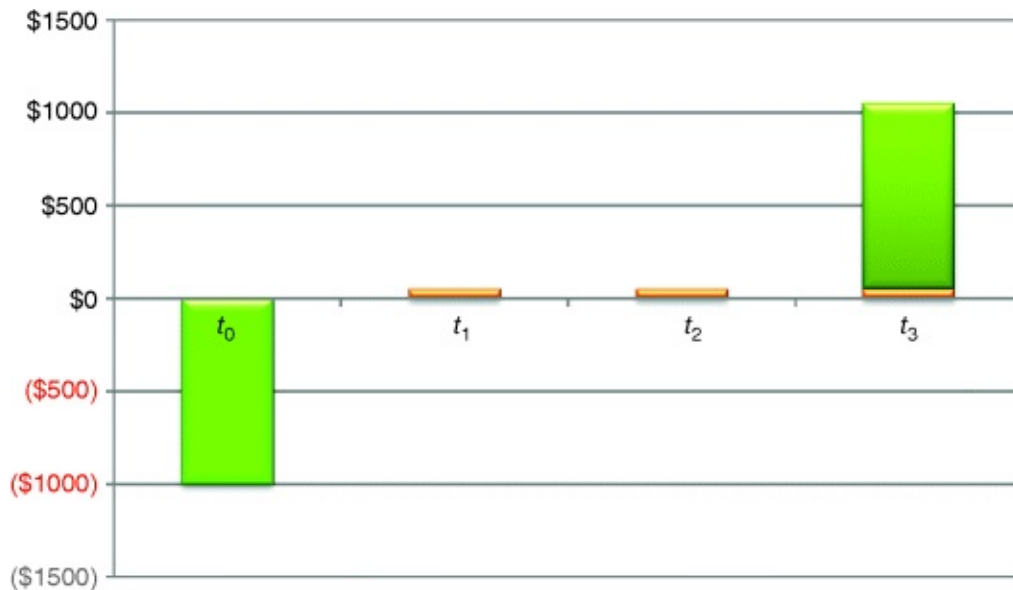
**FIGURE 3.12** Financial Engineering of Credit Default Swap

A CDS buyer takes the opposite position—short the risky bond, make a deposit in a default-free deposit account, and contract a receiver swap. We can obtain a hedge for a CDS by taking various positions in securities replicating a credit default swap.

### Defaultable Bond

A defaultable bond will pay out cash flows as in [Fig. 3.13](#) if it doesn't default. If it does default on any of its payments, then the scheduled coupon payments will discontinue and the par value will be repaid either partially or not at all.



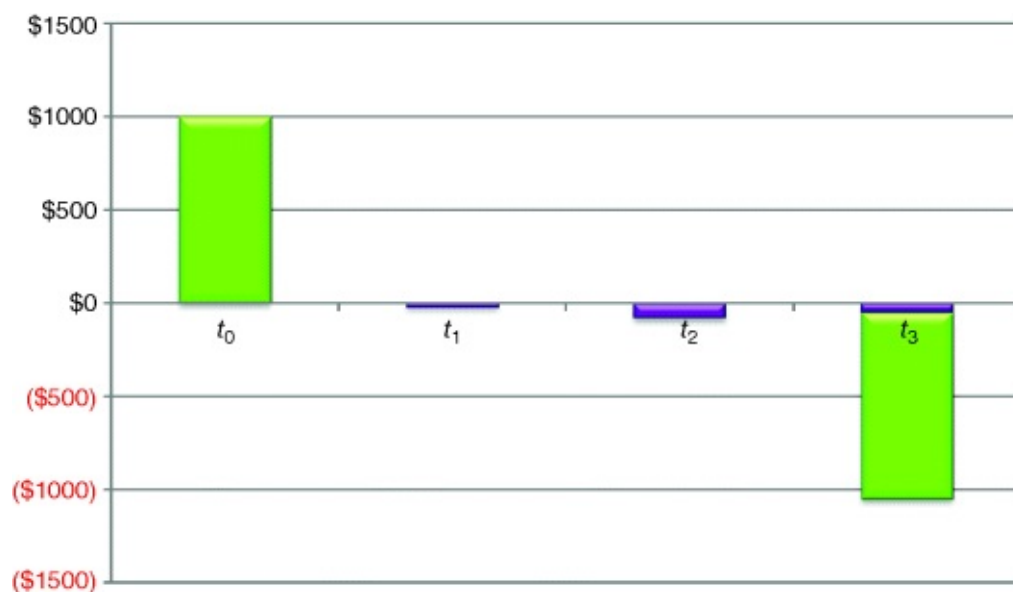


**FIGURE 3.13** Long Position in a Defaultable Bond

Figure 3.13 shows a cash flow diagram for a three-period defaultable bond. Its coupon payments are set at \$50 each. They will be paid at the end of each period. At the end of the life of the bond, assuming there is no default, the investor receives the last coupon payment along with the bond par value.

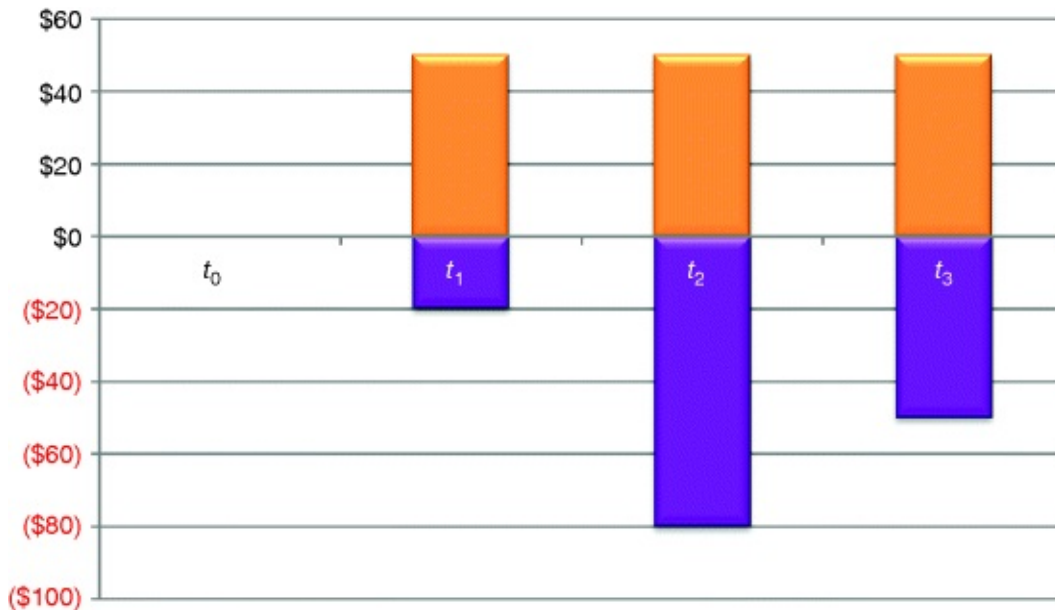
### Default-Free Loan

The second component in the synthetic is a default-free loan. This loan has floating, Libor-based, coupon payments (Fig. 3.14).



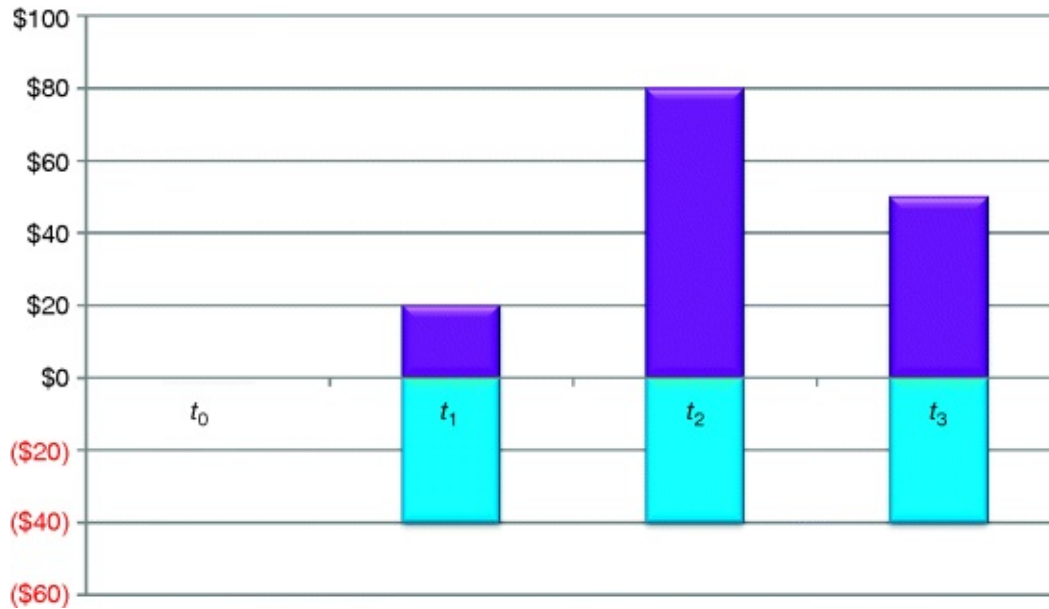
### FIGURE 3.14 Default-Free Floating Rate Loan

Figure 3.15 is obtained by adding information from Figs. 3.13 and 3.14. The notional principal payments cancel each other out and we are left with a swap. This swap exchanges Libor payments for the defaultable bond's coupon payments. Note that the payments have been magnified due to the reset of the vertical axis in Fig. 3.15.



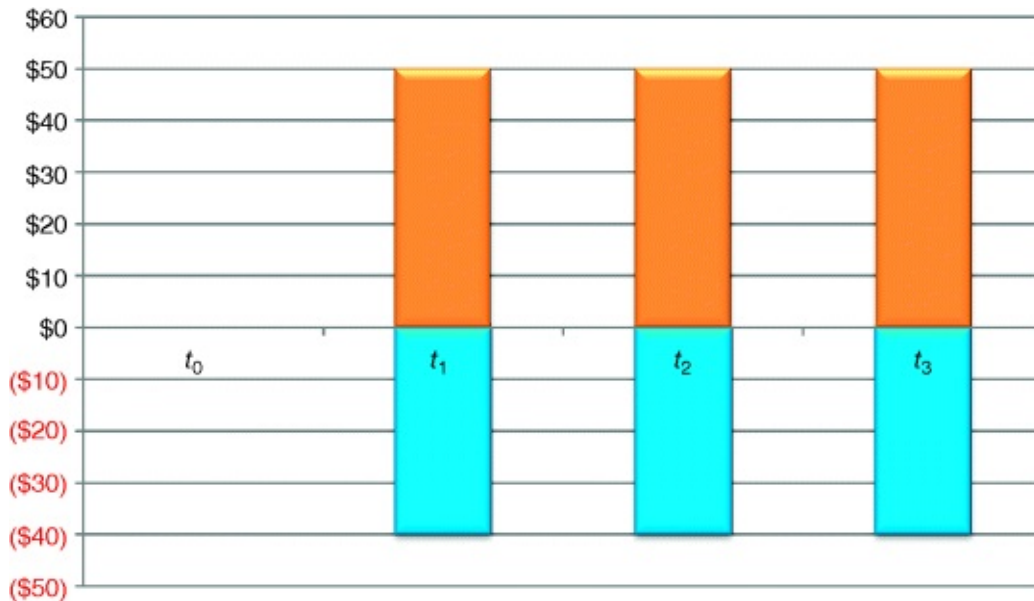
### FIGURE 3.15 Cash Flows from Combining Defaultable Bond with Default-Free Loan

Next we add a payer IRS into the mix. In this swap, we are exchanging a fixed interest swap rate for floating Libor-based payments (Fig. 3.16). Assume that the fixed swap payments are \$40 each.



**FIGURE 3.16** IRS: Pay Fixed, Receive Float

Finally, we will add [Figs. 3.15](#) and [3.16](#) to eliminate the floating Libor-based payments ([Fig. 3.17](#)). The resulting swap is paying the swap rate and receiving the coupon rate. It turns out that the coupon rate is greater than the swap rate. It is so because it contains all the issuing company's credit information. The larger the default probability, the larger the gap between the two rates.



**FIGURE 3.17** Combining Cash Flows from [Fig. 3.15](#) and Fixed Payer IRS in [Fig. 3.16](#)

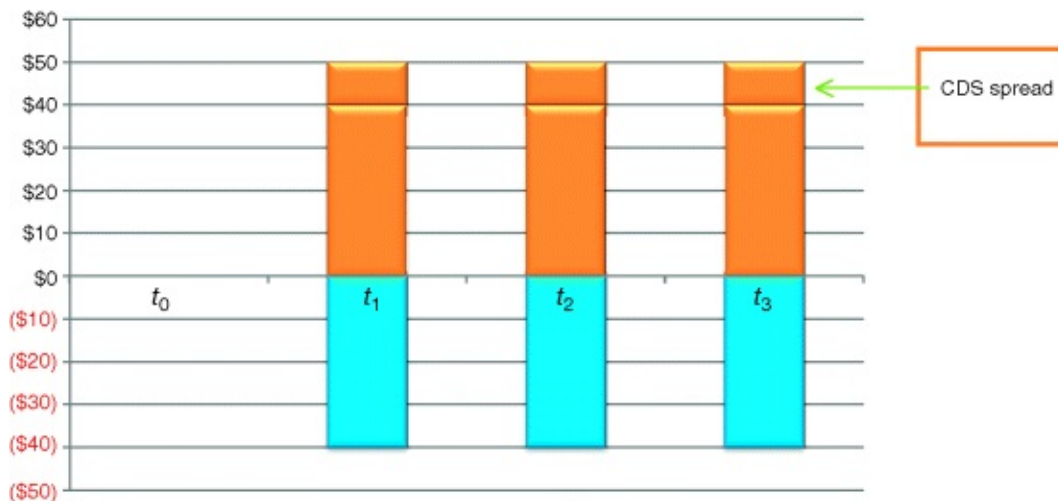
## Notation

$c_{t_0}$	Defaultable bond coupon rate
$s_{t_0}$	Swap fixed rate
$Spread^{CDS}$	“Insurance” premium on credit default swap

### Equation 3.33 : Credit default swap spread

$$Spread^{CDS} = c_{t_0} - s_{t_0}$$

It turns out actually that credit default swap spread is precisely the difference between the coupon rate and the interest rate swap rate ([Fig. 3.18](#)). In our case, it is \$10. This cash flow diagram shows the position of the protection seller. He/she is the one collecting the spread in the form of “insurance premium.”



**FIGURE 3.18** CDS Spread

## CDS Valuation

Valuing a CDS essentially means coming up with the appropriate CDS spread. One way to do so is described above as using a synthetic to back out the appropriate spread. Another way relies on default probabilities of the particular company's securities.

In order to value a credit default swap, we need to calculate both probability of default and probability of survival on each settlement date during the life of the swap. The probability of default within the next period is of crucial importance.

### Notation

$q_t$	Probability of survival until the next period
$p_t$	Probability of default within the next period
$EP_t$	Expected premium payment in time period $t$
$r$	Discount rate
$M$	Number of compounding periods within one year
$PL1$	First part of the payment leg of the CDS
$EAP_t$	Expected accrual payment in period $t$
$m_t$	Fraction of the settlement period, a time within the settlement period when default can occur
$PL2$	Second part of the payment leg reflecting accrual payments
$ED_t$	Expected payoff at default
$RR_t$	Recovery rate in period $t$
$DL$	Default leg of the CDS

Probability of survival is a function of the imminent probability of default.

### Equation 3.34 : Probability of survival

$$q_t = (1 - p_1)^t$$

Probability of default is a function of the initial probability of default in concert with the previous period probability of survival.

### Equation 3.35 : Probability of default

$$p_t = p_1 \times q_{t-1}$$

Valuing a CDS requires to come up with such a spread that will equalize the present value of the expected CDS premium payments with the present value of expected default cash flows should default occur. Let us think about it as a credit swap having two legs: a payment leg and a default leg.

#### Payment Leg: Premium Payment

Expected payment is the product of the CDS spread and the probability of survival.

### Equation 3.36 : Expected payment of CDS premium at time

$$EP_t = Spread^{CDS} \times q_t$$

In order to value a CDS, we need to find the present value of each expected payment.

### Equation 3.37 : Present value of expected payment : discrete

$$PV(EP_t) = \frac{1}{\left(1 + \frac{r}{M}\right)^{M \times t}} \times EP_t$$

Should we assume continuous compounding, then the present value is

calculated as follows:

Equation 3.38 : Present value of expected payment : continu

$$PV(EP_t) = e^{-rt} \times EP_t$$

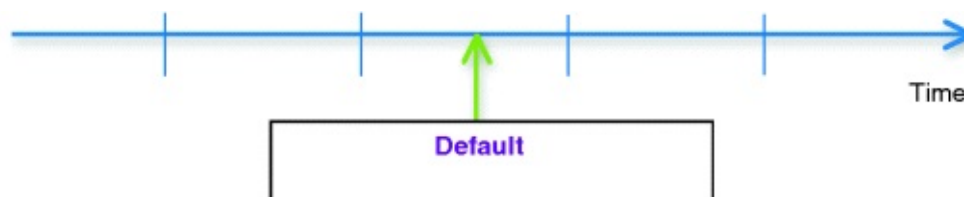
Summing up over the settlement dates gives us the present value of total expected premium payments.

Equation 3.39 : CDS payment leg : first part

$$PL1 = \sum_{t=1}^T PV(EP_t)$$

### Accrual Payments

In case a default occurs in-between settlement dates, we need to adjust the payment leg by adding any remaining accrual payments.



Expected accrual payment is calculated as the product of CDS spread, fraction of the settlement period, and probability of default.

Equation 3.40 : Expected accrual payment at time  $t$

$$EAP_t = Spread^{CDS} \times m_t \times p_t$$

Equation 3.41 : Present value of EAP : discrete time

$$PV(EAP_t) = \frac{1}{\left(1 + \frac{r}{M}\right)^{M \times t}} \times EAP_t$$

Equation 3.42 : Present value of EAP : continuous time

$$PV(EAP_t) = e^{-rt} \times EAP_t$$

Once again, we'll sum over time to get the PV of total expected accrual payments.

Equation 3.43 : CDS payment leg : second part

$$PL2 = \sum_{t=1}^T PV(EAP_t)$$

Equation 3.44 : CDS payment leg

$$PL = PL1 + PL2$$



### Default Leg: Default Payoff Payment

The expected payoff at default is a product of what is lost and cannot be recovered and the probability of default.

Equation 3.45 : Expected default payoff at time  $t$

$$ED_t = (1 - RR_t) \times p_t$$

Equation 3.46 : Present value of ED : discrete time

$$PV(ED_t) = \frac{1}{\left(1 + \frac{r}{M}\right)^{M \times t}} \times ED_t$$

Equation 3.47 : Present value of ED : continuous time

$$PV(ED_t) = e^{-rt} \times ED_t$$

Total default leg is the sum of the present values of total expected payoffs.

Equation 3.48 : CDS default leg

$$DL = \sum_{t=1}^T PV(ED_t)$$

Set the two legs equal to each other and solve for the CDS spread.

Equation 3.49 : CDS spread : discrete time

$$Spread^{CDS} = \frac{DL}{\sum_{t=1}^T \frac{1}{\left(1 + \frac{r}{M}\right)^{M \times t}} \times q_t + \sum_{t=1}^T \frac{1}{\left(1 + \frac{r}{M}\right)^{M \times t}} \times m_t \times p_t}$$

Equation 3.50 : CDS spread : continuous time

$$Spread^{CDS} = \frac{DL}{\sum_{t=1}^T e^{-rt} \times q_t + \sum_{t=1}^T e^{-rt} \times m_t \times p_t}$$

### Case Study 3.3.2-1

Consider a five-year credit default swap. Payments of the CDS premiums will be made annually. Suppose the immediate default probability for the underlying credit is 2% and the Libor discount rate is 5%. The default time is halfway through the settlement period. Thus, if default should occur it would be at six-month time into the particular year. Assume recovery rate at 40% irrespective of default timing. What is the appropriate CDS swap rate (i.e., CDS spread)? Assume notional principal of \$1.

**Answer**

Credit Default Swap										
Libor	0.05									
Current Probability of Default	0.02									
m	0.5									
PAYMENT LEG										
PV of Expected Premium Payments					PV of the Accrual Payment.					
Time	Probability of Default	Probability of Survival	Expected Payment	Discount Factor	PV of Expected Payment	Time in years	Probability of Default	Expected Accrual Payment	Discount Factor	PV of Expected Accrual Payment
1	0.020000	0.980000		0.98	0.951229425	0.5	0.020000	0.010000	0.975309912	0.00975310
2	0.019600	0.960400		0.9604	0.904837418	1.5	0.019600	0.009800	0.927743486	0.00909189
3	0.019208	0.941192	0.941192	0.860707976	0.810091462	2.5	0.019208	0.009604	0.882496903	0.00847550
4	0.018824	0.922368	0.92236816	0.818730753	0.755171178	3.5	0.018824	0.009412	0.839457021	0.00790090
5	0.018447	0.903921	0.903920797	0.778800783	0.700974224	4.5	0.018447	0.009224	0.798516219	0.00736526
<b>Total</b>					<b>4.070447557</b>	<b>Total</b>				<b>0.04258665</b>
DEFAULT LEG										
PV of Default Payoff										
Time in years	Probability of Default	Recovery Rate	Expected Payoff (our loss given recovery rate)	Discount Factor	PV of Expected Payoff					
0.5	0.020000	0.4	0.012000	0.975309912	0.011703719					
1.5	0.019600	0.4	0.011760	0.927743486	0.010910263					
2.5	0.019208	0.4	0.011525	0.882496903	0.01014706					
3.5	0.018824	0.4	0.011294	0.839457021	0.009481083	PV of sum of expected prmts x spread = PV of expected payoffs. Solve for spread.				
4.5	0.018447	0.4	0.011068	0.798516219	0.008838311					
<b>Total</b>					<b>0.051103977</b>	<b>CDS Spread</b>	<b>1.2425%</b>			

Credit Default Swap										
Libor	0.05									
Current Probability of Default	0.02									
m	0.5									
PAYMENT LEG										
PV of Expected Premium Payments					PV of the Accrual Payment.					
Time	Probability of Default	Probability of Survival	Expected Payment	Discount Factor	PV of Expected Payment	Time in years	Probability of Default	Expected Accrual Payment	Discount Factor	PV of Expected Accrual Payment
1	=S\$4	=(1-S\$4)*A9	0.98	=EXP(-S\$3*A9)	=C9*E9	0.5	=S\$4*(1-S\$4)*0	=I9*S\$5	=EXP(-S\$3*H9)	=I9*K9
2	=S\$4*C9	=(1-S\$4)*A10	0.9604	=EXP(-S\$3*A10)	=C10*E10	1.5	=S\$4*(1-S\$4)*A9	=I10*S\$5	=EXP(-S\$3*H10)	=I10*K10
3	=S\$4*C10	=(1-S\$4)*A11	0.941192	=EXP(-S\$3*A11)	=C11*E11	2.5	=S\$4*(1-S\$4)*A10	=I11*S\$5	=EXP(-S\$3*H11)	=I11*K11
4	=S\$4*C11	=(1-S\$4)*A12	0.92236816	=EXP(-S\$3*A12)	=C12*E12	3.5	=S\$4*(1-S\$4)*A11	=I12*S\$5	=EXP(-S\$3*H12)	=I12*K12
5	=S\$4*C12	=(1-S\$4)*A13	0.9039207968	=EXP(-S\$3*A13)	=C13*E13	4.5	=S\$4*(1-S\$4)*A12	=I13*S\$5	=EXP(-S\$3*H13)	=I13*K13
<b>Total</b>					=SUM(F9:F13)	<b>Total</b>				<b>0.0425866472159162</b>
DEFAULT LEG										
PV of Default Payoff										
Time in years	Probability of Default	Recovery Rate	Expected Payoff (our loss given recovery rate)	Discount Factor	PV of Expected Payoff					
0.5	=S\$4*(1-S\$4)*0	0.4	=B18*(1-C18)	=EXP(-S\$3*A18)	=D18*E18					
1.5	=S\$4*(1-S\$4)*A9	0.4	=B19*(1-C19)	=EXP(-S\$3*A19)	=D19*E19					
2.5	=S\$4*(1-S\$4)*A10	0.4	=B20*(1-C20)	=EXP(-S\$3*A20)	=D20*E20					
3.5	=S\$4*(1-S\$4)*A11	0.4	=B21*(1-C21)	=EXP(-S\$3*A21)	=D21*E21					
4.5	=S\$4*(1-S\$4)*A12	0.4	=B22*(1-C22)	=EXP(-S\$3*A22)	=D22*E22					
<b>Total</b>					=SUM(F18:F22)	<b>CDS Spread</b>	=F23/(F14+L14)			

## Problems

**3.3.2-1.** Revisit Case Study 3.3.2-1. Now that you have set up your model, please perform a sensitivity analysis of the obtained CDS spread. In a sensitivity analysis, we change one variable at a time and evaluate the resulting change in the spread. Please look at changing

- Libor rate;
- recovery rates:
  - set the same recovery rate for all time periods;
  - change recovery rates with changing time periods;

c. default probability.

Which one of these variables has the greatest effect on the CDS spread?

# 4 OPTIONS

## 4.1 Options Fundamentals

### 4.1.1 Basic Information

#### Definition

We recognize two types of options—puts and calls. A call represents the right of its holder to buy the underlying asset at the stated strike price at some time in the future. A put also reflects a right of its holder to trade the underlying asset at the predetermined strike price sometime in the future. This time, however, the right is to sell the underlying.

#### American or European?

At the outset, we will distinguish between two types of options—European and American. The names have no geographical significance. The difference lies in the time of exercise. European options can only be exercised on the expiry date. European options are typically valued using the Black–Scholes or Black model formula. The Black–Scholes model is an equation with a closed-form solution that has become standard in the financial community. We will spend more time on this in the later chapters.

American options can be exercised at any time up to and including the expiration date. There are no general formulas for valuing American options. However, we have a choice of models that approximate the price quite well such as the binomial model or Monte Carlo simulation-based models.

American options are rarely exercised early. This is because all options have a nonnegative time value and are usually worth more unexercised. Owners who wish to realize the full value of their options will mostly prefer to sell them rather than exercise them early and sacrifice some of the time value. An implication of this is that we can actually use the

Black–Scholes model for pricing. The reason is that American-type options will behave just like European if left unexercised up until maturity.

### Exchange-Traded or OTC?

Options that trade over-the-counter are customized to fit any particular needs an investor may have. Everything from price, strike price, expiration, underlying, or delivery terms can be customized. Just like other OTC derivatives, these options carry credit risk. The option holder faces the possibility that the option writer will not fulfill his/her obligations. At times, investors that are long OTC option require collateral from their counterparties.

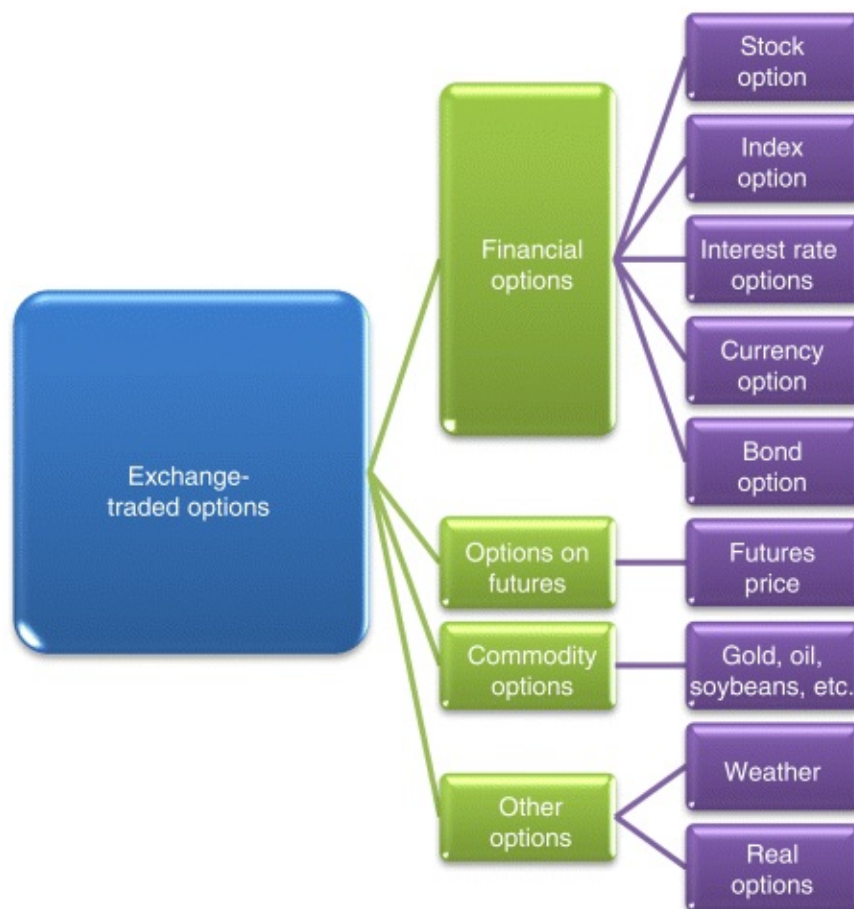
Most of the options we will be dealing with, however, are exchange-traded. These contracts are standardized in terms of the underlying asset, strike price, delivery method, and expiration time. The exchange establishes the minimum price quotation unit, whether the option is American or European, the contract size, and whether settlement is in cash or in the underlying asset. Each option contract is usually for a lot of 100 shares of the underlying. Trading volumes are the highest on options that are currently at-the-money. Neither deep-in-the-money options nor deep-out-of-the money options trade very often. Options that don't trade often will exhibit larger bid-ask spreads. Options for the most part are short-term instruments with most trading volume in expirations of less than one year. Also options can be bought and sold with ease. There is no need to hold it until expiration time and exercise the option. However, when options are in-the-money or at-the-money, they are frequently exercised.

### Moneyness of Options

The concept of moneyness describes whether an option is ITM, ATM, or OTM by comparing the position of strike vs. existing market price of the option's underlying security. Note that moneyness does not take into account option premiums.



## Types of Exchange-Traded Options



Financial options are the ones we'll focus mostly on in the remainder of

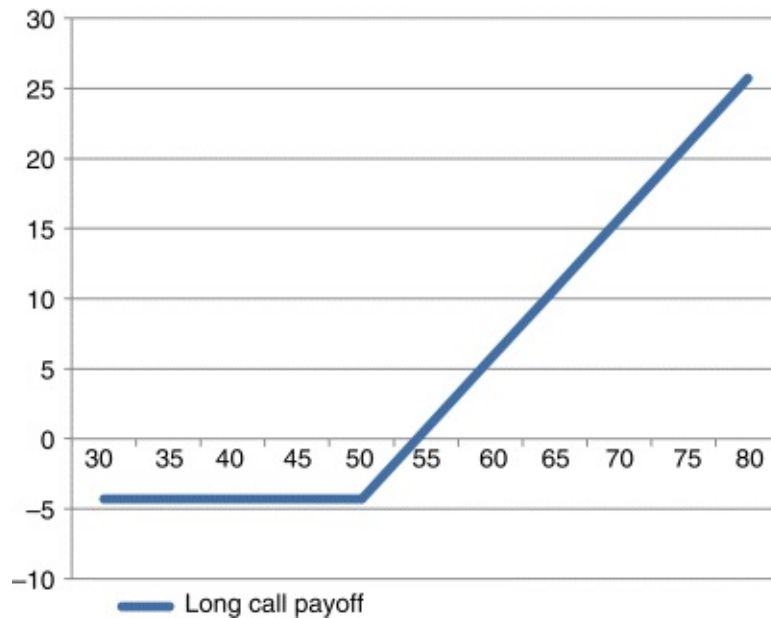
this book. A little bit of detail about each one of them is in order.

Stock Option	Index Option	Interest Rate Option	Currency Option	Bond Option
<ul style="list-style-type: none"><li>• These instruments are also known as equity options. They give the buyer the right to buy or sell an underlying stock at a certain strike price during a specific period of time or on a specific date.</li></ul>	<ul style="list-style-type: none"><li>• A call or put option on a financial index, such as the Nasdaq or S&amp;P 500.</li></ul>	<ul style="list-style-type: none"><li>• Interest rate options are European-style, cash-settled options on the yield of T-bills.</li><li>• If we expect an increase in interest rates, we'd buy a call option. If we expect interest rates to go down, we'd invest in a put.</li><li>• Interest rate options and other interest rate derivatives represent the largest portion of the derivatives market.</li></ul>	<ul style="list-style-type: none"><li>• A contract that grants the holder the right to buy or sell a certain currency at a specific strike price during a period of time. Investors use them to hedge against foreign currency risk.</li></ul>	<ul style="list-style-type: none"><li>• This is an option on a bond. If we expect a bond prices to increase, we'd buy a call. If, on the other hand, we expect interest rates to increase and bond prices decrease as a result, we'd invest in a put.</li></ul>

## Payoffs at Expiration

Options' payoffs at expiration time depend on the type of positions investors are holding. Investors can participate in options trading by taking on one the following four positions: long call, short call, long put, and a short put. The long positions give their holders the right to either buy or sell depending on whether they hold calls or puts. The short positions represent obligations. The obligation stems from their counterparty role to the long positions. If the long position decides to buy the underlying (i.e. exercise their call), the short position in this call has to sell it to them for the contractually binding strike price. The same applies in puts. If the long position holder sells the underlying for the strike price, it is the short position that has the obligation to buy it from them ([Fig. 4.1](#)).





**FIGURE 4.1** Long Call Option Payoff Function at Expiration Time

The payoff formulas for individual options follow.

*Notation*

$C$  Call option premium

$P$  Put option premium

$S_T$  Price of the underlying asset at expiration time

$X$  Strike price

The actual payoff at expiration time is calculated as the maximum of either zero or a positive difference between the price of the underlying and the strike price. For the net payoff, it is necessary to subtract the price we paid to get the option—in this case the call premium.

Equation 4.1 : Long call option payoff

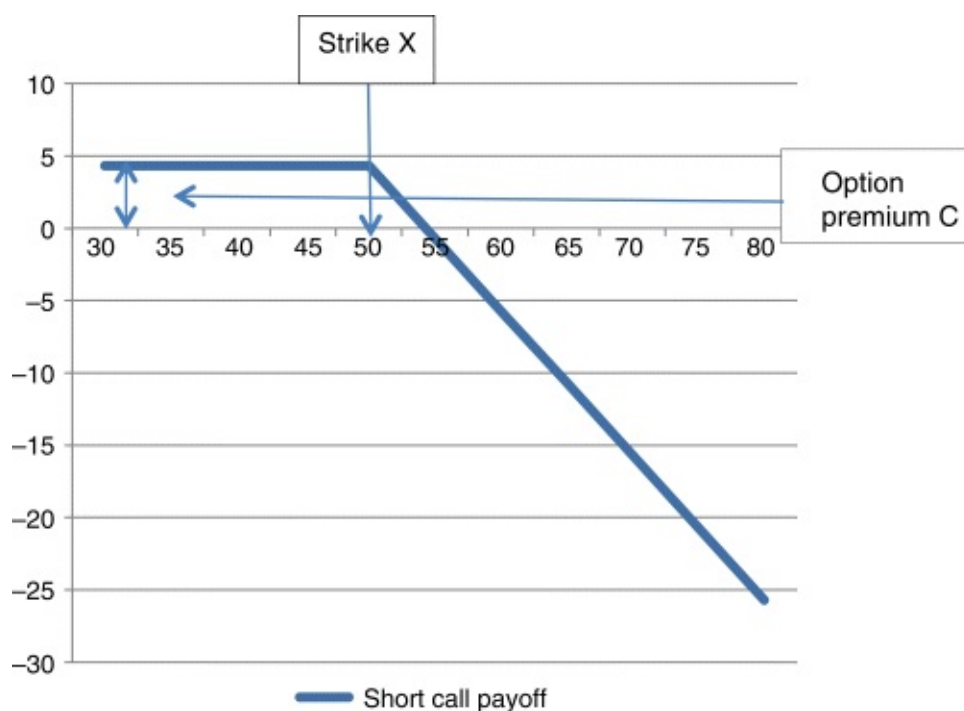
$$Payoff_{CALL}^{LONG} = [Max(0, S_T - X) - C]$$

Suppose in a call transaction we are the party that writes the call. In this

case, we hold the short position. Should the long position exercise their right to buy the underlying, the short position provides the asset for the agreed upon strike price. The short position also collects the option premium from the long position holder (Fig. 4.2).

### Equation 4.2 : Short call option payoff

$$Payoff_{CALL}^{SHORT} = -[Max(0, S_T - X) - C]$$

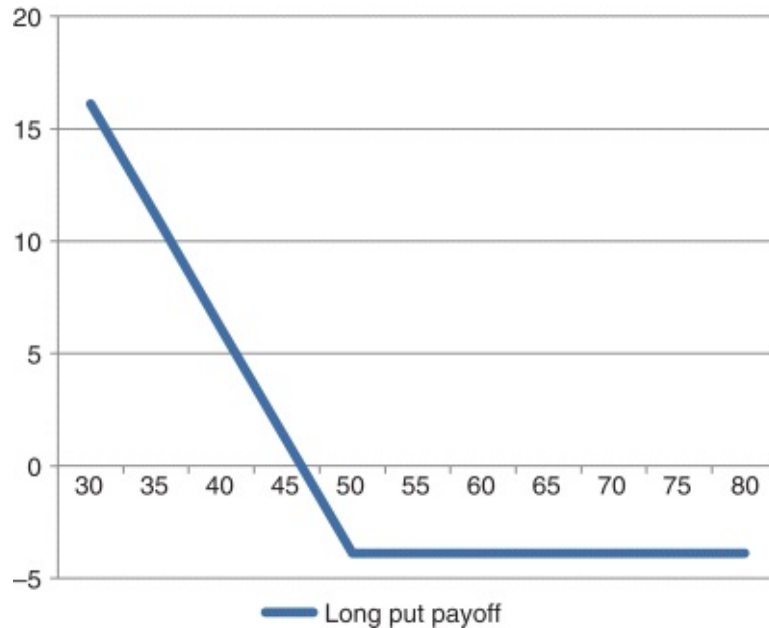


**FIGURE 4.2** Short Call Option Payoff Function at Expiration,  $X = \$50$ ,  $C = \$4.30$

Now let us turn to puts. A long put position has the right to sell the underlying asset at some time in the future for a prespecified strike price (Fig. 4.3). To obtain this right, the holder must pay a put option premium to the option writer.

### Equation 4.3 : Long put option payoff

$$\text{Payoff}_{PUT}^{LONG} = [\text{Max}(0, X - S_T) - P]$$

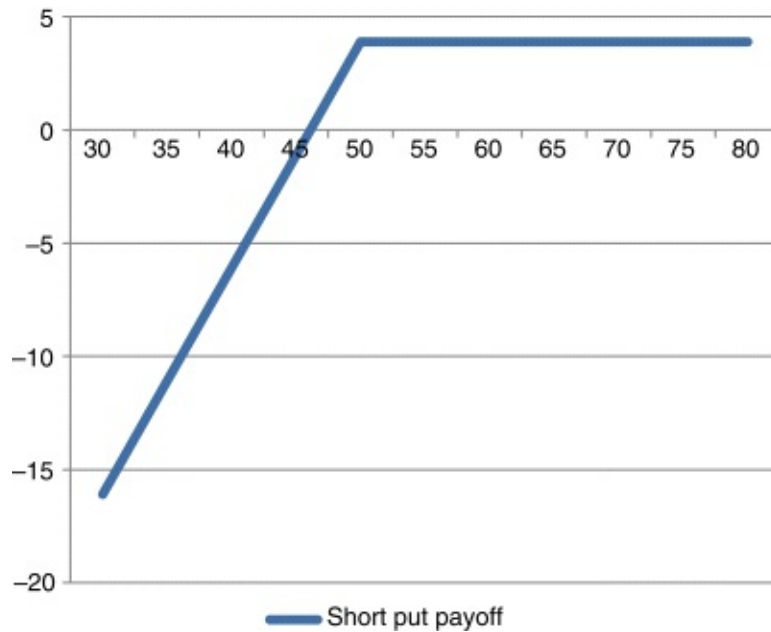


**FIGURE 4.3** Long Put Option Payoff Function at Expiration Time

Finally, the short put option payoff completes the set of four possible positions in options. The short put is a counterparty to the long put. Thus, as the long position exercises its right to sell the underlying, the short position must buy it from them, paying the strike price for the asset ([Fig. 4.4](#)).

### Equation 4.4 : Short put option payoff

$$\text{Payoff}^{SP} = -[\text{Max}(0, X - S_T) - P]$$



**FIGURE 4.4** Short Put Option Payoff Function at Expiration Time

The equation changes slightly for interest rate options depending on the tenure of the rate that is being used as the underlying in the contract.

Equation 4.5 : Interest rate option : long call payoff

$$Payoff_{CALL,IR}^{LONG} = \left[ \text{Max}(0, S_T - X) \times \left( \frac{T-t}{360} \right) - C \right]$$

Equation 4.6 : Interest rate option : long put payoff

$$Payoff_{PUT,IR}^{LONG} = \left[ \text{Max}(0, X - S_T) \times \left( \frac{T-t}{360} \right) - P \right]$$

Option's final payoffs at expiration are calculated by deducting the option premium paid from the intrinsic value of the option. Thus, an option could

produce a negative payoff if the premium is greater than the intrinsic value of the option.

The intrinsic value of an option is nonnegative for long positions in both puts and calls.

Equation 4.7 : Intrinsic value of a long call

$$\text{Intrinsic value}_{CALL}^{LONG} = \text{Max}(0, S_T - X)$$

Equation 4.8 : Intrinsic value of a long put

$$\text{Intrinsic value}_{PUT}^{LONG} = \text{Max}(0, X - S_T)$$

Options' overall value is a combination of the intrinsic value and its time value. We can define time value as any value of an option other than its intrinsic value. Time value is basically the risk premium that the seller requires to provide the option buyer with the right to buy or sell the underlying up to the expiration date.

Time value is related to the volatility of the option's underlying asset. It is low for a low-volatility underlying asset and high for a high-volatility underlying asset. It decreases as an option gets closer and closer to expiration. We will explore the reasons for this in more detail in the later chapters. This is why options are sometimes referred to as wasting assets. They “waste” their value as time approaches maturity. The reason is that the underlying asset has less and less time to move in a favorable direction for the option holder.

### Caps and Floors

Interest rate caps are used by borrowers in order to hedge against

floating rate risk. In an interest rate cap, the seller agrees to compensate the buyer for the amount by which an underlying floating interest rate exceeds a specified rate on a series of dates during the life of the contract. An interest rate cap is an instrument that caps your interest exposure. As a holder, you are guaranteed not to exceed the strike interest rate in your payments. A cap is actually a series of European calls, each of which expire on the date the floating loan rate will be reset. The individual parts of the call are called caplets. At each interest payment date, the holder decides whether to exercise or let that particular caplet expire.

Floors are often used to hedge against interest rates falling, and so it is a useful tool for lenders. They are similar to caps in that they consist of a series of European interest put options. Each part of the series is called a floorlet. In an interest rate floor, the seller agrees to compensate the buyer for a rate falling below the specified rate during the contract period. Using caps and floors we may construct a collar. A collar is a combination of a long (short) cap and short (long) floor, struck at different interest rates.

Equation 4.9 : Long cap payoff at expiration

$$Payoff_{CAP}^{LONG} = \left[ Max(0, S_T - X) \times NP \times \left( \frac{T-t}{360} \right) - C \right]$$

Equation 4.10 : Long floor payoff at expiration

$$Payoff_{FLOOR}^{LONG} = \left[ Max(0, X - S_T) \times NP \times \left( \frac{T-t}{360} \right) - P \right]$$

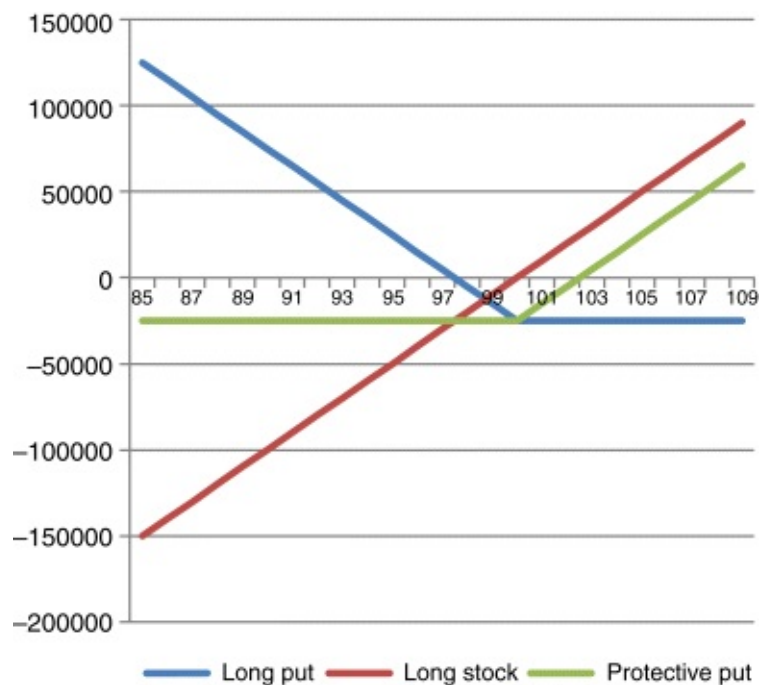
*Notation*

- $X$  Locked-in cap/floor rate
- $S_T$  Current market rate
- $C$  Option premium for a cap
- $P$  Option premium for a floor

### 4.1.2 Options Trading Strategies

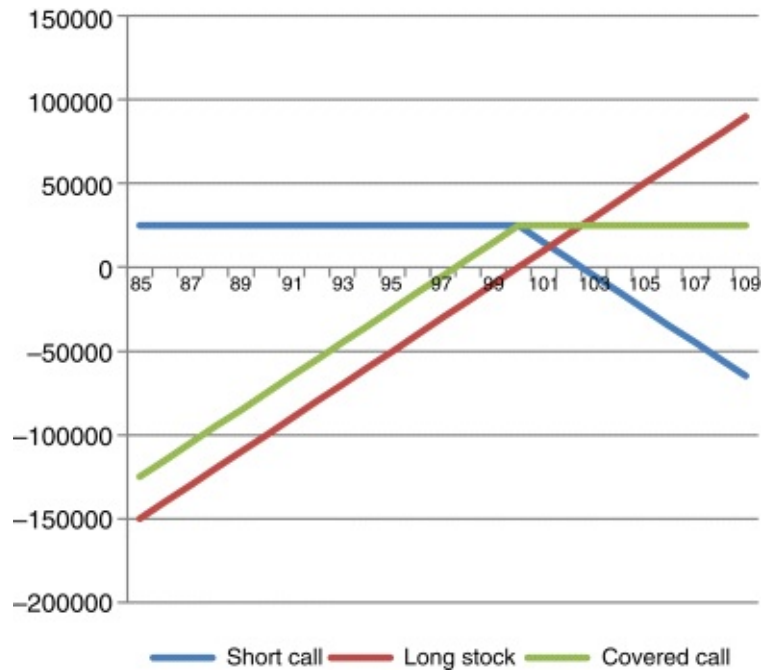
#### Protective Put & Covered Call

Protective put is a strategy combining a long position in the underlying and a long put struck near the purchasing price of the underlying asset ([Fig. 4.5](#)).



**FIGURE 4.5 Protective Put**

Covered call is a strategy joining a long position in the underlying with a short position in a call struck near the purchasing price of the underlying asset ([Fig. 4.6](#)).



**FIGURE 4.6 Covered Call**

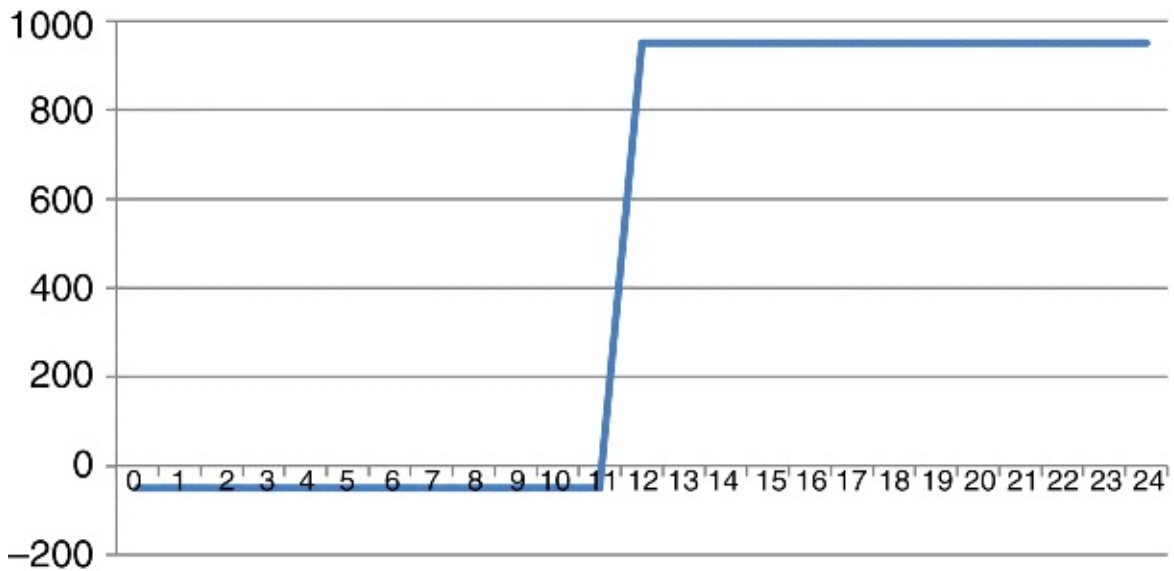
## Spreads

### Bull Spread

Bull spread is a strategy expecting a bull market in the underlying asset. It provides a way to cap the investor's downside exposure as well as limiting the upside. It can be created using a portfolio consisting of two puts or calls.

Consider a portfolio of two calls. One of them will be struck at  $\$X_1$  and the other one at  $\$X_2$ , where  $X_1 < X_2$ . We will take a long position in the first one and a short position in the second. The portfolio payoff is the bull spread ([Fig. 4.7](#)).



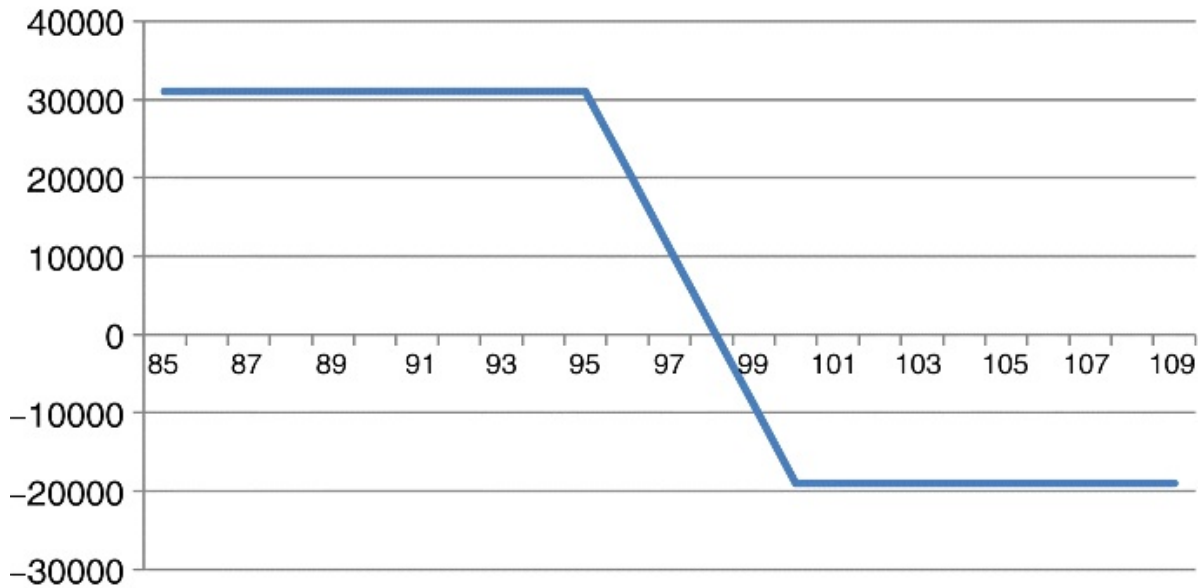


**FIGURE 4.7 Bull Spread**

### Bear Spread

Bear spread is a strategy profitable if the market for the underlying asset is going to be bearish. It provides a way to limit the investor's downside exposure as well as capping the upside. It can be created using a portfolio consisting of two puts or calls.

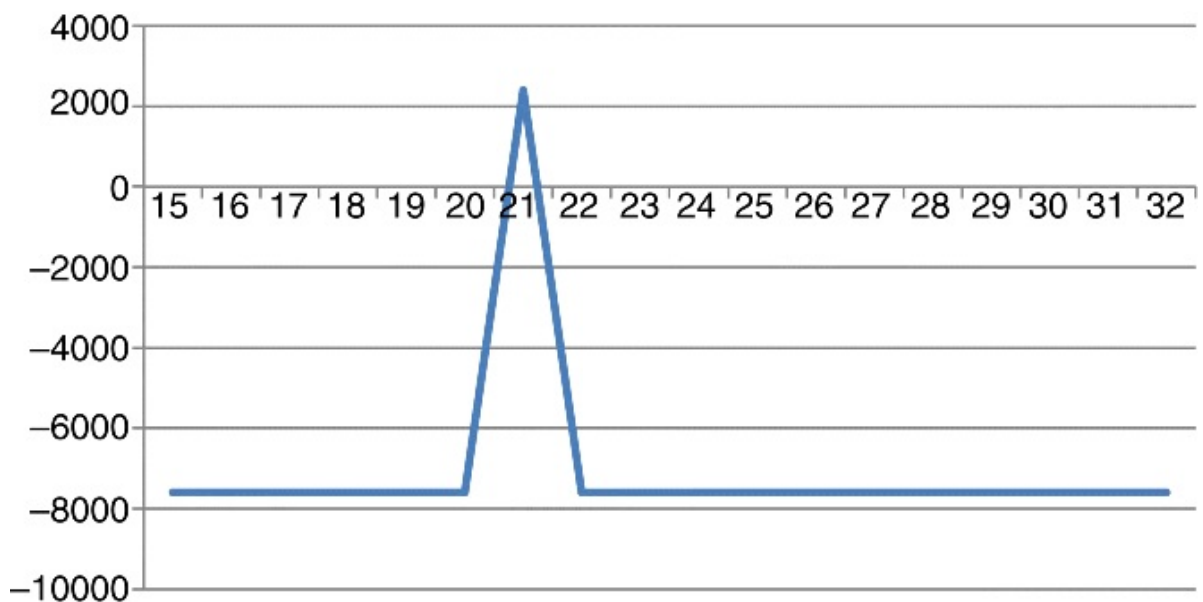
Consider a portfolio of two puts. One of them will be struck at  $\$X_1$  and the other one at  $\$X_2$ , again  $X_1 < X_2$ . We will take a short position in the first one and a long position in the second one. The portfolio payoff is the bear spread ([Fig. 4.8](#)).



**FIGURE 4.8** Bear Spread

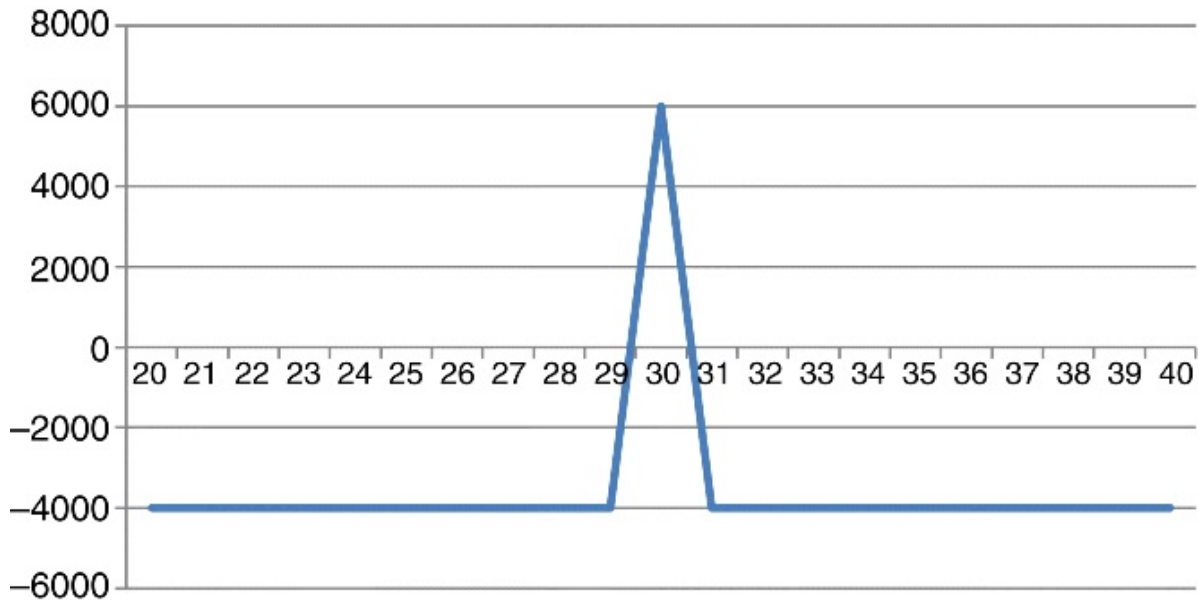
### Butterfly

Butterfly spread is a strategy consisting of three calls (or puts), each struck at different strike price. We want to be long the call with the lowest and the call with the highest strike price. The call position of the option in between the lowest and the highest strike price will be a short one, doubling up on the number of option contracts here (Fig. 4.9).



**FIGURE 4.9** Call Butterfly Spread, Underlying GM

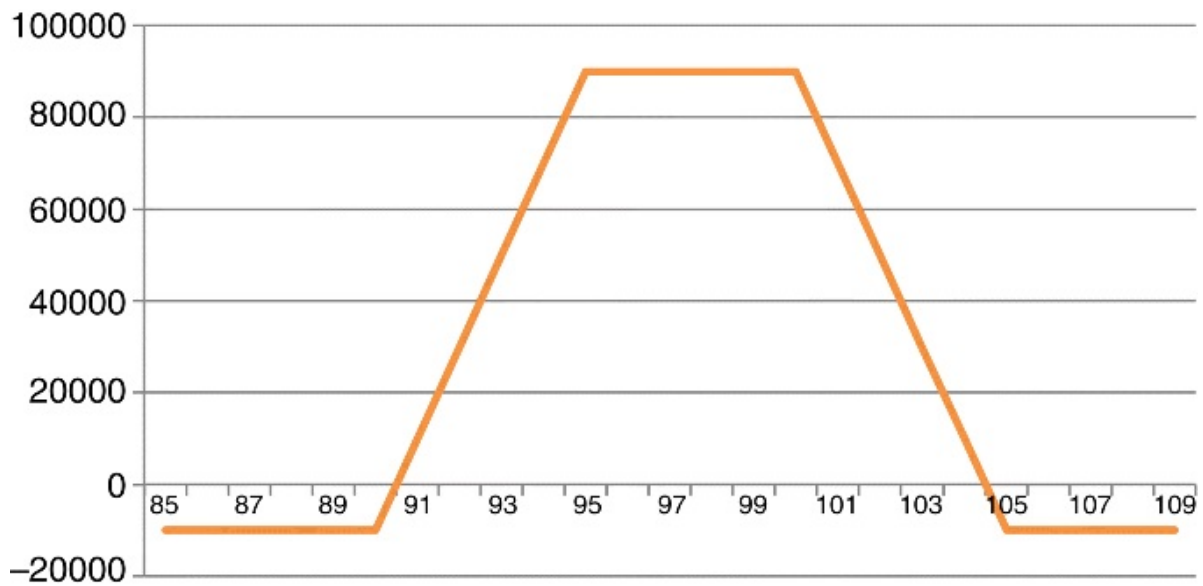
We can construct a butterfly using puts only as well ([Fig. 4.10](#)).



**FIGURE 4.10** Put Butterfly Spread, Underlying JCI

### Condor

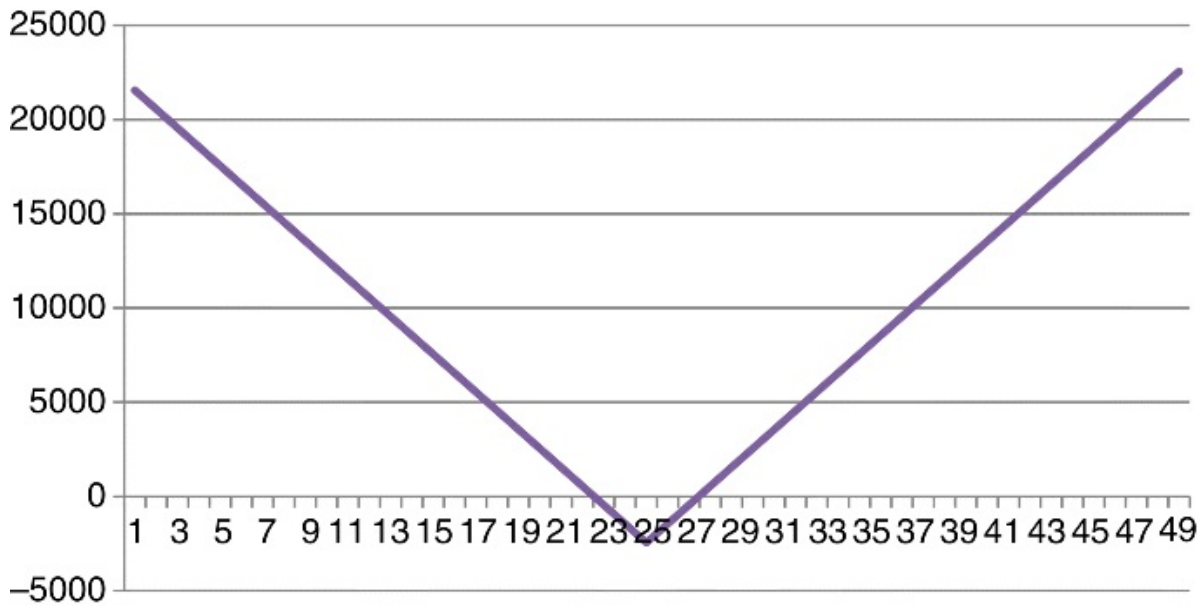
Condor spread is a portfolio consisting of four puts (or calls), with two short positions struck at different strike prices and two long positions struck at different strike prices. The condor offers a comfortable upside with a limited downside risk ([Fig. 4.11](#)).



## FIGURE 4.11 Condor

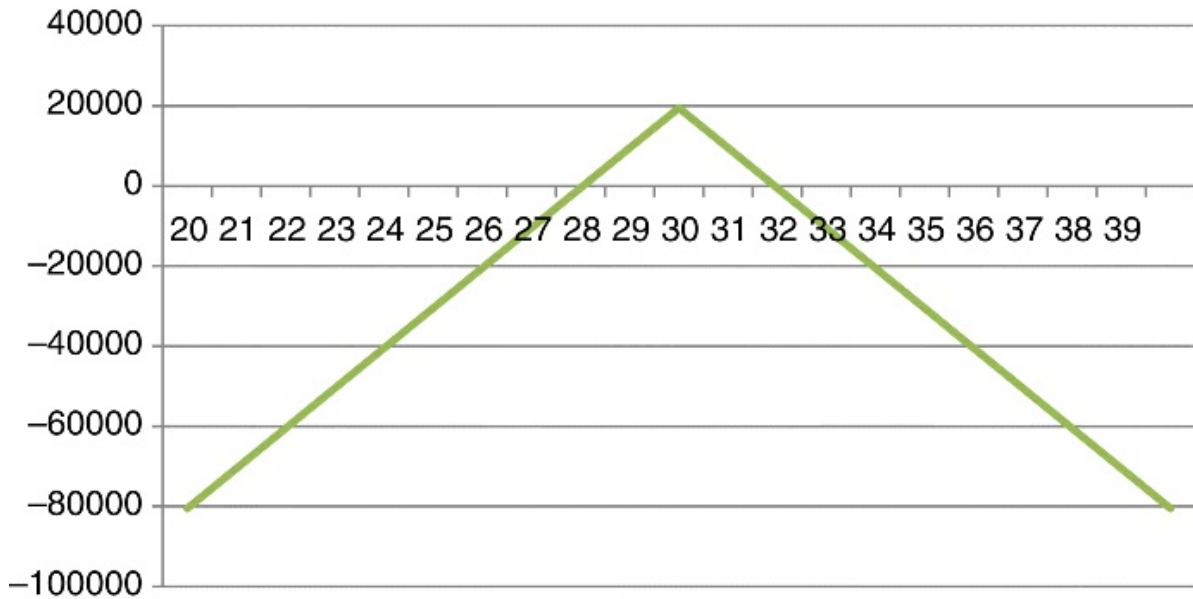
### Straddle

Straddle is essentially a volatility play. You can create a long straddle consisting of one long call and one long put, both struck at the same strike price. In a short straddle, you'll need to write (i.e., be short) one call and one put struck at the same strike price ([Fig. 4.12](#)).



## FIGURE 4.12 Long Straddle

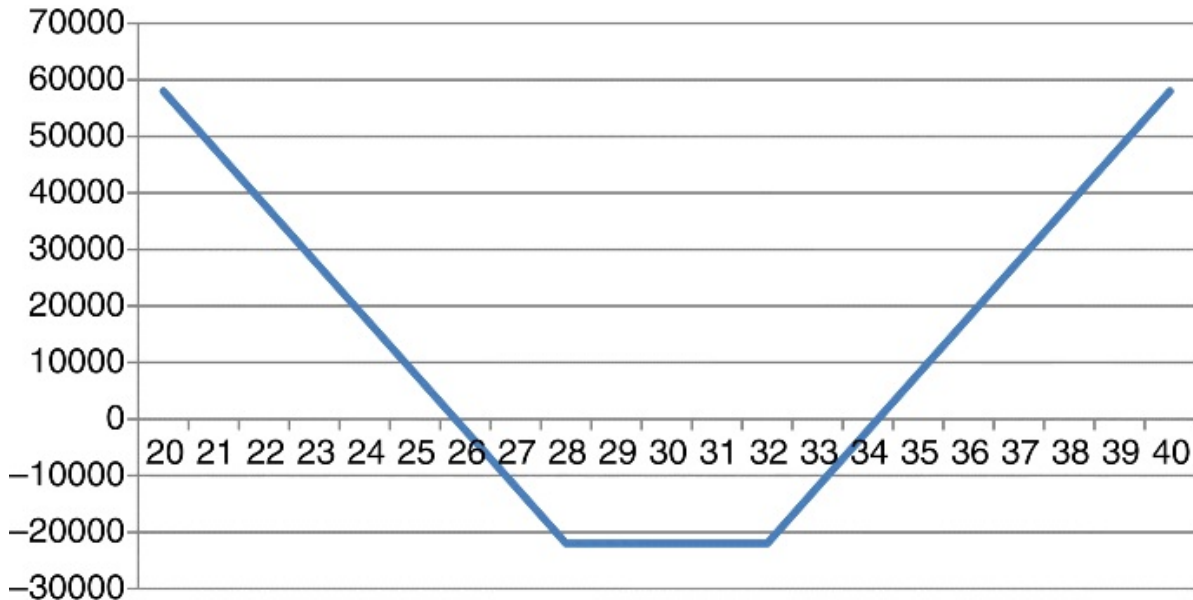
Long straddle is used when we expect volatility to increase. Short straddle on the other hand provides a positive payoff in times of low volatility ([Fig. 4.13](#)).



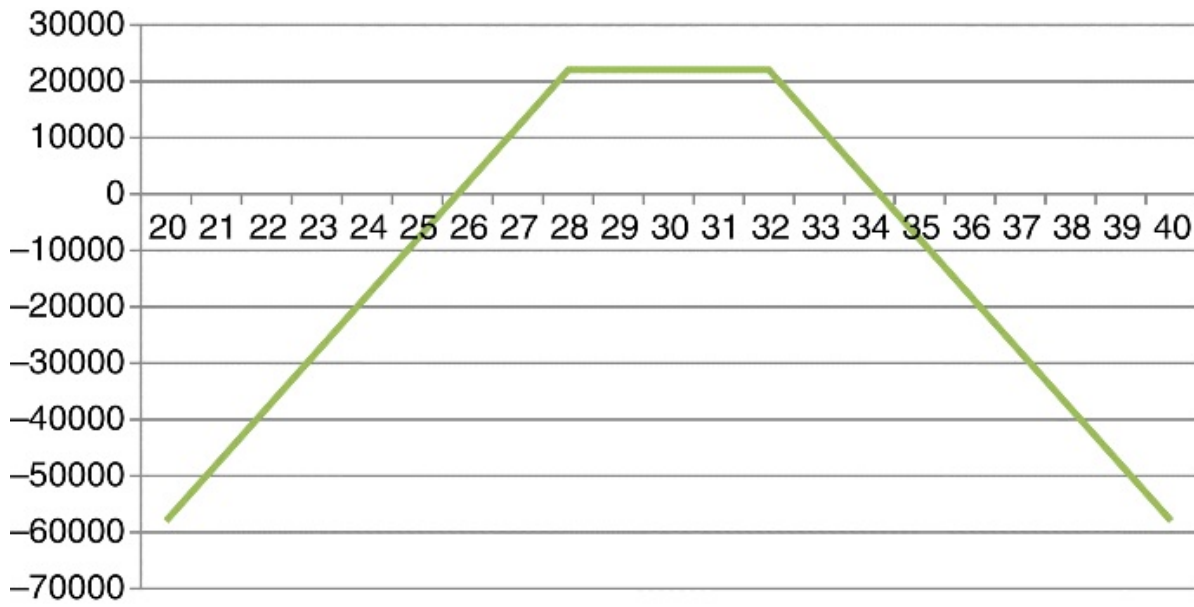
**FIGURE 4.13** Short Straddle

### Strangle

And finally strangle is very similar to straddle. In this case, however, the put strike price is always lower than the call strike price. This holds for both long and short strangle ([Figs. 4.14](#) and [4.15](#)).



**FIGURE 4.14** Long Strangle



**FIGURE 4.15** Short Strangle

### Case Study 4.1.2-1

Suppose it is May of 2012. You are considering your first options trade. You expect Morgan Stanley (MS) to do well in the upcoming months, that is, its stock price will likely rise. You'd like to take advantage of this opportunity using an option strategy. You are considering two approaches: entering into a plain vanilla call written on MS stock, or entering into a bull spread strategy written on MS stock.

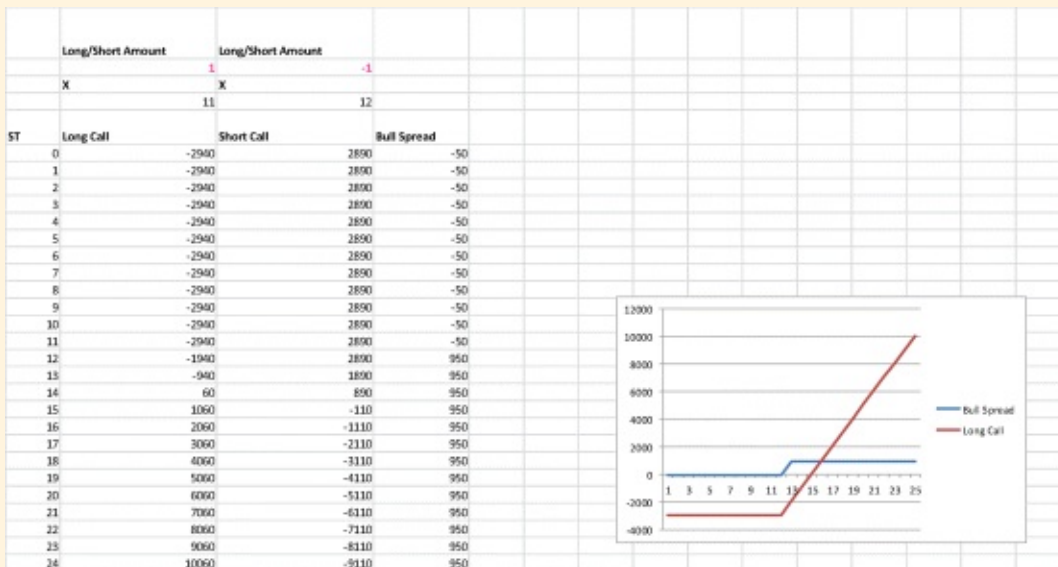
Using Excel spreadsheet please compare these two strategies in terms of gains/losses as a function of different future stock prices for MS stock.

Your time horizon for the anticipated stock price increase is approximately three months. You are selecting options with expiration date September 22, 2012. Given your budget you select a contract size of 10 contracts, each for 100 options. Thus, your position will consist of 1000 calls. At the time of the purchase, the September 22, 2012 call option premium is 2.94 (ask) for strike price of \$11, and 2.89 (bid) for strike price of \$12.

Using this information please calculate the payoffs, as functions of the underlying stock price, at the option's expiration time for

- a \$11 September 22, 2012 long call option position;
- a bull spread consisting of a long and a short position in the \$11 and \$12 September 22, 2012 calls;
- sketch your resulting position's payoffs on a diagram; and
- given that September 22, 2012 occurred in the past, using Yahoo Finance or other publicly available data sources we can find out the actual price of the underlying MS stock at that date. Once you have this information, please select the more favorable strategy, that is, plain call or the bull spread.

### Answer



Given the results a plain vanilla option would have been the strategy yielding greater gains. Both strategies, however, would lead to positive profits, and the bull spread has a lower downside risk.

	Contract Size	Contract Size	
	=100*10	=100*10	
	Call premium	Call premium	
	2.94	2.89	
	Long/Short Amount	Long/Short Amount	
	1	-1	
	X	X	
	11	12	
ST	Long Call	Short Call	Bull Spread
0	=B56*(MAX(0,SA10-B58)-B54))*B52	=C56*(MAX(0,SA10-C58)-C54))*C52	=SUM(B10:C10)
=A10+1	=B56*(MAX(0,A11-B58)-B54))*B52	=C56*(MAX(0,SA11-C58)-C54))*C52	=SUM(B11:C11)
=A11+1	=B56*(MAX(0,A12-B58)-B54))*B52	=C56*(MAX(0,SA12-C58)-C54))*C52	=SUM(B12:C12)
=A12+1	=B56*(MAX(0,A13-B58)-B54))*B52	=C56*(MAX(0,SA13-C58)-C54))*C52	=SUM(B13:C13)
=A13+1	=B56*(MAX(0,A14-B58)-B54))*B52	=C56*(MAX(0,SA14-C58)-C54))*C52	=SUM(B14:C14)
=A14+1	=B56*(MAX(0,A15-B58)-B54))*B52	=C56*(MAX(0,SA15-C58)-C54))*C52	=SUM(B15:C15)
=A15+1	=B56*(MAX(0,A16-B58)-B54))*B52	=C56*(MAX(0,SA16-C58)-C54))*C52	=SUM(B16:C16)
=A16+1	=B56*(MAX(0,A17-B58)-B54))*B52	=C56*(MAX(0,SA17-C58)-C54))*C52	=SUM(B17:C17)
=A17+1	=B56*(MAX(0,A18-B58)-B54))*B52	=C56*(MAX(0,SA18-C58)-C54))*C52	=SUM(B18:C18)
=A18+1	=B56*(MAX(0,A19-B58)-B54))*B52	=C56*(MAX(0,SA19-C58)-C54))*C52	=SUM(B19:C19)
=A19+1	=B56*(MAX(0,A20-B58)-B54))*B52	=C56*(MAX(0,SA20-C58)-C54))*C52	=SUM(B20:C20)
=A20+1	=B56*(MAX(0,A21-B58)-B54))*B52	=C56*(MAX(0,SA21-C58)-C54))*C52	=SUM(B21:C21)
=A21+1	=B56*(MAX(0,A22-B58)-B54))*B52	=C56*(MAX(0,SA22-C58)-C54))*C52	=SUM(B22:C22)
=A22+1	=B56*(MAX(0,A23-B58)-B54))*B52	=C56*(MAX(0,SA23-C58)-C54))*C52	=SUM(B23:C23)
=A23+1	=B56*(MAX(0,A24-B58)-B54))*B52	=C56*(MAX(0,SA24-C58)-C54))*C52	=SUM(B24:C24)
=A24+1	=B56*(MAX(0,A25-B58)-B54))*B52	=C56*(MAX(0,SA25-C58)-C54))*C52	=SUM(B25:C25)
=A25+1	=B56*(MAX(0,A26-B58)-B54))*B52	=C56*(MAX(0,SA26-C58)-C54))*C52	=SUM(B26:C26)
=A26+1	=B56*(MAX(0,A27-B58)-B54))*B52	=C56*(MAX(0,SA27-C58)-C54))*C52	=SUM(B27:C27)
=A27+1	=B56*(MAX(0,A28-B58)-B54))*B52	=C56*(MAX(0,SA28-C58)-C54))*C52	=SUM(B28:C28)
=A28+1	=B56*(MAX(0,A29-B58)-B54))*B52	=C56*(MAX(0,SA29-C58)-C54))*C52	=SUM(B29:C29)
=A29+1	=B56*(MAX(0,A30-B58)-B54))*B52	=C56*(MAX(0,SA30-C58)-C54))*C52	=SUM(B30:C30)
=A30+1	=B56*(MAX(0,A31-B58)-B54))*B52	=C56*(MAX(0,SA31-C58)-C54))*C52	=SUM(B31:C31)
=A31+1	=B56*(MAX(0,A32-B58)-B54))*B52	=C56*(MAX(0,SA32-C58)-C54))*C52	=SUM(B32:C32)
=A32+1	=B56*(MAX(0,A33-B58)-B54))*B52	=C56*(MAX(0,SA33-C58)-C54))*C52	=SUM(B33:C33)
=A33+1	=B56*(MAX(0,A34-B58)-B54))*B52	=C56*(MAX(0,SA34-C58)-C54))*C52	=SUM(B34:C34)

## Case Study 4.1.2-2

Suppose that you are making the following investment decision in May of 2012. You are considering entering into a short straddle or using a butterfly strategy. The underlying stock is Johnson Controls (JCI). The current price of the underlying is \$ 30.39 per share. You do not expect it to change very much in the upcoming month. You are selecting options with expiration date of June 15, 2012.

Using Excel spreadsheet please compare these two strategies in terms of gains/losses at expiration time.

Given your budget you select a contract size of 10 contracts, each for 100 options. Thus your position will consist of 1000 calls.

At the time of the purchase, we observe the following option premia:

\$29 June 15, 2012 put for \$0.85 (bid) - \$0.95 (ask)

\$30 June 15, 2012 put for \$1.15 (bid) - \$1.20 (ask)

\$30 June 15, 2012 call for \$1.25 (bid) - \$1.30 (ask)



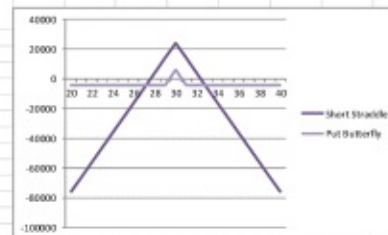
\$31 June 15, 2012 put for \$1.65 (bid) - \$1.75 (ask)

Using this information please calculate the payoffs, as functions of the underlying stock price, at the option's expiration time for

- a short straddle consisting of a short put and a short call;
- a butterfly spread;
- sketch your resulting position's payoffs on a diagram;
- using Yahoo Finance or other publicly available data sources please find out the actual price of the underlying JCI stock on June 15, 2012. Once you have this information, please point out the more profitable strategy.

### Answer

JCI	Expiration Date	Stock purchase price	Contract Size	Contract Size	Contract Size	Contract Size
	15-Jun-12	30.39	10000	10000	10000	10000
			Put	Put	Put	Put
			premium	premium	premium	premium
			1.25	0.95	1.15	1.75
			Long/Short Amount	Long/Short Amount	Long/Short Amount	Long/Short Amount
			-1	1	-2	1
			X	X1	X2	X3
			30	29	30	31
				Put-based Butterfly		
ST			Short Straddle		Put Butterfly	
20	12500	-88900	-76000	80500	-177000	92500
21	12500	-78900	-66000	70500	-157000	82500
22	12500	-68900	-56000	60500	-137000	72500
23	12500	-58900	-46000	50500	-117000	62500
24	12500	-48900	-36000	40500	-97000	52500
25	12500	-38900	-26000	30500	-77000	42500
26	12500	-28900	-16000	20500	-57000	32500
27	12500	-18900	-6000	10500	-37000	22500
28	12500	-8900	4000	500	-17000	12500
29	12500	1900	14000	-9500	3000	2500
30	12500	11900	24000	-9500	23000	-7500
31	2500	11900	14000	-9500	23000	-17500
32	-7500	11900	4000	-9500	23000	-17500
33	-17500	11900	-6000	-9500	23000	-17500
34	-27500	11900	-16000	-9500	23000	-17500
35	-37500	11900	-26000	-9500	23000	-17500
36	-47500	11900	-36000	-9500	23000	-17500
37	-57500	11900	-46000	-9500	23000	-17500
38	-67500	11900	-56000	-9500	23000	-17500
39	-77500	11900	-66000	-9500	23000	-17500
40	-87500	11900	-76000	-9500	23000	-17500



In this case, the more profitable strategy would have been the short straddle rather than the butterfly spread.

JCI	Expiration Date	Stock purchase price								
	4/27/15	30.39								
	Contract Size	Contract Size			Contract Size	Contract Size	Contract Size	Contract Size	Contract Size	
	=100*100	=100*100			=100*100	=100*100	=100*100	=100*100	=100*100	
	Call premium	Put premium			Put premium	Put premium	Put premium	Put premium	Put premium	
	1.25	1.15			0.95	1.15	1.15	1.75		
	Long/Short Amount	Long/Short Amount			Long/Short Amount	Long/Short Amount	Long/Short Amount	Long/Short Amount	Long/Short Amount	
	-1	-1			1	-2	1	1		
	X	X			X1	X2	X3	X3		
	30	30			29	30	31	31		
										Put-based Butterfly
ST										Butterfly
=A14+1	=B59*(MAX(0, SA14-B511)-B57)*B55	=C59*(MAX(0, CS11- SA14)-C57)*C55	=B14+C14		=(G59*(MAX(0, GS11- SA14)-GS7)*G55	=(H59*(MAX(0, HS11- SA14)-HS7)*H55	=(I59*(MAX(0, IS11- SA14)-IS7)*I55	=SUM(G14:H14)		
=A15+1	=B59*(MAX(0, SA15-B511)-B57)*B55	=C59*(MAX(0, CS11- SA15)-C57)*C55	=B15+C15		=(G59*(MAX(0, GS11- SA15)-GS7)*G55	=(H59*(MAX(0, HS11- SA15)-HS7)*H55	=(I59*(MAX(0, IS11- SA15)-IS7)*I55	=SUM(G15:H15)		
=A16+1	=B59*(MAX(0, SA16-B511)-B57)*B55	=C59*(MAX(0, CS11- SA16)-C57)*C55	=B16+C16		=(G59*(MAX(0, GS11- SA16)-GS7)*G55	=(H59*(MAX(0, HS11- SA16)-HS7)*H55	=(I59*(MAX(0, IS11- SA16)-IS7)*I55	=SUM(G16:H16)		
=A16+1	=B59*(MAX(0, SA17-B511)-B57)*B55	=C59*(MAX(0, CS11- SA17)-C57)*C55	=B17+C17		=(G59*(MAX(0, GS11- SA17)-GS7)*G55	=(H59*(MAX(0, HS11- SA17)-HS7)*H55	=(I59*(MAX(0, IS11- SA17)-IS7)*I55	=SUM(G17:H17)		
=A17+1	=B59*(MAX(0, SA18-B511)-B57)*B55	=C59*(MAX(0, CS11- SA18)-C57)*C55	=B18+C18		=(G59*(MAX(0, GS11- SA18)-GS7)*G55	=(H59*(MAX(0, HS11- SA18)-HS7)*H55	=(I59*(MAX(0, IS11- SA18)-IS7)*I55	=SUM(G18:H18)		
=A18+1	=B59*(MAX(0, SA19-B511)-B57)*B55	=C59*(MAX(0, CS11- SA19)-C57)*C55	=B19+C19		=(G59*(MAX(0, GS11- SA19)-GS7)*G55	=(H59*(MAX(0, HS11- SA19)-HS7)*H55	=(I59*(MAX(0, IS11- SA19)-IS7)*I55	=SUM(G19:H19)		
=A19+1	=B59*(MAX(0, SA20-B511)-B57)*B55	=C59*(MAX(0, CS11- SA20)-C57)*C55	=B20+C20		=(G59*(MAX(0, GS11- SA20)-GS7)*G55	=(H59*(MAX(0, HS11- SA20)-HS7)*H55	=(I59*(MAX(0, IS11- SA20)-IS7)*I55	=SUM(G20:H20)		
=A20+1	=B59*(MAX(0, SA21-B511)-B57)*B55	=C59*(MAX(0, CS11- SA21)-C57)*C55	=B21+C21		=(G59*(MAX(0, GS11- SA21)-GS7)*G55	=(H59*(MAX(0, HS11- SA21)-HS7)*H55	=(I59*(MAX(0, IS11- SA21)-IS7)*I55	=SUM(G21:H21)		
=A21+1	=B59*(MAX(0, SA22-B511)-B57)*B55	=C59*(MAX(0, CS11- SA22)-C57)*C55	=B22+C22		=(G59*(MAX(0, GS11- SA22)-GS7)*G55	=(H59*(MAX(0, HS11- SA22)-HS7)*H55	=(I59*(MAX(0, IS11- SA22)-IS7)*I55	=SUM(G22:H22)		
=A22+1	=B59*(MAX(0, SA23-B511)-B57)*B55	=C59*(MAX(0, CS11- SA23)-C57)*C55	=B23+C23		=(G59*(MAX(0, GS11- SA23)-GS7)*G55	=(H59*(MAX(0, HS11- SA23)-HS7)*H55	=(I59*(MAX(0, IS11- SA23)-IS7)*I55	=SUM(G23:H23)		
=A23+1	=B59*(MAX(0, SA24-B511)-B57)*B55	=C59*(MAX(0, CS11- SA24)-C57)*C55	=B24+C24		=(G59*(MAX(0, GS11- SA24)-GS7)*G55	=(H59*(MAX(0, HS11- SA24)-HS7)*H55	=(I59*(MAX(0, IS11- SA24)-IS7)*I55	=SUM(G24:H24)		
=A24+1	=B59*(MAX(0, SA25-B511)-B57)*B55	=C59*(MAX(0, CS11- SA25)-C57)*C55	=B25+C25		=(G59*(MAX(0, GS11- SA25)-GS7)*G55	=(H59*(MAX(0, HS11- SA25)-HS7)*H55	=(I59*(MAX(0, IS11- SA25)-IS7)*I55	=SUM(G25:H25)		
=A25+1	=B59*(MAX(0, SA26-B511)-B57)*B55	=C59*(MAX(0, CS11- SA26)-C57)*C55	=B26+C26		=(G59*(MAX(0, GS11- SA26)-GS7)*G55	=(H59*(MAX(0, HS11- SA26)-HS7)*H55	=(I59*(MAX(0, IS11- SA26)-IS7)*I55	=SUM(G26:H26)		
=A26+1	=B59*(MAX(0, SA27-B511)-B57)*B55	=C59*(MAX(0, CS11- SA27)-C57)*C55	=B27+C27		=(G59*(MAX(0, GS11- SA27)-GS7)*G55	=(H59*(MAX(0, HS11- SA27)-HS7)*H55	=(I59*(MAX(0, IS11- SA27)-IS7)*I55	=SUM(G27:H27)		
=A27+1	=B59*(MAX(0, SA28-B511)-B57)*B55	=C59*(MAX(0, CS11- SA28)-C57)*C55	=B28+C28		=(G59*(MAX(0, GS11- SA28)-GS7)*G55	=(H59*(MAX(0, HS11- SA28)-HS7)*H55	=(I59*(MAX(0, IS11- SA28)-IS7)*I55	=SUM(G28:H28)		
=A28+1	=B59*(MAX(0, SA29-B511)-B57)*B55	=C59*(MAX(0, CS11- SA29)-C57)*C55	=B29+C29		=(G59*(MAX(0, GS11- SA29)-GS7)*G55	=(H59*(MAX(0, HS11- SA29)-HS7)*H55	=(I59*(MAX(0, IS11- SA29)-IS7)*I55	=SUM(G29:H29)		
=A29+1	=B59*(MAX(0, SA30-B511)-B57)*B55	=C59*(MAX(0, CS11- SA30)-C57)*C55	=B30+C30		=(G59*(MAX(0, GS11- SA30)-GS7)*G55	=(H59*(MAX(0, HS11- SA30)-HS7)*H55	=(I59*(MAX(0, IS11- SA30)-IS7)*I55	=SUM(G30:H30)		
=A30+1	=B59*(MAX(0, SA31-B511)-B57)*B55	=C59*(MAX(0, CS11- SA31)-C57)*C55	=B31+C31		=(G59*(MAX(0, GS11- SA31)-GS7)*G55	=(H59*(MAX(0, HS11- SA31)-HS7)*H55	=(I59*(MAX(0, IS11- SA31)-IS7)*I55	=SUM(G31:H31)		
=A31+1	=B59*(MAX(0, SA32-B511)-B57)*B55	=C59*(MAX(0, CS11- SA32)-C57)*C55	=B32+C32		=(G59*(MAX(0, GS11- SA32)-GS7)*G55	=(H59*(MAX(0, HS11- SA32)-HS7)*H55	=(I59*(MAX(0, IS11- SA32)-IS7)*I55	=SUM(G32:H32)		
=A32+1	=B59*(MAX(0, SA33-B511)-B57)*B55	=C59*(MAX(0, CS11- SA33)-C57)*C55	=B33+C33		=(G59*(MAX(0, GS11- SA33)-GS7)*G55	=(H59*(MAX(0, HS11- SA33)-HS7)*H55	=(I59*(MAX(0, IS11- SA33)-IS7)*I55	=SUM(G33:H33)		
=A33+1	=B59*(MAX(0, SA34-B511)-B57)*B55	=C59*(MAX(0, CS11- SA34)-C57)*C55	=B34+C34		=(G59*(MAX(0, GS11- SA34)-GS7)*G55	=(H59*(MAX(0, HS11- SA34)-HS7)*H55	=(I59*(MAX(0, IS11- SA34)-IS7)*I55	=SUM(G34:H34)		

# Problems

**4.1.2-1.** Suppose you took a long position in a June 2014 75 put three months prior to its expiration. You held it until maturity. The price per share of the underlying stock was \$73.20 at expiration. The put premium you paid at the end of March 2014 was \$1.35. How much have you made/lost per contract?

**4.1.2-2.** Suppose you have written a put expiring in September 2014. The strike price is \$50. The price of the underlying stock turned out to be \$55.35 at expiration. The put option premium was \$2.25. What is your gain/loss per contract at expiration time?

**4.1.2-3.** Sketch a covered call strategy using a payoff diagram.

**4.1.2-4.** Sketch a condor spread using a payoff diagram. How many different options do you need to create a condor?

**4.1.2-5.** When would you use a bull spread? Under what circumstances would a bear spread make sense?

**4.1.2-6.** You expect Reef Industries, Inc. to do really well in the upcoming year. You would like to enter into a bull spread strategy written on Reef Industries, Inc. stock. Using Excel spreadsheet please construct this strategy and create a payoff chart showing gains/losses under different Reef Industries, Inc. stock prices.

Given your budget you select a contract size of 10 contracts, each

for 100 options, for each part of the spread strategy. At the time of the purchase, the June 22, 2014 call option premium is 0.55 (ask) for strike price of \$10, and 0.28 (bid) for strike price of \$11.

Using this information please calculate the payoffs, as functions of the underlying stock price, at the strategy's expiration time for

- a. a bull spread consisting of a long and a short position in the \$10 and \$11 June 22, 2014 calls;
- b. create an Excel chart showing your results—that is, payoff diagram; and
- c. What would your payoff look like if the June 22 stock price ends up being \$10.65?

**4.1.2-7.** Using information from 4.1.2-6 please construct a bear spread.

Using this information please calculate the payoffs as functions of the underlying stock price at the strategy's expiration time for this bear spread. At the time of the purchase, the June 22, 2014 put option premium is 0.45 (ask) for strike price of \$11, and 0.28 (bid) for strike price of \$10.

Create an Excel chart showing your results—that is, payoff diagram.

What would your payoff look like if the June 22 stock price ends up being \$10.65?

**4.1.2-8.** Suppose you have a put and a call. Both have the same strike price and expiration date. The strike price is \$25. The put premium is \$1.15 (bid) and \$1.20 (ask). The call premium is \$1.20 (bid) and \$1.25 (ask).

Using Excel please calculate payoffs for a long Straddle position as a function of the underlying stock price. Assume stock price ranges from \$1 to \$50. Please chart your answer.

How would your answer change if you held a short Straddle position?

## 4.2 Pricing

### 4.2.1 Binomial Tree Option Pricing Model

Option pricing is a challenging task. This book will present two fundamental approaches to the task. One is the Black–Scholes pricing formula, which will be introduced later in the text. Right now, let us focus on the binomial tree pricing model.

The binomial pricing model is based on the assumption that an asset price at any given moment can either go up or down or stay the same. For simplicity of exposure, in the following we'll assume that at each time node the price moves either up or down limiting our options to two possible states of the world instead of three—hence the name binomial. Furthermore, we assume that our “trees” have recombining “branches”.

The recombining quality of the trees is apparent once we introduce more than one future time period. Suppose, we want to analyze two future time periods. If the price path is up at time 1 and down at time 2, that price will be identical as if we had proceeded down at time 1 and up at time 2. Let me explain this in greater detail once we elaborate on binomial models with two or more periods.

In the following I will assume that the options we are pricing are of European type. I do that for simplicity of exposition. But, what about the American type of options? The American options pricing procedure is essentially identical—we start with the expiration-time payoffs at each possible state of the world and work our way through time to the present. What is different for the American type of options is that at all earlier times before expiration time the value of such an option will either be determined by (a) applying the binomial pricing model or (b) will be equal to the actual payoff from early exercise. In contrast, the European option value at all earlier times is equal to the model-determined value.

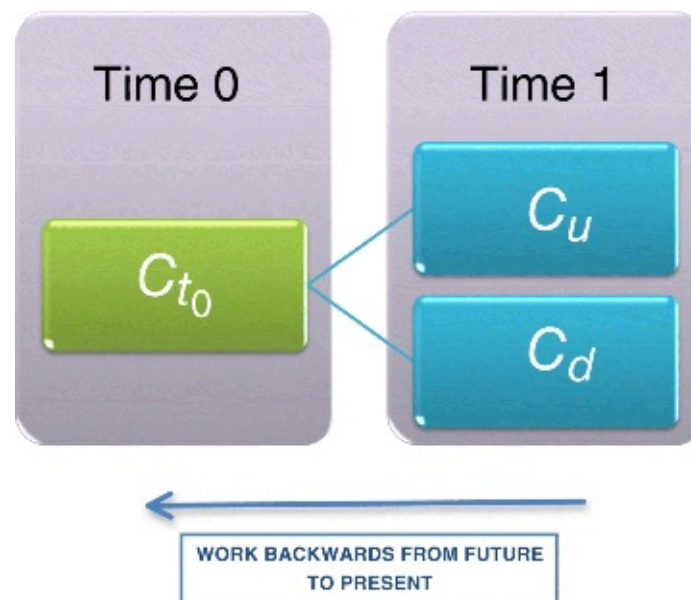
Now let us zoom on a very important concept in pricing derivative securities such as options—the notion of a risk-neutral world. In a risk-neutral world, all individuals are indifferent to risk, and so they require no compensation for risk. Their expected return on all securities is the risk-free interest rate. In pricing options, it is legitimate to assume that the world is risk neutral. The resulting option prices are accurate not only in a risk-neutral world, but also in the real world.

Whenever we move from the real world to the risk-neutral world the expected growth rates in variables (such as stock returns) change but their volatilities remain the same. This important concept is known as the

Girsanov theorem. Switching between the risk-neutral and real worlds is often referred to as changing the measure.

### One-Period Binomial Model

Let us start the exposition by introducing the one-period binomial model. Once the basic structure is in place and well understood we can extend the model to any number of time periods. This model lets us calculate the current “fair” price of a call (or put) option, relying on information from future option payoffs. At each time, we assume that the respective price can either move up or down by a certain amount with a certain probability ([Fig. 4.16](#)).



**FIGURE 4.16** One-Period Binomial Tree

#### Notation

- $u$  Positive growth parameter for the underlying asset if the price move is up
- $d$  Negative growth parameter for the underlying asset if the price moves down
- $S_u$  Price of the underlying asset after an up move
- $S_d$  Price of the underlying asset after a down move
- $S_{t_0}$  Current price of the underlying asset
- $C_u$  Call price in the up state

- $C_d$  Call price in the down state  
 $C_{t_0}$  Call price at present  
 $X$  Strike price on the option  
 $h$  Hedge ratio, often referred to as Delta  
 $\sigma$  Stock return volatility  
 $\Delta t$  Length of a time step

In a one-period binomial model, the expiration-time payoff is calculated as the maximum of either a positive difference between the price of the underlying and the strike price, or zero.

Equation 4.11 : Call option payoff at expiration

$$\begin{aligned} C_u &= \text{Max}(0, S_u - X) \\ C_d &= \text{Max}(0, S_d - X) \end{aligned}$$

The down parameter and the up parameter are of crucial importance and also have to satisfy the following condition:

$$d \leq 1 + r \leq u$$

How can one determine these two parameters? In practice, these parameters are chosen to match the stock return volatility. A widely used way of determining these is  $u = e^{\sigma\sqrt{\Delta t}}$  for the up and  $d = e^{-\sigma\sqrt{\Delta t}}$  for the down parameter.

The price of the underlying asset is calculated as the current price multiplied by the up/down factor.

Equation 4.12 : Future price of the underlying

$$\begin{aligned} S_u &= S_{t_0} \times u \\ S_d &= S_{t_0} \times d \end{aligned}$$

The Delta—that is, the hedge ratio—is a ratio of the change in the price of the option to the change in price of the underlying stock. It represents the number of units of the underlying asset one should hold to cover his short option exposure. This is called delta hedging. If we hedge our portfolio, it will always yield the risk-free rate.

Thus, their expected return on all securities is the risk-free interest rate. An option's value can be calculated as its expected payoff discounted at a risk-free rate.

Equation 4.13 : Hedge ratio

$$h = \frac{C_u - C_d}{S_u - S_d}$$

A hedge portfolio consists of a long position in  $h$  units of stock and short position in one call. Thus, the value of the hedge portfolio can be calculated as the difference between  $h$  units of the underlying and the current call price.

Equation 4.14 : Hedge portfolio value

$$V_0 = h \times S_{t_0} - C_{t_0}$$

Equation 4.15 : Future value of the hedge portfolio

$$V_u = h \times S_u - C_u$$

$$V_d = h \times S_d - C_d$$

In order to achieve same return regardless of movements in the underlying asset, these have to equal.

$$V_u = V_d$$

$$h \times S_u - C_u = h \times S_d - C_d$$

$$h \times S_u - h \times S_d = C_u - C_d$$

$$h \times (S_u - S_d) = C_u - C_d$$

Solving for h yields, the above-mentioned formulas are used for calculating the hedge ratio.

The initial hedge portfolio grows at a risk-free rate. So, its value at expiration time (i.e., one period from present) will be  $V_0(1 + r)$ .

$$V_0(1 + r) = V_u = V_d$$

$$(h \times S_{t_0} - C_{t_0})(1 + r) = h \times S_u - C_u$$



Simplify

$$h = \frac{C_{t_0} + C_{t_0} \times r - C_u}{S_{t_0} + S_{t_0} \times r - S_u}$$

Plug in  $h = \frac{C_u - C_d}{S_u - S_d}$  and solve for  $C_{t_0}$ .

The current—that is, time 0—price of the call option is calculated as a weighted average of future prices discounted to present. We may assume discrete or continuous discounting.

Equation 4.16 : Call option price at time 0

$$C_{t_0} = \frac{p \times C_u + (1-p) \times C_d}{(1+r)}$$

The probability of an up/down move is given by the following:

Equation 4.17 : Probability of an up move

$$p = \frac{(1+r) - d}{u - d}$$

Equation 4.18 : Probability of a down move

$$(1 - p) = \frac{u - (1+r)}{u - d}$$

Probability of an up/down move is a risk-neutral probability. Reflecting on [Equation 4.16](#)—the value of the option is its expected payoff in a risk-neutral world discounted at the risk-free rate. This result is an example of an important general principal in option pricing known as the risk-neutral valuation.

Should we assume calculating option prices in continuous time, we will need to modify [Equations 4.16](#) and [4.17](#) somewhat.

Equation 4.19 : Continuous time call price & probability of an up move

$$C_{t_0} = e^{-rT} (p \times C_u + (1 - p) \times C_d)$$

Equation 4.20 : Continuous time probability of an up move

$$p = \frac{e^{rT} - d}{u - d}$$

## Case Study 4.2.1-1

Set up a one-period binomial tree and find out the current price of a call option using the information below.

$$S_{t_0} = 100$$

$$X = 100$$

$$u = 1.25$$

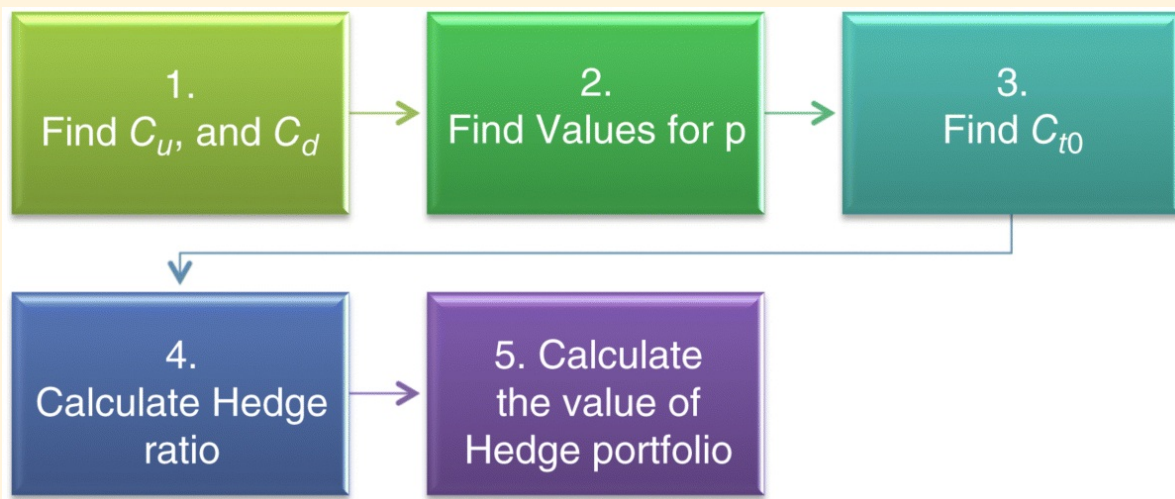
$$d = 0.8$$

$$r = 0.07$$

What is the current value of a hedge portfolio consisting of a short call and  $h$  units of the underlying stock?

**Answer**

In this exercise we'll follow this sequence of steps.

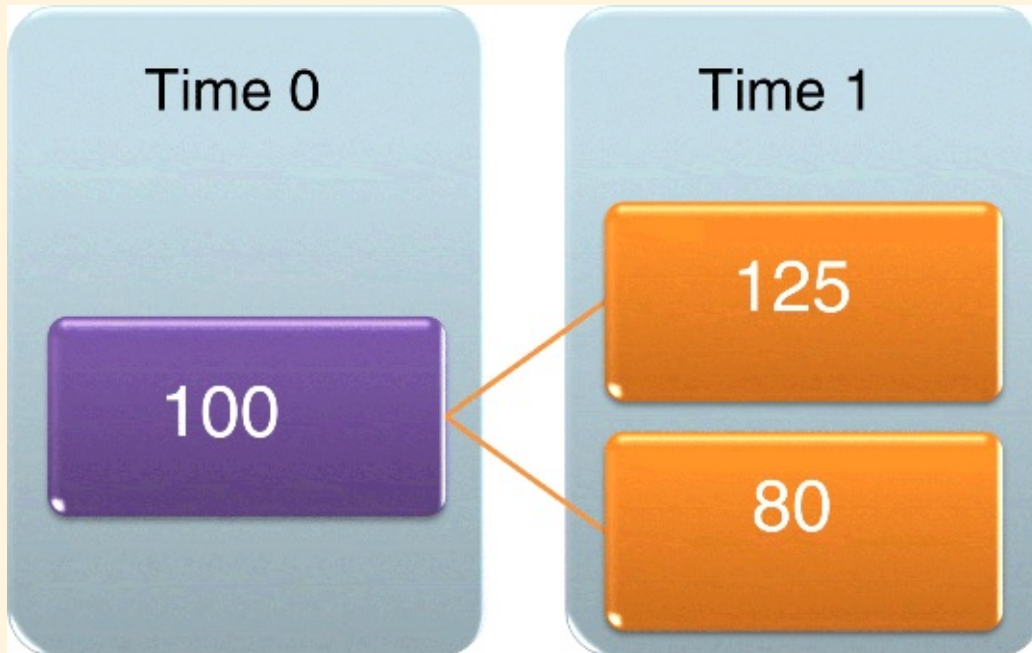


In order to find the values of the call options at the option expiration time, we need to calculate the future values of the underlying asset first. Use [Equation 4.12](#).

$$S_u = 100 \times 1.25 = 125$$

$$S_d = 100 \times 0.8 = 80$$

We can observe the current stock price of the underlying, in fact, that information has been given to us. The resulting binomial tree shows two possible state of the world in the next time period ([Fig. 4.17](#)).

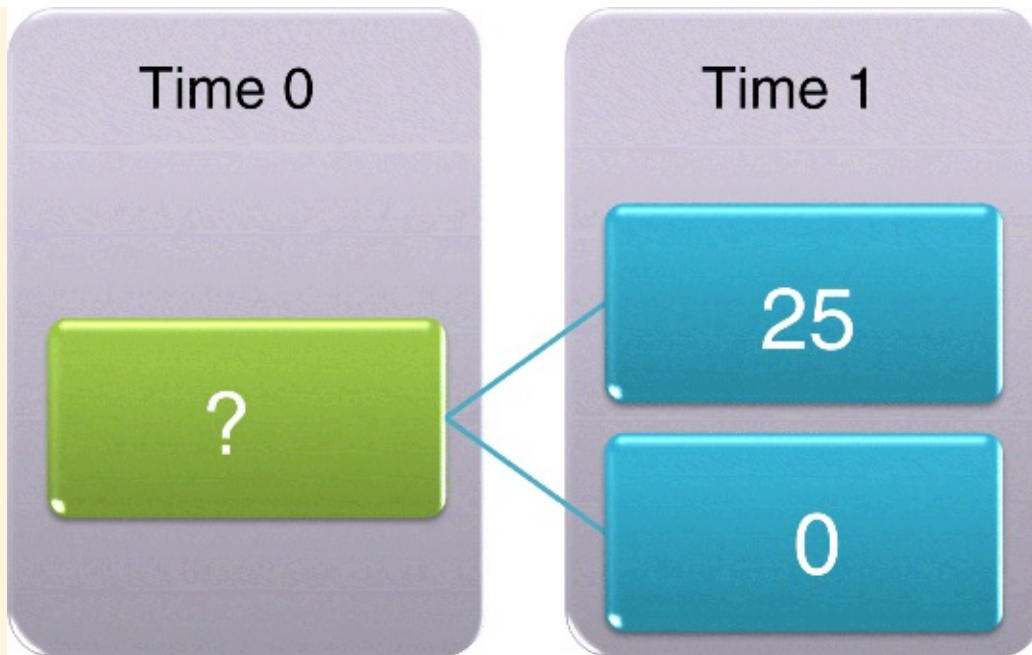


**FIGURE 4.17** One-period Asset Price Binomial Tree

Next, we'll calculate the option prices at expiration time utilizing [Equation 4.11](#) ([Fig. 4.18](#)).

$$C_u = \text{Max}(0, 125 - 100) = 25$$

$$C_d = \text{Max}(0, 80 - 100) = 0$$



**FIGURE 4.18** The Current Call Option Price is the Probability-weighted Present Value of the Future Call Prices

Given the call prices in the two possible states of the world in one time period from now, we are ready to calculate the present value of the call option. In order to do that, we need to first figure out how likely it is that each of the two states of the world will occur. We do this by calculating the probabilities of the up and down moves.

$$p = \frac{1+0.07-0.8}{1.25-0.8} = 0.6$$

$$1 - p = 0.4$$

Given these probabilities we can now derive the price of this call option using [Equation 4.16](#).

$$C_{t_0} = \frac{0.6 \times 25 + 0.4 \times 0}{1.07} = 14.02$$

The current price of the call option should be \$14.02. The following binomial tree depicts the full information set on this call price ([Fig. 4.19](#)).



**FIGURE 4.19** One Period Binomial Tree for a Call Option

Our next task is to calculate the value of the hedge portfolio. In order to do that, firstly we need to determine the hedge ratio using [Equation 4.13](#).

$$h = \frac{25 - 0}{125 - 80} = 0.556$$

Hedge portfolio investment value utilizing [Equations 4.14](#) and [4.15](#) yields the following:

$$V_0 = 0.556 \times 100 - 14.02 = 41.58$$

If we are indeed fully hedged, the value of the portfolio grows at the risk-free rate no matter which state of the world actually happens. Let us verify this fact.

$$V_u = 0.556 \times 125 - 25 = 44.50$$

$$r_u = \frac{44.50 - 41.58}{41.58} = 0.07$$

The value of the portfolio in the down state is fairly close to that of the up state. There might not be precisely the same.

$$V_d = 0.556 \times 80 - 0 = 44.48$$

$$r_d = \frac{44.48 - 41.58}{41.58} = 0.069 \approx 0.07$$

## Case Study 4.2.1-2

Suppose that the market price for this call option deviates from the calculated \$14.02. Let us examine cases where the option is either overvalued or undervalued. What are the effects on the return of the hedge portfolio?

### **Answer**

Suppose that currently the market quotes \$14.50 for this call. If we compare it to the fair value of \$14.02, it is overvalued. Anything that is overvalued relative to its true value will experience increased selling activity ([Fig. 4.20](#)).

$$V_0 = 0.556 \times 100 - 14.50 = 41.10$$

$$V_u = 0.556 \times 125 - 25 = 44.50$$

$$V_d = 0.556 \times 80 - 0 = 44.48$$

$$r_u = \frac{44.50 - 41.10}{41.10} = 0.0827$$

$$r_d = \frac{44.48 - 41.10}{41.10} = 0.0822$$

The hedge ratio return is higher at 8% than the risk-free rate of 7%. This creates an arbitrage opportunity, which will result in increased selling of the call. The selling will continue until the call price drops to its intrinsic value of \$14.02. The trading activity ceases at that price.



**FIGURE 4.20** Overvalued Call



At other times we observe that the option is selling at a price lower than \$14.02. This situation results in increasing buying activity ([Fig. 4.21](#)).

$$V_0 = 0.556 \times 100 - 13.50 = 42.10$$

$$V_u = 0.556 \times 125 - 25 = 44.50$$

$$V_d = 0.556 \times 80 - 0 = 44.48$$

$$r_u = \frac{44.50 - 42.10}{42.10} = 0.0570$$

$$r_d = \frac{44.48 - 42.10}{42.10} = 0.0565$$

In the event, a call option has been undervalued, the hedge portfolio return will be smaller than the risk-free rate. Again, an arbitrage opportunity arises. This time traders will be buying the call until the price is bid up sufficiently for the trading activity to cease and the hedge portfolio return to equalize with the risk-free rate. That will happen at \$14.02 per call option.



**FIGURE 4.21** Undervalued Call

## Problems

**4.2.1-1.** Set up the Case Study 4.2.1-1 in an Excel spreadsheet. Verify the pricing and hedge portfolio values at time 1.

**4.2.1-2.** Suppose that the annual observed stock return volatility is at 38%. We would like to construct a binomial tree with time steps equal to one day. Please calculate the appropriate up and down parameters.

$$\sigma = 38\%$$

$$\Delta t = \frac{1}{250}$$

**4.2.1-3.** Explain the no-arbitrage and risk-neutral valuation approaches to valuing a European option using a one-step binomial tree.

**4.2.1-4.** Set up a one-period binomial tree for a put option in an Excel spreadsheet. Use the following information about the underlying stock price and the risk-free interest rate.

What is the current value of the hedge portfolio? Does it grow at a risk-free rate over time?

$$S_{t_0} = 59$$

$$X = 60$$

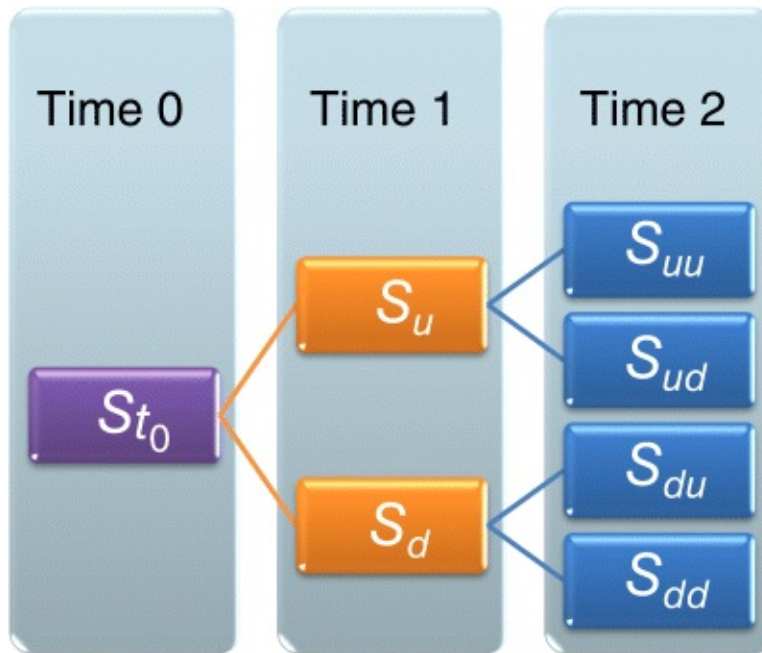
$$u = 1.15$$

$$d = 0.85$$

$$r = 0.05$$

### N-Period Binomial Model

In this chapter, we will take a closer look at the real world of pricing option via Binomial models. We start with two time periods and later expand the model to any number of periods. In fact, if we increase the number of periods to infinity the model yields a price that is identical to the price obtained by using the Black–Scholes formula ([Fig. 4.22](#)).



**FIGURE 4.22** Stock Price Two-Period Binomial Tree

In a two-period binomial model, the stock price will have  $2^2 = 4$  possible values at the option expiration time. We are assuming that there are only these four possible states of the world. In order to calculate the option prices, we need to determine the prices of the underlying asset first.

Equation 4.21 : Stock price after moving up and up

$$S_{uu} = S_{t_0} \times u^2$$

Equation 4.22 : Stock price after moving down and down

$$S_{dd} = S_{t_0} \times d^2$$

As the binomial trees presented in this text are recombining, it doesn't matter whether the stock price path goes up and down or down and up in the second time period. In both cases, we arrive at the same value for the price of the underlying ([Fig. 4.23](#)).

Equation 4.23 : Stock price in time – period two after moving down or down and up

$$S_{ud} = S_{du} = S_{t_0} \times u \times d$$

In order to calculate the current call option price, we need to start working from the expiration time to present ([Fig. 4.24](#)).

Equation 4.24 : Call price at expiration for all possible states

$$C_{uu} = \text{Max}(0, S_{uu} - X)$$

$$C_{ud} = \text{Max}(0, S_{ud} - X)$$

$$C_{du} = \text{Max}(0, S_{du} - X)$$

$$C_{dd} = \text{Max}(0, S_{dd} - X)$$

Equation 4.25 : Call prices at time 1

$$C_u = \frac{p \times C_{uu} + (1-p) \times C_{ud}}{(1+r)}$$

$$C_d = \frac{p \times C_{du} + (1-p) \times C_{dd}}{(1+r)}$$

$$C_{t_0} = \frac{p \times C_u + (1-p) \times C_d}{(1+r)}$$

Equation 4.26 : Put prices at expiration

$$P_{uu} = \text{Max}(0, X - S_{uu})$$

$$P_{ud} = \text{Max}(0, X - S_{ud})$$

$$P_{du} = \text{Max}(0, X - S_{du})$$

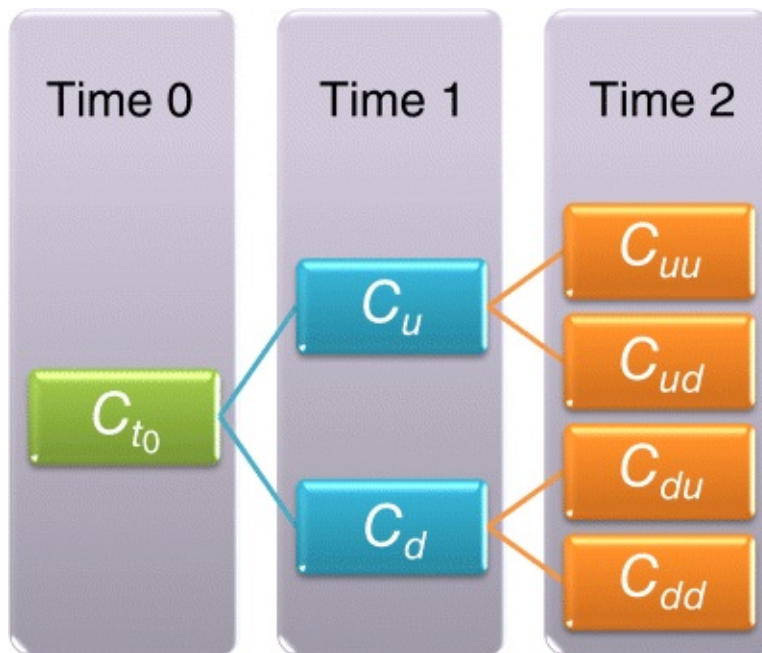
$$P_{dd} = \text{Max}(0, X - S_{dd})$$

Equation 4.27 : Put prices at time 1

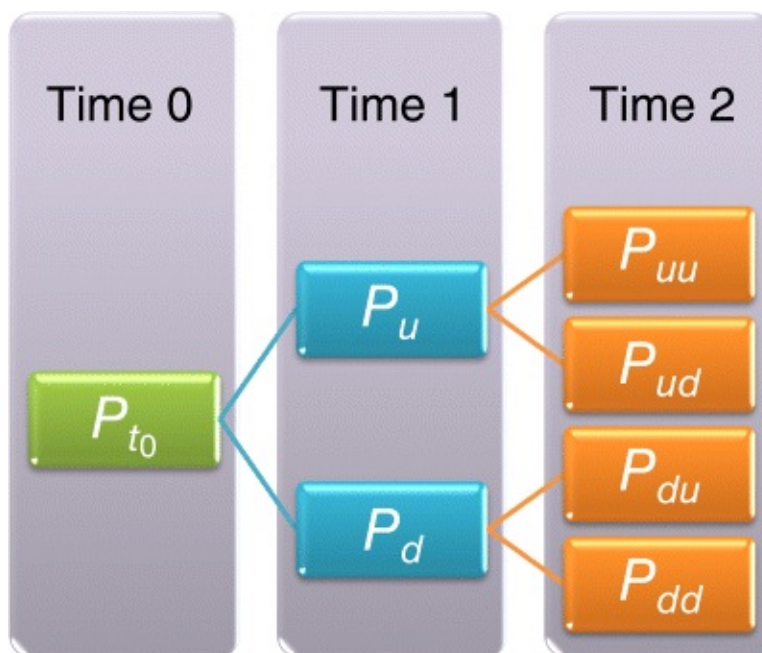
$$P_u = \frac{p \times P_{uu} + (1-p) \times P_{ud}}{(1+r)}$$

$$P_d = \frac{p \times P_{du} + (1-p) \times P_{dd}}{(1+r)}$$

$$P_{t_0} = \frac{p \times P_u + (1-p) \times P_d}{(1+r)}$$



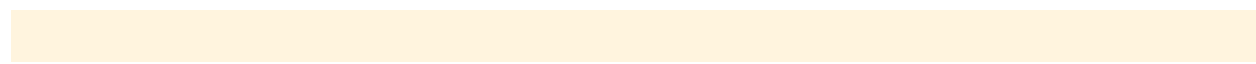
**FIGURE 4.23** Call Price Tree



**FIGURE 4.24** Put Price Tree

Hedge Ratio: Two-Period Binomial Model

Hedge Ratio for Calls





$$h = \frac{C_u - C_d}{S_u - S_d}$$

Suppose that the underlying asset price moves up (down) in the first time period. We will recalculate the hedge ratios for call options according to the following ([Fig. 4.25](#)):

Equation 4.28 : Hedge ratios applicable between times one a

$$\begin{aligned} h_u &= \frac{C_{uu} - C_{ud}}{S_{uu} - S_{ud}} \\ h_d &= \frac{C_{du} - C_{dd}}{S_{du} - S_{dd}} \end{aligned}$$

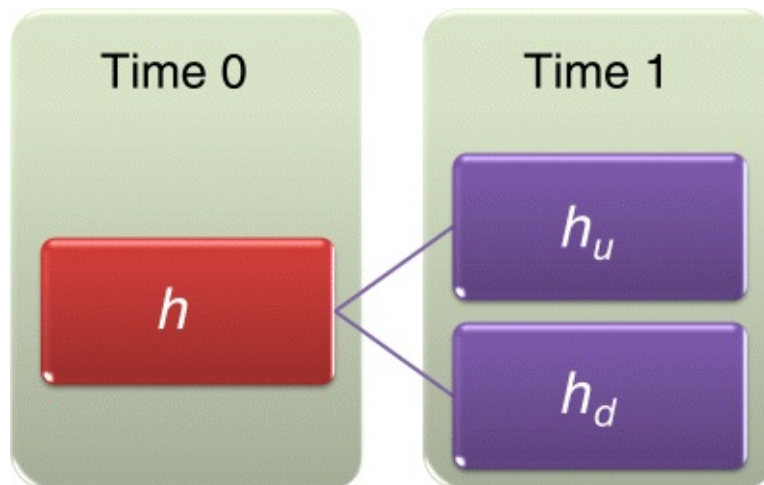
Equation 4.29 : Change in the hedge ratio

$$\begin{aligned} \Delta h_u &= h_u - h \\ \Delta h_d &= h_d - h \end{aligned}$$

Once the hedge ratios change we need to reflect that change in the hedge portfolios. Once again we will manipulate the amount of stock used to hedge our call/put position. We refer to this activity as rehedging. Once we have rehedged, we check our outcomes by calculating the portfolios returns and verifying that they have indeed grown by the amount of the risk-free rate.

Equation 4.30 : Hedge portfolio value after re – hedging

$$\begin{aligned} V_u^{Call} &= S_u(h_u - \Delta h_u) - C_u \\ V_d^{Call} &= S_d(h_d - \Delta h_d) - C_d \end{aligned}$$



**FIGURE 4.25** Hedge Ratio Tree

### Hedge Portfolio for Puts

The hedge portfolio for put position consists of  $h$  units of the underlying asset and a long position in puts. So, in this case both the asset and the put are long positions.

Equation 4.31 : Initial hedge ratios for puts

$$h = \frac{P_u - P_d}{S_u - S_d}$$

Equation 4.32 : Put hedge ratios calculated after time 1

$$h_u = \frac{P_{uu} - P_{ud}}{S_{uu} - S_{ud}}$$
$$h_d = \frac{P_{du} - P_{dd}}{S_{du} - S_{dd}}$$

Equation 4.33 : Put hedge portfolio value at time 0

$$V_{t_0} = P_{t_0} + hS_{t_0}$$

Equation 4.34 : Put hedge portfolio value at time 1

$$V_u = P_u + hS_u$$
$$V_d = P_d + hS_d$$

### Rehedging of Puts

In the case of puts unlike calls whenever we end up with a positive change in the hedge ratio we will sell the underlying. If the change in the hedge ratio for puts is negative, we'll end up buying the underlying asset.

Equation 4.35 : Hedge portfolio value after re – hedging : P1

$$\begin{aligned} V_u^{Put} &= S_u(|h_u| + \Delta h_u) + P_u \\ V_d^{Put} &= S_d(|h_d| + \Delta h_d) + P_d \end{aligned}$$

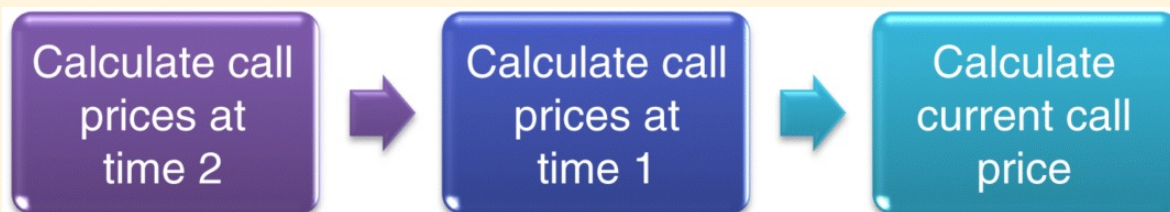
### Case Study 4.2.1-3

Let us expand the one-period binomial model to two periods using the same initial inputs as in Case Study 4.2.1-1. Please calculate the underlying stock prices for time 1 and 2. Find out the call prices in time periods 2, 1, and 0. What is the current time hedge ratio? How will the ratio change if the stock price moves up/down in time period 1? What is the value of the hedge portfolio at times 0, 1, and 2? Verify that at each time the portfolio indeed grows at the risk-free rate.

#### **Answer**

In order to be able to calculate the desired option price we first need to obtain future stock prices for all four possible states of the world. Once we have the information above, we can start calculating the option's values.

Just like in the one-period binomial model, we start with calculating the payoffs at expiration time for all states of the world. After getting results for time 2, we move on to obtain call values for time 1. At each step, we calculate the price as a discounted weighted average of the prices from the previous period. In the case of a two-period, we go through the process only twice—for time periods 1 and 0 ([Fig. 4.26](#)).



$$C_{uu} = \text{Max}(0, 156.25 - 100) = 56.25$$

$$C_{ud} = \text{Max}(0, 100 - 100) = 0$$

$$C_{dd} = \text{Max}(0, 64 - 100) = 0$$

$$C_u = \frac{0.6 \times 56.25 + 0.4 \times 0}{1.07} = 31.54$$

$$C_d = \frac{0.6 \times 0 + 0.4 \times 0}{1.07} = 0$$

$$C_{t_0} = \frac{0.6 \times 31.54 + 0.4 \times 0}{1.07} = 17.66$$

The current call price is \$17.66 ([Fig. 4.27](#)).



**FIGURE 4.26** Stock Price Tree for Two Periods



**FIGURE 4.27** Call Option Price Tree

We will calculate the hedge ratios for time 0 and figure out the hedge portfolio values as well. Secondly, we focus on the next time period. The underlying asset price has moved either up or down and we need to re-hedge as a consequence.

$$h = \frac{31.54 - 0}{125 - 80} = 0.701$$

$$V_0 = 0.701 \times 100 - 17.66 = 52.44$$

Hedge portfolio value at time 1

$$V_u = 0.701 \times 125 - 31.54 = 56.09$$

$$V_d = 0.701 \times 80 - 0 = 56.08$$

Return calculations over the first period are equal to the risk-free rate.

$$r_u = \frac{56.09}{52.44} - 1 = 0.07$$

$$r_d = \frac{56.08}{52.44} - 1 = 0.07$$

Portfolio reheding

In this next section, we will calculate the hedge ratios for the up state and the down state

Up state

$$h_u = \frac{56.25 - 0}{156.25 - 100} = 1$$

In the case of the up move, in order to re-hedge our portfolio, we need to buy additional units of stock equal to the difference between 1 and 0.701.

Buying additional stock with borrowed money means the following changes to the value of the hedge portfolio.

$$\Delta h_u = 1 - 0.701 = 0.299$$

$$V_u = 125 \times (1 - 0.299) - 31.54 = 56.09$$

Down state

$$h_d = \frac{0 - 0}{100 - 64} = 0$$

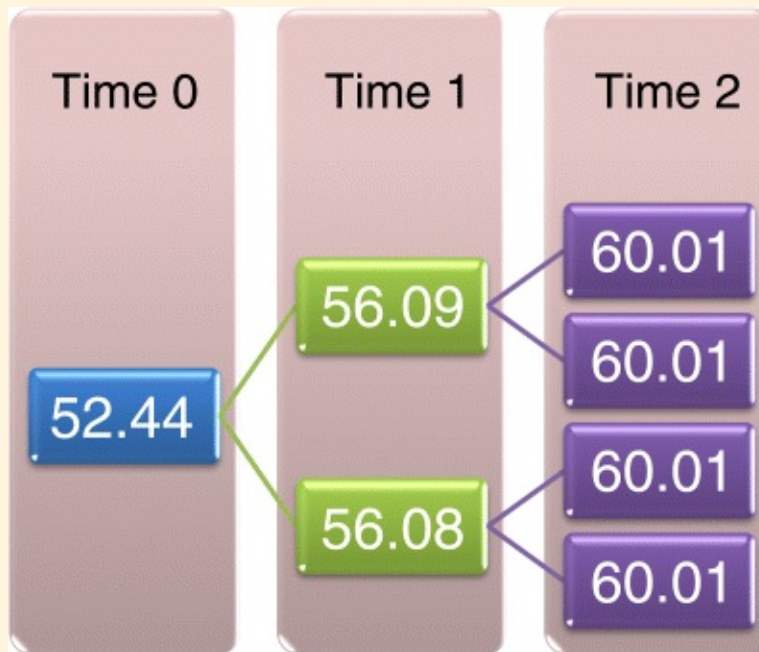
$$\Delta h_d = 0 - 0.701 = -0.701$$

If the underlying asset price moves down than our new hedge ratio is equal to zero. That means that we will sell all of our units of the underlying asset—0.701 and use the proceeds to buy a risk-free bond.



$$V_d = 80 \times (0 + 0.701) - 0 = 56.08$$

At time 2, we have four possible states of the world. If we had rehedge correctly, the hedge portfolio values will be identical for each state of the world ([Fig. 4.28](#)).



**FIGURE 4.28** Hedge Portfolio Tree

Up-up state

$$V_{uu} = 1 \times 156.25 - 37.375 \times (1 + 0.07) - 56.25 = 60.01$$

Up-down state

$$V_{ud} = 1 \times 100 - 37.375 \times (1 + 0.07) - 0 = 60.01$$

Down-up state

$$V_{du} = 0 \times 100 + 56.08 \times (1.07) - 0 = 60.01$$

Down-down state

$$V_{dd} = 0 \times 80 + 56.08 \times (1.07) - 0 = 60.01$$

Return calculations over the second period should yield, once again, the risk-free interest rate.

$$r_u = \frac{60.01}{56.09} - 1 = 0.07$$

$$r_d = \frac{60.01}{56.08} - 1 = 0.07$$

## Case Study 4.2.1-4

Let us again expand the one-period binomial model to two periods using the same initial inputs as in Case Study 4.2.1-1. Please

calculate the underlying stock prices for time 1 and 2. Find out the put prices in time periods 2, 1, and 0.

What is the current time hedge ratio? How will the ratio change if the stock price moves up/down in time period 1? Verify that at each time the portfolio indeed grows at the risk-free rate. Using Excel spreadsheet calculate the hedge portfolio value at time 0, 1, and 2. In time period 1, re-hedge and calculate the value of the portfolio at time 2. If your re-hedge consist of selling securities, reinvest the cash into a risk-free bond. If you end up buying additional units of the underlying asset, assume that you get a loan priced at the risk-free rate for the purchase.

### **Answer**

While pricing a put in a two-period binomial model, we find a number of similarities to calls, yet we need to pay attention primarily to the differences. Just like previously we work the binomial tree backwards—first calculating the payoffs at expiration time and then proceeding to present ([Fig. 4.29](#)).



$$P_{uu} = \text{Max}(0, 100 - 156.25) = 0$$

$$P_{ud} = \text{Max}(0, 100 - 100) = 0$$

$$P_{du} = \text{Max}(0, 100 - 100) = 0$$

$$P_{dd} = \text{Max}(0, 100 - 64) = 36$$

$$P_u = \frac{0.6 \times 0 + 0.4 \times 0}{1.07} = 0$$

$$P_d = \frac{0.6 \times 0 + 0.4 \times 36}{1.07} = 13.46$$

$$P_{t_0} = \frac{0.6 \times 0 + 0.4 \times 13.46}{1.07} = 5.03$$



**FIGURE 4.29** Put Option Price Tree

Utilizing Excel spreadsheet we may want to set up a short put pricing

module.

Once all the functions are properly entered, you should be getting these following numerical results. If some numbers end up being different, go back and double-check your entered formulas. If all is well you can now perform a number of scenario and sensitivity analyses.

Binomial Trees			
Two Period			
Put Price			
u		1.25	
d		0.8	
X		100	
r (d<1+r<u)		0.07	
Probability p		0.6	
	Time 0	Time 1	Time 2
			156.25
		125	
Stock Price	100		100
		80	
			64
			0
		0.00	
Put Price	5.03		0
		13.46	
			36
			0
Hedge Ratio	-0.299		-1.000
Hedge Portfolio			
long h shares of stock			
long a put			
			37.38
			0.00
	29.91	37.38	
	5.03		
Hedge Portfolio Value	34.94	23.93	
		13.46	
		37.38	
Rehedging			
sell remaining stock, invest in a risk free bond			
			0
			0
		0	40
		0	40
		37.38	0
		37.38	0
			40
			40
buy remaining stock with borrowed funds			
			100
			0
		80	40
		13.46	-60
		37.38	64
		-56.07	36
			40
			-60

Binomial Trees			
Two Period			
Put Price			
u	1.25		
d	0.8		
X	100		
r (d<1+rcu)	0.07		
Probability p	$=(1+B9-B7)/(B6-B7)$		
	Time 0	Time 1	Time 2
Stock Price	100	$=B20*B6$	$=C19*B6$
		$=B20*B7$	$=C19*B7$
			$=C21*B7$
Put Price	$=(B511*C27+(1-B511)*C29)/(1+B59)$	$=(B511*D26+(1-B511)*D28)/(1+B59)$	$=MAX(0,5858-D18)$
		$=(B511*D28+(1-B511)*D30)/(1+B59)$	$=MAX(0,5858-D20)$
			$=MAX(0,5858-D22)$
Hedge Ratio	$=(C27-C29)/(C19-C21)$	$=(D26-D28)/(D18-D20)$	
		$=(D28-D30)/(D20-D22)$	
Hedge Portfolio			
long h shares of stock		$=C19*ABS(B34)$	
long a put		$=C27$	
	$=B20*ABS(B34)$	$=SUM(C40:C41)$	
	$=B28$		
Hedge Portfolio Value	$=SUM(B42:B43)$	$=C21*ABS(B34)$	
		$=C29$	
		$=SUM(C44:C45)$	
Rehedging			
sell remaining stock, invest in a risk free bond			$=ABS(C33)*D18$
			$=D26$
		$=ABS(C33)*C19$	$=SUM(D50:D51)+D53$
		$=C27$	$=C55*(1+B59)$
		$=SUM(C52:C53)+C55$	$=ABS(C33)*D20$
		$=(C33-B34)*C19$	$=D28$
			$=SUM(D54:D55)+D57$
			$=C55*(1+B59)$
buy remaining stock with borrowed funds			$=ABS(C35)*D20$
			$=D28$
		$=ABS(C35)*C21$	$=SUM(D60:D61)+D63$
		$=C29$	$=C56*(1+B59)$
		$=SUM(C62:C63)+C65$	$=ABS(C35)*D22$
		$=(C35-B34)*C21$	$=D30$
			$=SUM(D64:D65)+D67$
			$=C56*(1+B59)$

## N-Period Binomial Model

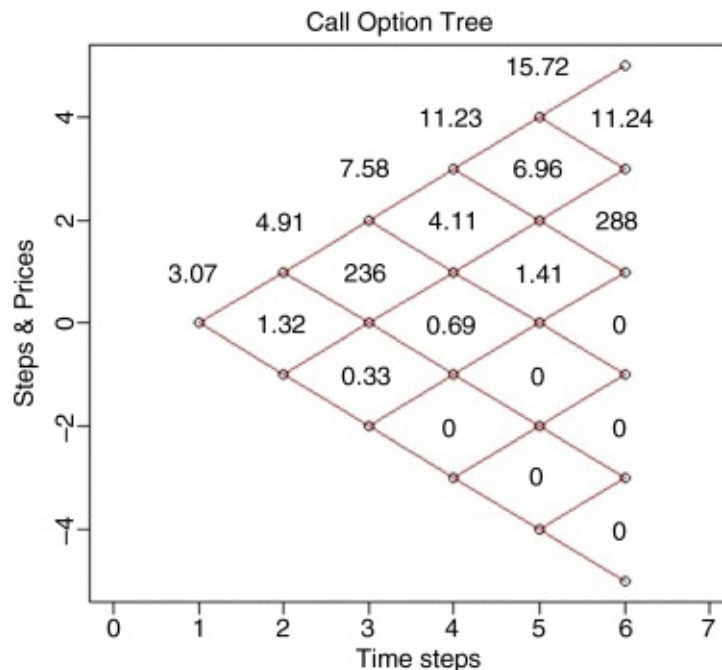
Equation 4.36 : Call price formula for any number of periods

$$C_{t_0} = \frac{\sum_{j=0}^n \left[ \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \text{Max} (0, u^j d^{(n-j)} S_T - X) \right]}{(1+r)^n}$$

Equation 4.37 : Put price formula for any number of periods

$$P_{t_0} = \frac{\sum_{j=0}^n \left[ \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \text{Max} (0, u^j d^{(n-j)} X - S_T) \right]}{(1+r)^n}$$

The following chart is an example of a call option binomial tree with five time periods. We are able to construct such trees for any number of periods easily using software, such as R.





## Case Study 4.2.1-5

Suppose you'd like to find the current “fair” value of a call option and a put option written on the same underlying stock, the same strike price and expiration time. The options expire five months from now. All the information below reflects a month-long time period (i.e., the risk-free rate per month is 1%).

Using an Excel spreadsheet, please find the correct prices.

$$S_{t_0} = 79$$

$$X = 80$$

$$u = 1.015$$

$$d = 0.98$$

$$r = 0.01$$

**Answer**

<b>Binomial Trees</b>		
<b>N Period</b>		
<b>Call &amp; Put Price</b>		
u	1.015	
d	0.98	
X	80	
r	0.01	
<b>Probability p</b>	0.857142857	
	<b>Time 0</b>	<b>Time 1</b>
		80.185
<b>Stock Price</b>	79	
		77.42
<b>j</b>	<b>Call Price</b>	<b>Put Price</b>
0	0.00	0.00
1	0.00	0.00
2	0.00	0.00
3	0.00	0.69
4	0.84	1.62
5	2.36	3.32
<b>Sum</b>	3.20	5.64
<b>Discount Factor</b>	0.95	0.95
<b>Price</b>	<b>3.04</b>	<b>5.36</b>



$$S_{t_0} = 52.23$$

$$X = 50$$

$$u = 1.33$$

$$d = 0.74$$

$$r = 0.03$$

Using Excel spreadsheet, please calculate the underlying stock prices for time 1 and 2, and find out the call prices in time periods 2, 1, and 0.

- What is the current time hedge ratio?
- How will the ratio change if the stock price moves up/down in time period 1?
- What is the value of the hedge portfolio at times 0, 1, and 2? Show that at each time the portfolio indeed grows at the risk-free rate of 3%.

**4.2.1-6.** Consider the following information on a put option:

$$S_{t_0} = 53.48$$

$$X = 50$$

$$u = 1.23$$

$$d = 0.79$$

$$r = 0.03$$

Using Excel spreadsheet, please determine the underlying stock prices for time 1 and 2, and find out the put prices in time periods 2, 1, and 0.

- What is the current time hedge ratio?
- How will the ratio change if the stock price moves up/down in time period 1?
- What is the value of the hedge portfolio at times 0, 1, and 2? Show that at each time the portfolio indeed grows at the risk-free rate of 3%.

**4.2.1-7.** Suppose you'd like to find the current “fair” value of a call option written on some underlying stock. The option expires four months from now. All the information below reflects a month-long time period (i.e., the risk-free rate per month is 0.1%).

$$\begin{aligned}
 S_{t_0} &= 80 \\
 X &= 80 \\
 u &= 1.02 \\
 d &= 0.98 \\
 r &= 0.001
 \end{aligned}$$

Using an Excel spreadsheet, please find the correct price for this option.

**4.2.1-8.** Please find the current “fair” value of a put option? Assume data from problem 4.2.1-7 hold.

How would the option premiums change if the current stock prices are 78.35 and 81.60?

## 4.2.2 Black–Scholes Formula

Origins of the Black–Scholes, Merton formula go back to the works of Brown, Bachelier, Einstein, and Ito.

Robert Brown was a Scottish scientist who observed and quantified the motion of pollen in water. He noticed that it followed no distinct pattern and moved completely randomly. This phenomenon became known as

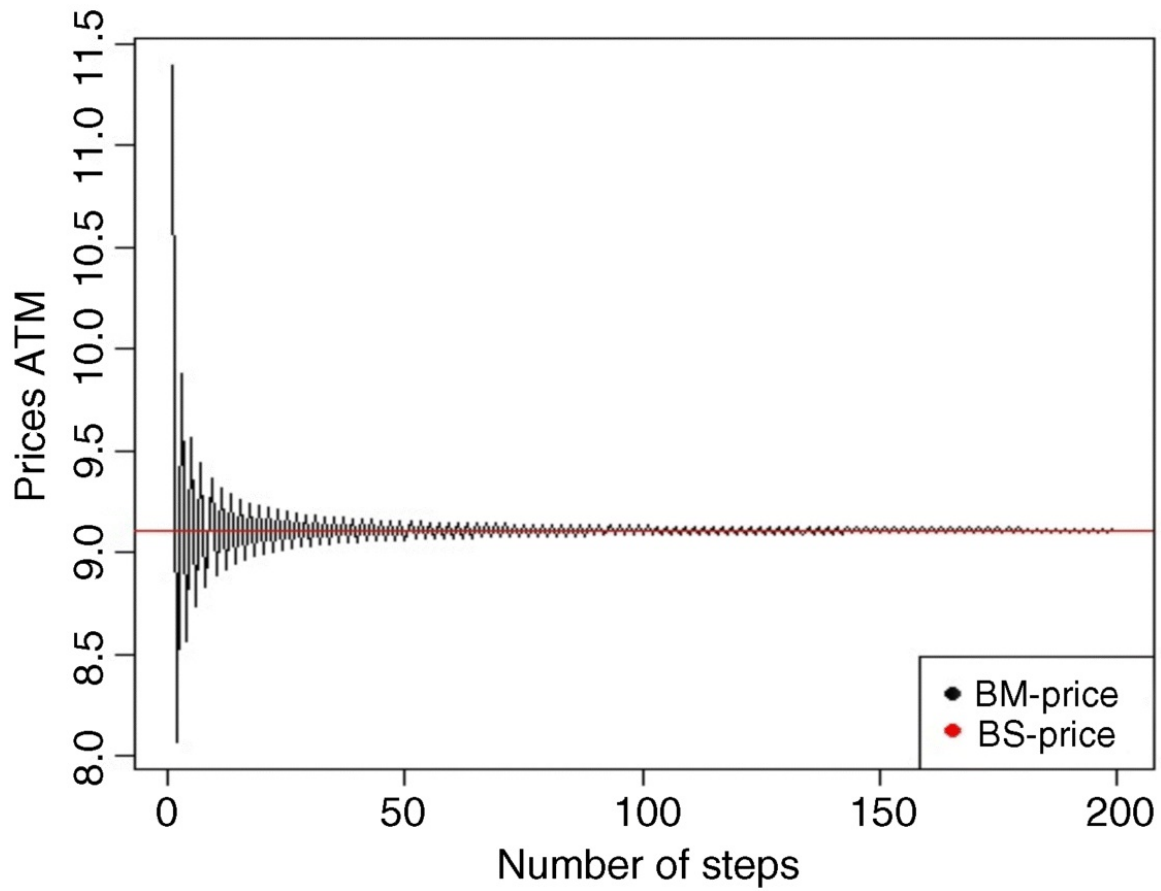
the Brownian motion.

Luis Bachelier was a French doctoral student who developed a model strikingly similar to the BSM. His thesis advisor back then discouraged him from further research, as he deemed pricing of a financial instrument as impractical. Several decades later, Albert Einstein used Brownian motion to explain movements of molecules while working on the foundations of his theories of relativity.

Finally in 1951, Kiyoshi Ito, a Japanese mathematician, developed what is known as Ito's Lemma. This is the pivotal mathematical result that 20 years later made it possible to find an option's price. This task was completed in 1973 by Fisher Black, Myron Scholes, and Robert Merton. In 1997, they were awarded the Nobel Prize in Economics for their discovery.

The Black–Scholes–Merton model (BSM) is, in fact, the limiting case of the binomial model. The binomial model is a discrete time model. The BSM is a continuous-time model. In the case of the discrete models, the life of the option is divided into a specific number of finite units of time. Stock's movements can be characterized as being either up the smallest possible increment—an uptick, or down the smallest possible decrement—a downtick. As we decrease the length of each time step, in the limit, it converges to continuous time and a specific option value. This value is the Black–Scholes–Merton price.

[Figure 4.30](#) shows the price convergence process for an at-the-money (ATM) option. The convergence process is symmetric here. The convergence process for an out-of-the-money (OTM) option is quite a bit more asymmetric. With sufficient number of time steps, however, even this one converges to the Black–Scholes price depicted by solid horizontal line.



**FIGURE 4.30** ATM Option Price—Binomial Model vs. Black-Scholes-Implied Price

### Limit of the Binomial Model

Equation 4.38 : Discrete compounding

$$\left(1 + \frac{r}{n}\right)^{nT}$$

### Equation 4.39 : Continuous compounding

$$e^{rT}$$

Increasing compounding periods leads us to the limiting case, where discrete time becomes continuous as soon as we assume an infinite number of compounding periods.

### Equation 4.40 : Limiting case of discrete is continuous

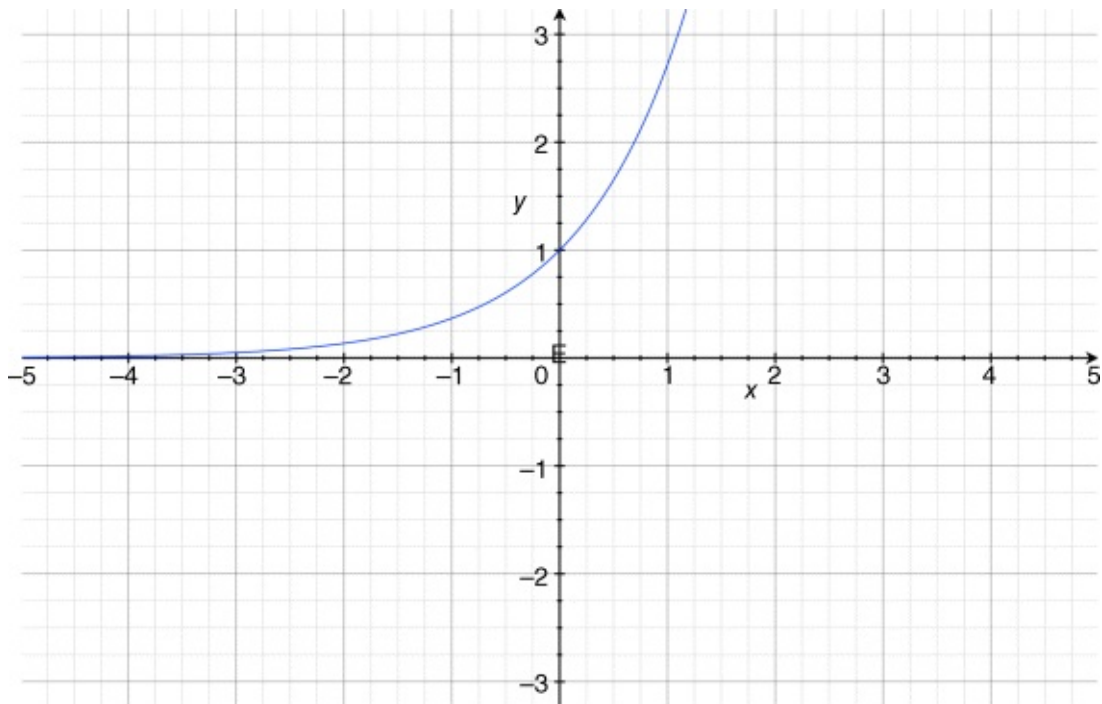
$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{nT} = e^{rT}$$

The exponential function  $e^x$  captures the value of an investment compounded continuously ([Fig. 4.31](#)).



## Equation 4.41 : Exponential function

$$f(x) = e^x$$



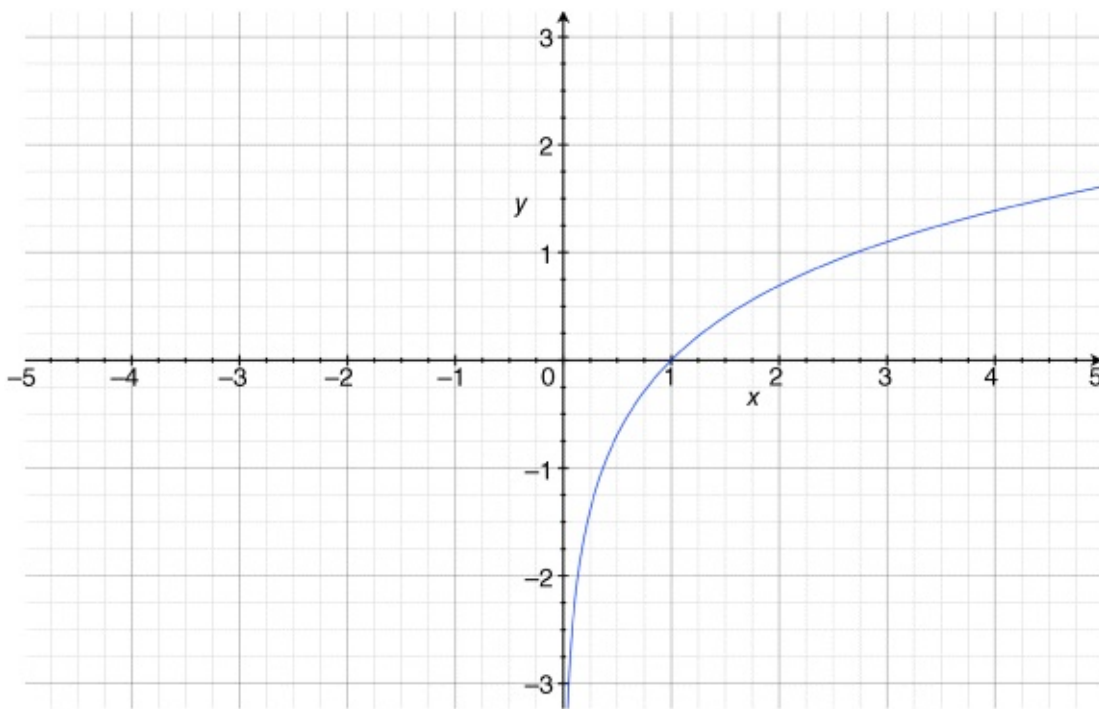
**FIGURE 4.31** Exponential Function

The inverse of the exponential function is the natural logarithm of  $x$  ([Fig. 4.32](#)).

$$g(x) = \frac{1}{f(x)} = f(x)^{-1}$$

## Equation 4.42 : Logarithmic function

$$g(x) = \ln(x)$$



**FIGURE 4.32** Logarithmic Function

### Log-Normal Distribution of Stock Returns

Stock returns are log-normally distributed. What that means is that once we take natural logarithm of returns on stocks the resulting distribution will be normal.

Discrete time equity return is calculated as a ratio of the later to the earlier price in the holding period less one.

Equation 4.43 : Discrete time return

$$r_D = \left( \frac{S_{t_1}}{S_{t_0}} \right) - 1$$

Assuming continuous time, we use the natural logarithm of the ratio of the two prices to come up with the appropriate return in continuous time.

Equation 4.44 : Continuous time return

$$r_C = \ln(1 + r_D)$$

The relationship between the time 0 price and time 1 price is an exponential function of the continuously compounded rate of return.

## Equation 4.45 : Growth in the price of underlying

$$S_{t_1} = S_{t_0} \times e^{r_C}$$

### Case Study 4.2.2-1

Suppose that a stock has a value of  $S_{t_0} = 100$  at the beginning of a time period and  $S_{t_1} = 112$  at the end of a holding period. Calculate the continuous time return on this investment.

#### **Answer**

The discrete return over the holding period is

$$r_D = \frac{112}{100} - 1$$
$$r_D = 12\%$$

Now, let us assume continuous compounding. The holding period return is somewhat lower.

$$r_C = \ln\left(\frac{112}{100}\right)$$

$$\ln(1.12) = 0.1133$$

$$r_C = 11.33\%$$

The continuously compounded return over one time period is 11.33%.

Check that the answer is correct by plugging the return into the exponential function.

$$112 = 100 \times e^{0.1133}$$

### Pricing a Call Option Using the Black–Scholes–Merton Model

The Black–Scholes formula is a result that gives us the arbitrage-free price of a call or a put option.

Underlying assumptions:

1. The risk-free interest rate is constant.
2. The underlying stock dynamics are described by a continuous-time stochastic differential equation.

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

3. The stock pays no dividends, and there are no other corporate actions, such as stock splits...
4. There are no transaction costs and no bid-ask spreads.

## Notation

$S_t$	Stock price at time $t$
$\frac{T-t}{365}$	Time to maturity as a fraction of a year
$(T - t)$	Number of days to maturity
$X$	Strike price
$r_c$	Continuously compounded risk-free rate
$\sigma$	Constant volatility parameter
$N(d_1)$	Normal CDF at cut-off point $d_1$
$N(d_2)$	Normal CDF at cut-off point $d_2$
$C_t$	Call option premium at time $t$
$P_t$	Put option premium at time $t$

In the Black–Scholes formula,  $X, \sigma, r_c$ , and  $T$  are considered parameters. The Black–Scholes formula holds in the following form only when these components are kept constant. As the volatility of the underlying has to be kept constant during the life of the option, we need to make a special effort in estimating it as correctly as possible. Note that the formula would not hold for time-varying volatility.

## The Black–Scholes Formula

Equation 4.46 : Call option premium at time  $t$

$$C_t = S_t N(d_1) - X e^{-r_c ((T-t)/365)} N(d_2)$$

Recall the Put–call parity.

Equation 4.47 : Put– call parity

$$P_t = C_t + Xe^{-r_c((T-t)/365)} - S_t$$

And hence the price of a put option is

$$P_t = Xe^{-r_c((T-t)/365)} [1 - N(d_2)] - S_t[1 - N(d_1)]$$

Equation 4.48 : Put option premium at time t

$$P_t = Xe^{-r_c((T-t)/365)} N(-d_2) - S_t N(-d_1)$$

Cutoff points for the normal distribution CDF used in the formula above are calculated using the following formulas:

Equation 4.49 : Cutoff point d1

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r_C + \frac{\sigma^2}{2}\right) \times \left(\frac{T-t}{365}\right)}{\sigma \sqrt{\left(\frac{T-t}{365}\right)}}$$

### Equation 4.50 : Cutoff point d2

$$d_2 = d_1 - \sigma \sqrt{\frac{T-t}{365}}$$

This same result can also be obtained using the following equation. This is useful if you don't have a result for d1 beforehand.

### Equation 4.51 : Cutoff point d2

$$d_2 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r_C - \frac{\sigma^2}{2}\right) \times \left(\frac{T-t}{365}\right)}{\sigma \sqrt{\left(\frac{T-t}{365}\right)}}$$

These points—d1 and d2—serve as cutoff points in a probability distribution. The Black–Scholes formula uses standard normal probability distribution in its calculation.

Whenever we get a cutoff point that is a negative number, in order to be able to look up the result using distribution tables we need to note the following.



Equation 4.52 : Remainder probability of  $N(x)$

$$N(-x) = 1 - N(x)$$

## Case Study 4.2.2-2

Please price a call option using the Black–Scholes formula. You have the following information:

$$S_t = 102.25$$

$$X = 100$$

$$r_D = 1\%$$

$$\frac{T-t}{365} = 45/365$$

$$\sigma = 0.7$$

### **Answer**

First let us calculate  $r_C$

$$r_C = \ln(1 + r_D)$$

$$r_C = \ln(1.01) = 0.00995$$

$$r_C = 0.995\%$$

$$d_1 = \frac{\ln\left(\frac{102.25}{100}\right) + \left(0.00995 + \frac{(0.7)^2}{2}\right) \times 0.12329}{0.7\sqrt{0.12329}}$$

$$d_1 = 0.21841$$

$$d_2 = 0.21841 - 0.7\sqrt{0.12329}$$

$$d_2 = -0.02737$$

We can use probability distribution tables to look up results, or perhaps more readily available is *NORMSDIST()* function in Excel.

$$N(d_1) = 0.58645 \rightarrow 58.65\%$$

$$N(-0.02737) = 1 - N(0.02737)$$

$$N(d_2) = 1 - N(0.02737)$$

$$N(d_2) = 1 - 0.51092$$

$$N(d_2) = 0.48908 \rightarrow 48.91\%$$

Finally, let us calculate the appropriate call price.

$$C_t = (102.25)(0.5865) - (100)(e^{-(0.00995)(0.12329)})(0.4891)$$

$$C_t = 11.12$$

### Case Study 4.2.2-3

Now, let us look at an example where we will price a put option with identical strike price and time to maturity, written on the exact same underlying stock.

**Answer**

Recall that

$$d_1 = 0.21841$$

$$d_2 = -0.02737$$

$$N(d_1) = 0.58645$$

$$N(d_2) = 0.48908$$

$$N(-d_1) = 1 - N(d_1)$$

$$N(-d_1) = 1 - 0.5865$$

$$N(-d_1) = 0.4135$$

$$N(-d_2) = 1 - N(d_2)$$

$$N(-d_2) = 1 - 0.4891$$

$$N(-d_2) = 0.5109$$

$$P_t = (100)e^{-(0.00995) \left(\frac{0.12329}{365}\right)} (0.5109) - (102.25)(0.4135)$$

$$P_t = 8.74$$

BSM Formula as Time to Maturity and Volatility Respectively Converge to Zero

$$\frac{(T - t)}{365} \rightarrow 0$$

Recall that

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r_C + \frac{\sigma^2}{2}\right) \times \left(\frac{T-t}{365}\right)}{\sigma \sqrt{\left(\frac{T-t}{365}\right)}}$$

Divide the formula into two parts.

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right)}{\sigma \sqrt{\left(\frac{T-t}{365}\right)}} + \frac{\left(r_C + \frac{\sigma^2}{2}\right) \times \left(\frac{T-t}{365}\right)}{\sigma \sqrt{\left(\frac{T-t}{365}\right)}}$$

As the time to maturity declines the second part of the formula converges to zero.

$$\frac{\left(r_C + \frac{\sigma^2}{2}\right) \times \left(\frac{T-t}{365}\right)}{\sigma \sqrt{\left(\frac{T-t}{365}\right)}} \rightarrow 0$$

Depending on the value of the underlying stock price relative to the strike price, we have either

$$\frac{S_t}{X} > 1$$
$$\ln\left(\frac{S_t}{X}\right) > 0$$

$d_1 \rightarrow +\infty$  and  $d_2 \rightarrow +\infty$ , which implies that  $N(d_1), N(d_2) \rightarrow 1$   
or

$$\frac{S_t}{X} \leq 1$$
$$\ln\left(\frac{S_t}{X}\right) \leq 0$$
$$d_1 \rightarrow -\infty$$

and  $d_2 \rightarrow -\infty$ , which implies that  $N(d_1), N(d_2) \rightarrow 0$ .

Thus, as time to maturity approaches zero, and if  $S_T > X$ , then the call price formula converges to

$$C_T = S_T N(d_1) - X e^{-r_c((T-t)/365)} N(d_2)$$
$$C_T = S_T(1) - X e^0(1)$$
$$C_T = S_T - X$$

For  $S_T \leq X$ :

$$C_T = S_T N(d_1) - X e^{-r_c((T-t)/365)} N(d_2)$$

$$C_T = S_T(0) - X(1)(0)$$

$$C_T = 0$$

Combining the two possible states of the world we get the call option intrinsic value formula.

$$C_T = \text{Max}(0, S_T - X)$$

BSM formula as volatility converges to zero

$$\sigma \rightarrow 0$$

Divide the formula into two parts.

$$d_1 = \frac{\ln\left(\frac{S_t}{X}\right)}{\sigma \sqrt{\left(\frac{T-t}{365}\right)}} + \frac{\left(r_C + \frac{\sigma^2}{2}\right) \times \left(\frac{T-t}{365}\right)}{\sigma \sqrt{\left(\frac{T-t}{365}\right)}}$$

Isolate the volatility parameter in the second part as follows:

$$d_1 = \frac{\ln\left(\frac{S_t}{Xe^{-rC((T-t)/365)}}\right)}{\sigma\sqrt{\left(\frac{T-t}{365}\right)}} + \frac{\left(\frac{\sigma^2}{2}\right) \times \left(\frac{T-t}{365}\right)}{\sigma\sqrt{\left(\frac{T-t}{365}\right)}}$$

This part converges to zero.

$$\frac{\left(\frac{\sigma^2}{2}\right) \times \left(\frac{T-t}{365}\right)}{\sigma\sqrt{\left(\frac{T-t}{365}\right)}} \rightarrow 0$$

$$d_1 = \frac{\ln\left(\frac{S_t}{Xe^{-rC((T-t)/365)}}\right)}{\sigma\sqrt{\left(\frac{T-t}{365}\right)}}$$

as

$$\sigma \rightarrow 0$$

If



$$S_t > Xe^{-rc((T-t)/365)}$$

then

$$d_1 \rightarrow +\infty \text{ and } d_2 \rightarrow +\infty, \text{ which implies that } N(d_1), N(d_2) \rightarrow$$

and

$$C_t = S_t - Xe^{-rc((T-t)/365)}$$

If

$$S_t \leq Xe^{-rc((T-t)/365)}$$

then

$$d_1 \rightarrow -\infty \text{ and } d_2 \rightarrow -\infty, \text{ which implies that } N(d_1), N(d_2) \rightarrow$$

$$C_t = 0$$

Combining these two outcomes yields

$$C_t = \text{Max} \left( 0, S_t - X e^{-r_C ((T-t)/365)} \right)$$

### Case Study 4.2.2-4

Assume that all Black–Scholes assumptions hold. Also assume no dividends. Consider a one-year European call option that is ATM. If the volatility is zero ( $\sigma = 0$ ), what is the value of this call?

**Answer**

Volatility of returns is the average deviation from expected returns. With no volatility, the stock is riskless. In the absence of arbitrage opportunities, the stock must offer an expected return equal to the risk-free rate. This is true in both the real world and the theoretical risk-neutral world. With no volatility, the value of the option at any time  $t$  is just the discounted expected payoff in a risk-neutral world at maturity time.

$$C_t = \text{Max} \left( 0, S_t - X e^{-r_C ((T-t)/365)} \right)$$

Another way to look at this in the absence of volatility is

$$C_t = e^{-r_C((T-t)/365)} \text{Max}(0, S_T - X)$$

However, at time  $t$  we don't know what the price of the underlying asset will be at maturity time. Given no volatility, our best guess is that it will grow at the risk-free rate.

$$S_T = S_t e^{r_C((T-t)/365)}$$

This is a good place to mention a connection between options and futures. Suppose that  $S_t$  is the price at time  $t$  of a stock that pays no dividends. In the absence of volatility, the expected time  $T$  stock price is just the futures price. So we have

$$F = S_t e^{r_C((T-t)/365)}$$

As soon as volatility is introduced, the distribution of terminal stock price  $S_T$  becomes wider and the above wouldn't hold any longer.

$$C_t = e^{-r_C((T-t)/365)} \text{Max}(0, F - X)$$

## Problems

**4.2.2-1.** Suppose that a stock has a value of  $S_{t_0} = 52.23$  at the

beginning of a time period and  $S_{t_1} = 54.06$  at the end of a holding period. Calculate the continuous time return on this investment.

**4.2.2-2.** Suppose that a stock has a value of  $S_{t_0} = 52.23$  at the beginning of a time period and  $S_{t_1} = 50.16$  at the end of a holding period. Calculate the continuous time return on this investment.

**4.2.2-3.** Suppose that a stock has a value of  $S_{t_0} = 50.00$  at the beginning of a time period and the continuously compounded return is 11.33%. What should  $S_{t_1}$  be at the end of the holding period?

**4.2.2-4.** Please confirm that the put–call parity holds in the following example. If you find that it doesn't hold, an arbitrage opportunity exists. If that is the case, please suggest a strategy to eliminate the arbitrage opportunity.  $e^{-r_c((T-t)/365)} = 0.9985$ ,  $X = 50$ ,  $S_0 = 55$ ,  $P = 3.55$ ,  $C = 6.50$ .

**4.2.2-5.** Using the Black–Scholes–Merton option pricing formula, please find (a) cutoff points  $d_1$  and  $d_2$  and (b) current price of the put option.

$$S_0 = 76.50$$

$$T = \frac{45}{360}$$

$$\sigma = 0.7$$

$$r_c = 0.06$$

$$X = 75$$

$$N(d_1) = 59.22\%$$

$$N(d_2) = 49.43\%$$

**4.2.2-6.** Using the information from Problem 4.2.2-5 please calculate the put prices for a range of different stock prices. Assume stock prices range between \$50 and \$100. Plot your answer. Hint: use Excel spreadsheet.

**4.2.2-7.** How would your answer to Problem 4.2.2-6 change if the option in question is a call option and you are holding (a) long position and (b) short position?

## 4.3 Greeks

This chapter surveys the Black–Scholes formula still a little deeper. We are going to look at sensitivities of option prices to their underlying variables and parameters. These sensitivity measures are denoted using a number of Greek letters, hence the name—Greeks.

### 4.3.1 Delta

Delta is the options' price sensitivity to changes in the underlying asset price. For simplicity, let us assume that the asset in question is equity, so the underlying variable will be a stock price. Our initial exposition will be focused firstly on calls and secondarily on puts.

Delta ranges from 0 to 1.

$$\delta_C = \frac{\partial C}{\partial S}$$

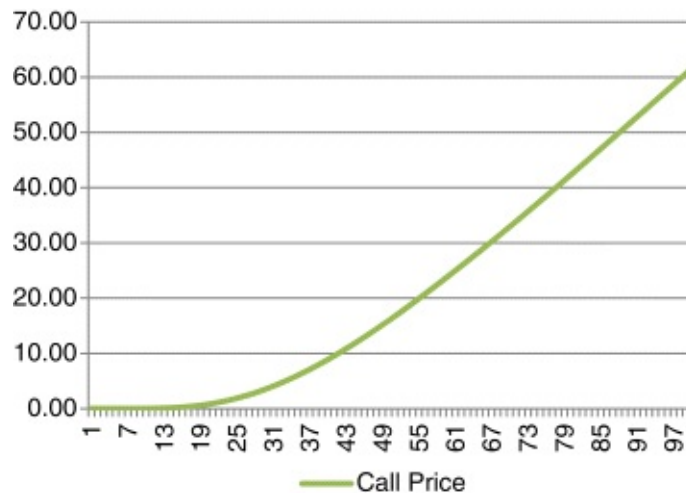
Equation 4.53 : Delta of a call option

$$\delta_C = N(d_1)$$

The following chart shows the sensitivity of the Black–Scholes call price to changes in the underlying stock price. This is actually our basic call price profile at any arbitrarily selected time prior to the option's expiration.

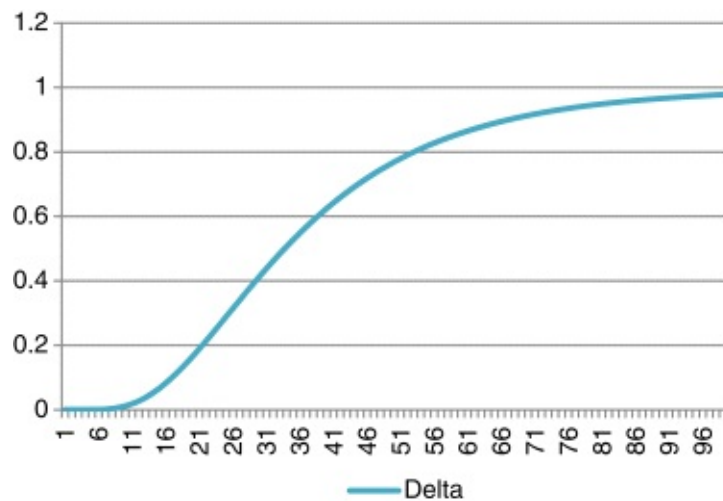
We are assuming the following information about this call: its strike price

is at \$40, time to maturity as a fraction of the year is 0.589—thus a little more than six months. Volatility of the underlying is assumed to be 0.7 and the risk-free rate is at 4.5% (Fig. 4.33).



**FIGURE 4.33** Call Price Preexpiration

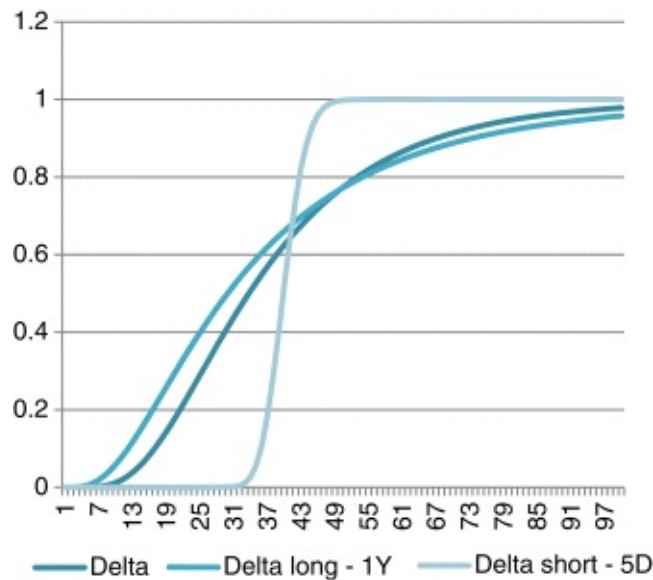
Figure 4.34 shows the values of delta as a function of the underlying stock prices. The stock prices range from 1 to 100, and the call delta ranges from 1 to 1. The shape of the graph depends on our assumptions for the underlying parameters.



**FIGURE 4.34** Delta of a Call Option

Let us change one of the underlying parameters and observe the changes in the shape of the delta function. In Figure 4.35, I have added two delta profiles to the original one, which assumed time to maturity of 0.589. Delta long assumes that we are looking at the delta function right at the

very issuance date of the call option—time to maturity is one year. Delta short assumes that there are only five days left before the option reaches its expiration (i.e., maturity) date.



**FIGURE 4.35** Delta Function at Varying Times to Expiration

Recall that delta measures the call price sensitivity to changes in the underlying stock price. This makes it one of the most crucial variables for options traders. High values of delta suggest that the call price will be very reactive to even a tiny change in the price of the underlying stock. Low values of delta show the opposite. In the case of the call option above—if the call is out of the money, we will see low values of delta and thus cannot expect large changes in the price of the option. As soon as the option becomes in-the-money, the delta approaches values closer to 1 and the reactivity of the call price increases markedly.

Secondly, it is important to consider the time to expiration of the option. As we get closer to the maturity time, the delta profile shows a rapid change from 0 to 1 occurring for stock price values very near the strike price. This tells us that as we see the option switching from being OTM to ITM we will experience large jumps in the option price reactivity to changes in the underlying stock price. Traders are aware that one needs to act fast in this environment.

Most traders hedge their short call positions with an underlying asset hedge. Delta hedging works well only for small changes in the underlying asset. Larger changes would produce an imprecise hedge. In such a

case, we refer to Gamma for adjustments of delta.

### Case Study 4.3-1

Suppose that  $\delta_C = 0.5692$ . That means a \$1 change in the underlying stock price will cause a \$0.57 change in the price of this call option, provided that all other inputs remain unchanged.

How large is the call price change if the underlying stock price decreases by \$0.25?

**Answer**

Equation 4.54 : Change in call premium as a function of de

$$\Delta C = \delta_C \times \Delta S$$

$$\Delta C = 0.5692 \times -0.25$$

$$\Delta C = -0.1423$$

The call price will decrease by \$0.14.

For puts:



$$\delta_P = \frac{\partial P}{\partial S}$$

Equation 4.55 : Delta of a put option

$$\delta_P = N(d_1) - 1$$

Note that the put delta is negative, thus the range is  $-1$  to  $0$ . Also a delta-neutral position will require a long stock position and a long put position ([Fig. 4.36](#)).

### Case Study 4.3-2

Suppose that  $\delta_P = -0.44$ . That means that a \$1 increase in the underlying stock price will cause a \$0.44 decrease in the price of this put option, all other inputs remain unchanged.

Equation 4.56 : Put premium change as a function of delta

$$\Delta P = \delta_P \times \Delta S$$

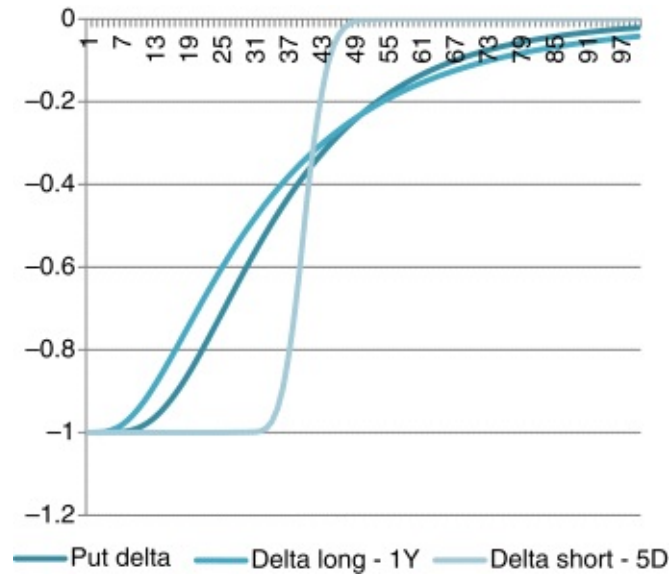
How much will the put price change if the underlying stock price decreases by \$0.35?

**Answer**

$$\Delta P = (-0.44) \times (-0.35)$$

$$\Delta P = 0.154$$

The put price will increase by \$0.15 due to a \$0.35 decrease in the price of the underlying.



**FIGURE 4.36** Delta of a Put Option

### 4.3.2 Gamma

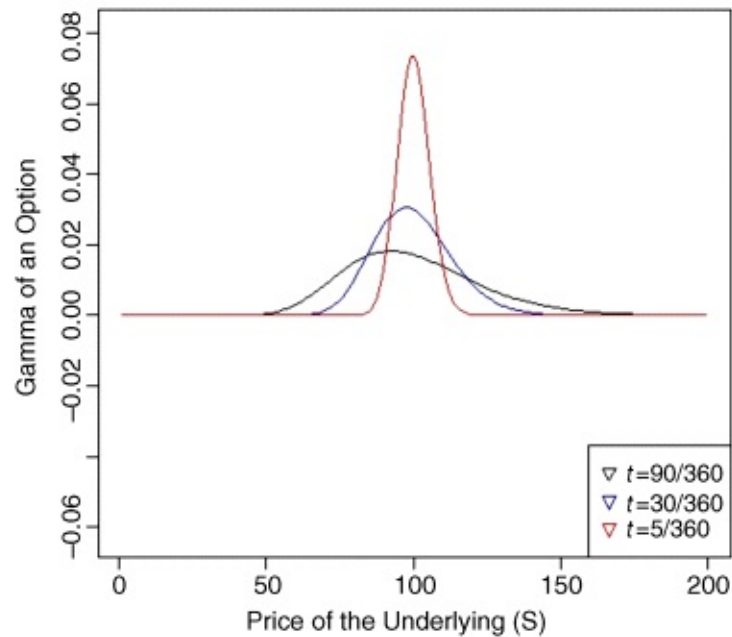
Gamma measures the change of the change. Gamma is the highest at times right before maturity and the lowest at issuance time. Note that gamma for a call option is the same as gamma for a put option.

$$\Gamma = \frac{\partial^2 C}{\partial S^2}$$

Equation 4.57 : Gamma for puts and calls

$$\Gamma = \frac{\frac{1}{\sqrt{2\pi}} e^{-(1/2)d_1^2}}{S\sigma\sqrt{\left(\frac{T-t}{365}\right)}}$$

Let us take a closer look at the Gamma of a call option that is struck at \$100, the time to maturity is assumed to be 90/360, 30/360, and 5/360, respectively. Its sigma is 0.46 and the interest rate is 2% (Fig. 4.37).



**FIGURE 4.37** Gamma Function as a Function of the Underlying Asset Price Showing Varying Expiration Times

Gamma is also used for hedging. Delta hedging works well only for small changes in the underlying stock price. For larger changes, gamma works well.

Equation 4.58 : Delta adjustment as a function of gamma

$$\delta_1 = \delta_0 + \Delta S \times \Gamma$$

Where

Equation 4.59 : Price change in the underlying

$$\Delta S = S_1 - S_0$$

The resulting change in the call price is calculated using this familiar formula.

Equation 4.60 : Change in option premium as a function of a

$$\Delta C = \delta_1 \times \Delta S$$

For American-style options, the formula above will not produce reliable results. You will probably want to choose to calculate gamma using numerical methods.

### 4.3.3 Rho

Rho represents the sensitivity of the option price to the continuously compounded risk-free rate.

Calculation in this case varies between calls and puts. In the case of a call, the formula is as follows:

$$\rho = \frac{\partial C}{\partial r_c}$$

Equation 4.61 : Rho of a call option

$$\rho_C = \left( \frac{T-t}{365} \right) X e^{-r_c((T-t)/365)} N(d_2)$$

Put option:

$$\rho = \frac{\partial P}{\partial r_c}$$

Equation 4.62 : Rho of a put option

$$\rho_P = - \left( \frac{T-t}{365} \right) X e^{-r_c((T-t)/365)} (1 - N(d_2))$$

Next, we calculate  $\rho_C$  or  $\rho_P$ . Note that we will use the original risk-free rate  $r_{c,o}$  in the formula. Suppose we experience a change in the continuously compounded risk-free rate.

Equation 4.63 : Change in risk – free rate

$$\Delta r_c = r_{c,1} - r_{c,o}$$

Combining the change in interest rate with the sensitivity parameter yields the change in option price.

Equation 4.64 : Change in call premium as a function of rho

$$\Delta C = \Delta r_c \times \rho_C$$

The new call option price is then

Equation 4.65 : New call option premium

$$C_1 = C_0 + \Delta C$$

### Case Study 4.3-3

Consider a call option. During our holding period the T-bill rate decreases from 3.5 to 2.75%, time to maturity is 0.1,  $X = \$15$  and  $N(d_2) = 0.45$ .

- Calculate the sensitivity of this call option to the change in the T-bill rate.
- How much will the call price change as a result?
- What is new call price  $C_1$  if  $C_0 = \$1.35$ ?

**Answer**

	Time to Maturity	N(d2)	$r_C$ time 0	$r_C$ time 1	X	Call Premium time 0
	0.1	0.45	0.035	0.0275	15	1.35
Rho Call						
0.67264163						
delta (i.e. change) $r_C$						
-0.0075						
delta (i.e. change) Call						
-0.005044812						
Call Premium time 1						
1.34						

	Time to Maturity	N(d2)	$r_C$ time 0	$r_C$ time 1	X	Call Premium time 0
	0.1	0.45	0.035	0.0275	15	1.35
Rho Call						
=B2*F2*EXP(-D2*B2)*C2						
delta (i.e. change) $r_C$						
=E2-D2						
delta (i.e. change) Call						
=A6*A4						
Call Premium time 1						
=G2+A8						

Rho call represents the answer to part (a) the option will experience a miniscule price decline of \$0.005. The new call price will become \$1.34.

### 4.3.4 Vega



Option prices are very sensitive to one particular underlying parameter—the volatility. It is quite unfortunate because Sigma, representing standard deviation of returns on the underlying asset, is the only unobservable variable in the Black–Scholes model.

Vega shows the option price sensitivity to changes in the underlying volatility.

$$Vega = \frac{\partial C}{\partial \sigma}$$

Equation 4.66 : Vega for puts and calls

$$Vega = \frac{S \sqrt{\left(\frac{T-t}{365}\right)} e^{-(1/2) d_1^2}}{\sqrt{2\pi}}$$

Vega, just like gamma, uses the same formula for both, put and call, options.

Equation 4.67 : Change in volatility

$$\Delta\sigma = \sigma_1 - \sigma_0$$

Equation 4.68 : Change in call premium as a function of vega

$$\Delta C = \Delta\sigma \times Vega$$

$$C_1 = C_0 + \Delta C$$

### Case Study 4.3-4

Consider a put option. During our holding period the volatility increases from 0.55 to 0.62, time to maturity is 45 days (assume 360-

day year), the price of the underlying  $S = \$120$  and  $d1 = 0.21$ . At time zero, the put premium is \$6.69.

- Please calculate this option's vega.
- What is the put option price change as a result of the increase in volatility? What is the new put premium?

### Answer

	Time to Maturity	d1	Sigma time 0	Sigma time 1	S	Put Premium time 0
	0.125	0.21	0.55	0.62	120	6.69
<b>Vega Put</b>						
<b>16.55654132</b>						
<b>delta Sigma</b>						
<b>0.07</b>						
<b>delta Put</b>						
<b>1.158957892</b>						
<b>Put Premium time 1</b>						
<b>7.85</b>						

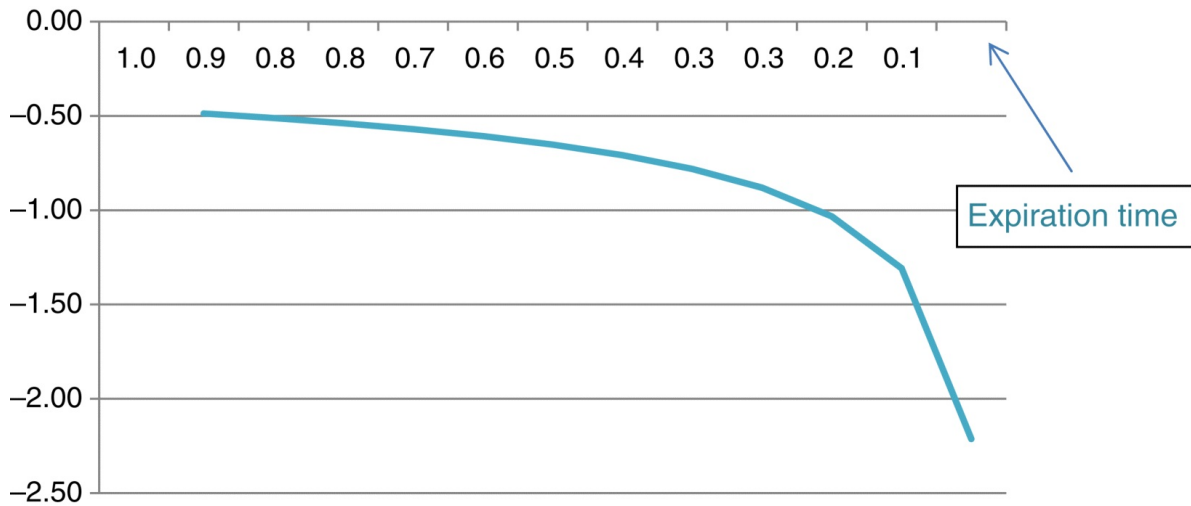
	Time to Maturity	d1	Sigma time 0	Sigma time 1	S	Put Premium time 0
	=45/360	0.21	0.55	0.62	120	6.69
Vega Put						
= $(F2*\text{SQRT}(B2)*\text{EXP}(-0.5*C2^2))/\text{SQRT}(2*3.1416)$						
delta Sigma						
=E2-D2						
delta Put						
=A6*A4						
Put Premium time 1						
=G2+A8						

This put option has a vega of 16.56. The new put premium is 7.85 as a result of a 7% increase in volatility, which in turn increased the put premium by \$1.16.

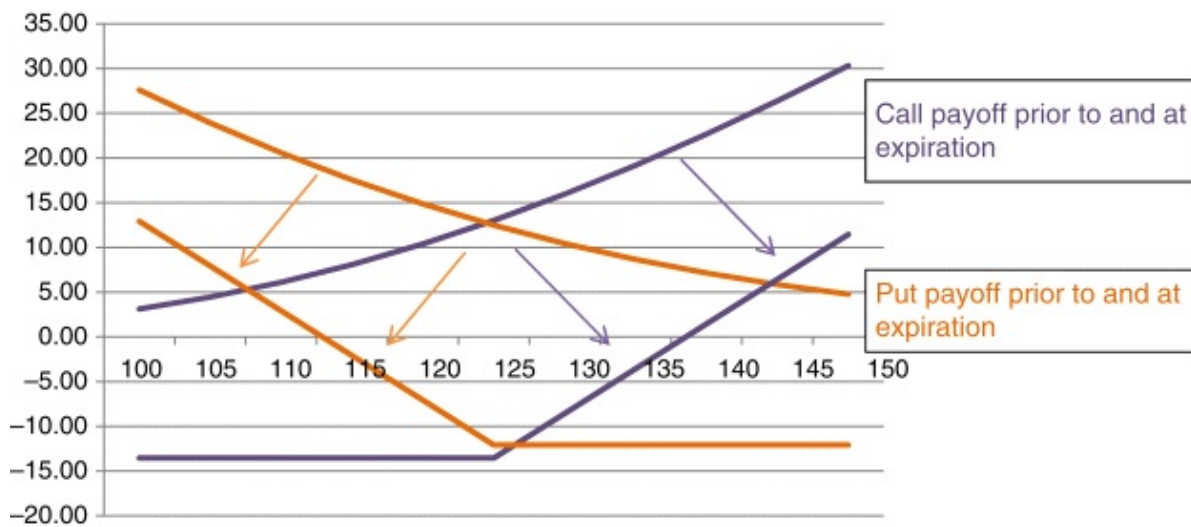
### 4.3.5 Theta

Theta represents the sensitivity of the option price to changes in the time to maturity. Option value decays toward the kinked final payoff function as expiration approaches. This time decay is called theta. Although, theta is negative for plain vanilla European calls, and for American calls and puts, a deep in-the-money European put decays upward in value. Thus,

deep ITM European puts have positive theta (Figs. 4.38 and 4.39).



**FIGURE 4.38** Theta Approaching Expiration Time



**FIGURE 4.39** Time Value Decay for Puts and Calls

Calls:

$$\theta_C = \frac{\partial C}{\partial \left( \frac{T-t}{365} \right)}$$

Equation 4.69 : Theta for a call option

$$\theta_C = -\frac{S\sigma e^{-(1/2)d_1^2}}{2\sqrt{2\pi\left(\frac{T-t}{365}\right)}} - r_c X e^{-r_c((T-t)/365)} N(d_2)$$

Puts:

$$\theta_P = \frac{\partial P}{\partial\left(\frac{T-t}{365}\right)}$$

Equation 4.70 : Theta for a put option

$$\theta_P = -\frac{S\sigma e^{-(1/2)d_1^2}}{2\sqrt{2\pi\left(\frac{T-t}{365}\right)}} + r_c X e^{-r_c((T-t)/365)} (1 - N(d_2))$$

Equation 4.71 : Change in time to expiration

$$\Delta\left(\frac{T-t}{365}\right) = \left(\frac{T-t}{365}\right)_1 - \left(\frac{T-t}{365}\right)_0$$

Equation 4.72 : Change in call premium as a function of the

$$\Delta C = \Delta\left(\frac{T-t}{365}\right) \times \theta_C$$

$$C_1 = C_0 + \Delta C$$

### Case Study 4.3-5

Consider two options—a call and a put. They are written on the same underlying, with strike price  $X = \$115$ . The price of the underlying asset at present is  $S = \$120$ . Volatility is 0.55,  $d1 = 0.21$ , and  $N(d2) = 0.5518$ . The present call premium is \$11.83, and the present-time put premium is \$6.69.

Currently, we have 45 days until maturity. How much will the options' prices change once we get twenty days closer to maturity? Please assume a 365-day year.

**Answer**

	$r_C$	$d_1$	(T-t/365) time 0	(T-t/365) time 1	S	Call Premium time 0	Put Premium time 0	X	$N(d_2)$	Sigma
	0.01	0.21	0.123288	0.06849315	120	11.83	6.69	115	0.5518	0.55
Theta Call										
	-57.3115315759									
Theta Put										
	-28.3774385623									
delta Time to Maturity										
	0.054794521									
delta Call										
	-3.140357895									
delta Put										
	-1.55492814									
Call Premium time 1										
	8.69									
Put Premium time 1										
	5.14									

	$r_C$	$d_1$	(T-t/365) time 0	(T-t/365) time 1	S	Call Premium time 0	Put Premium time 0	X	$N(d_2)$	Sigma
	0.01	0.21	=45/365	=25/365	120	11.83	6.69	115	0.5518	0.55
Theta Call										
	=(-(\$F\$2*\$K\$2*EXP(-0.5*\$C\$2^2))/(2*SQRT(2*3.1416*\$D\$2))-(\$B\$2*\$I\$2*EXP(-\$B\$2*\$D\$2)*\$J\$2))									
Theta Put										
	=(-(\$F\$2*\$K\$2*EXP(-0.5*\$C\$2^2))/(2*SQRT(2*3.1416*\$D\$2))+(\$B\$2*\$I\$2*EXP(-\$B\$2*\$D\$2)*(1-\$J\$2)))									
delta Time to Maturity										
	=ABS(E2-D2)									
delta Call										
	=A8*A4									
delta Put										
	=A8*A6									
Call Premium time 1										
	=G2+A10									
Put Premium time 1										
	=H2+A12									

Due to the effects of the time value decay, moving forward in time toward maturity time causes both the put and the call lose some of their value. In the case of the call, the premium declines by \$3.14. The put's price also declines but only by \$1.55 in this case.

## Problems

**4.3-1.** Suppose that we are long a put option. During our holding period the T-bill rate increases from 1.5 to 2.25%, time to maturity is 0.08,  $X = \$100$ , and  $N(d_2) = 0.45$ .

- Calculate the sensitivity of this put option to the change in the T-bill rate.
- How much will the put price change as a result?

**4.3-2.** Suppose that  $\delta_P = -0.51$ . That means that a \$1 increase in the underlying stock price will cause a \$0.51 decrease in the price of this put option. How much will the put premium change if the underlying stock price decreases by \$2.55?

**4.3-3.** Consider two options—a call and a put. They are written on the same underlying, with strike price  $X = \$80$ . The price of the underlying asset at present is  $S = \$82$ . Volatility is 0.57,  $d1 = 0.32$ ,  $N(d1)$  is 0.49, and  $N(d2) = 0.59$ . The present call premium is \$7.90, and the present-time put premium is \$7.45. Currently, we have 30 days until maturity.

- a. Please calculate theta and gamma for both the put and the call.
- b. How much will the options' prices change once we get twenty days closer to maturity? Please assume a 365-day year. All other inputs are constant.
- c. What are the options' price changes if the underlying stock price decreases by \$5.50?

**4.3-4.** Consider a call option. During our holding period the volatility increases from 0.35 to 0.52, time to maturity is 15 days (assume a 365-day year), the price of the underlying  $S = \$97$  and  $d1 = 0.29$ . At time 0, the call premium is \$5.42.

- a. Please calculate this option's vega.
- b. What is the new call price (also known as premium)?

**4.3-5.** Suppose that two standard options have exactly the same features, except that one has a long maturity and the other has a short maturity. Which one would you expect to have the higher gamma?

**4.3-6.** Compare similarities and differences between gamma and theta

## 4.4 Volatility

### 4.4.1 Delta Hedging

Delta is a “Greek” that reflects option price sensitivity to the movements of the underlying asset price. Delta of an option price is the slope of the price curve function at the current price of the underlying. Possible values for delta range from 0 to 1. If the option is currently OTM, the delta will be low, approaching 0. If it is ITM, delta will be approaching 1. ATM delta is very close to 0.5.

Delta hedging is a tactic that enables us to hedge an option's value



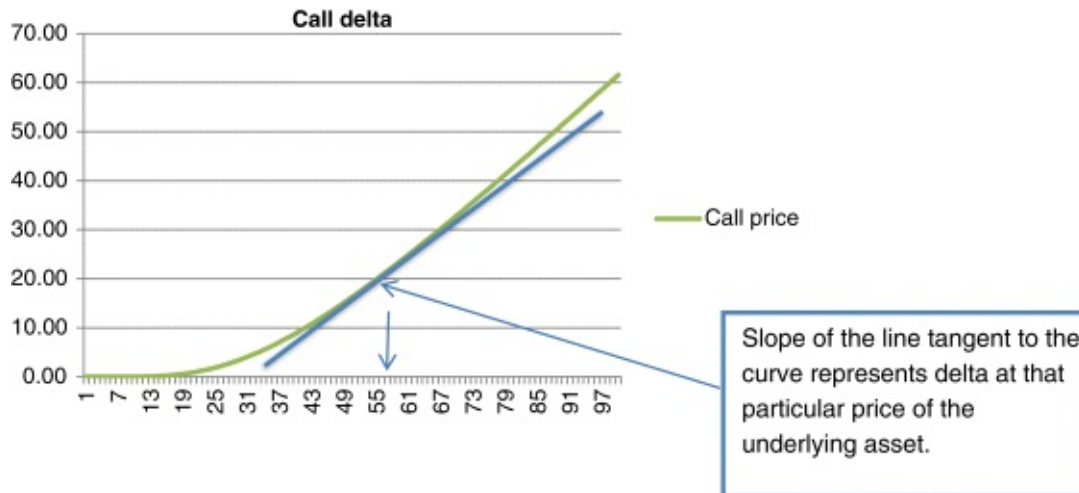
exposure by buying or selling the underlying asset. Options are convex (i.e., nonlinear) instruments and their replication requires dynamic hedging. Hedging a nonlinear asset with a linear asset will create an imperfect hedge. Continuous reheding is ideal but in reality unattainable. Thus, it is done often but only periodically.

With delta hedging an option position we always end up selling the underlying at the higher price and buying it back at a relatively lower price. Repeating this hedging exercise leads to convexity gains over time. Note that this example omits bid ask spreads, transactions fees, and interest charges.

### Dynamic Hedging

Even though continuous hedging would be the most desirable, in reality hedging adjustments are done periodically, but the frequency is subject to considerations such as transaction fees. The time elapsing between reheding times may be of variable length. In any case, hedging occurs dynamically over time.

Suppose, we desire to hedge a long call position with a short position in the underlying. Let's assume that we are six days away from expiration time. The call is currently at the money with a delta of 0.5. Within the next six days, until the expiration time, the price will oscillate around the strike price changing Delta accordingly. When the price moves into the money—delta will increase and when the option moves OTM the delta will decrease below the initial 0.5. We will take advantage of this by short selling when delta is high and buying to cover the short sale when delta, and also the price of the underling, is lower. Such continuous buying low and selling high produce “convexity” gains. These gains stem from the fact that we are hedging a convex instrument—the call—with a linear instrument—the underlying.



The same situation occurs with put hedging. We end up buying low and selling at a higher price each time we re-hedge our position. The hedge for a long put consists of a long position in the underlying.

As market prices of the underlying change over time, and options' deltas do so as well. Let us record an option's delta at each price point of interest. In the next step, we will measure the change in the option's delta from one reading date to the other. The change in delta determines the relevant cash flows. Considering a put position, for example, a negative change in delta will result in negative cash flows, a positive change in delta results in positive cash flow. A positive cash flow signals selling, a negative cash flow reflects buying additional units of the underlying asset. Note that in the case of long puts, delta is a negative number.

### Case Study 4.4.1-1

Suppose a trader buys an ATM call option (long position) with current delta of 0.5. Once he makes the purchase, we will simultaneously want to hedge his exposure by selling (short position) 0.5 units<sup>1</sup> of the underlying asset. Now, suppose that the ATM price was 10/share. The trader rehedges his exposure every day. The next day the price increased to 11/share. The options delta is now at 0.7. The trader has to increase his short position by 0.2<sup>2</sup> units. Thus, he sells additional 20 shares at the higher price of 11/share. Suppose that on day three the price drops to 9.50/share. The corresponding delta is 0.45. Rehedging now consists of decreasing our short position in the underlying asset. That means that we will reverse the short selling by

buying 0.25 units of the underlying. This will be bought at the now lower price of 9.50/share.

Please calculate the net gains from hedging a long call position. How would the process change for a long put position? Will the net gains from a put be the same as from the call? why?

**Answer**

Long Call Option						
Time in days	Underlying Price	Delta	Change in Delta	Short Position in Underlying	Cash Flow	
1	10	0.5		50	500	Cash inflow from the intial short sale
2	11	0.7	0.2	70	220	Cash inflow from the additional short sale
3	9.5	0.45	-0.25	45	-237.5	Cash outflow from closing a part of the short sale
4	9	0.4	-0.05	40	-45	...
5	8.5	0.25	-0.15	25	-127.5	
6	10	0.5	0.25	50	250	
				Sum Total	560	Convexity gains generated by delta hedging adjustments.
				Net	60	Unwinding the intitial short sale hedge yields net gains

Note that the underlying price process is exactly identical to the one used in the call calculation.

Long Put Option						
Time in days	Underlying Price	Delta	Change in Delta	Long Position in Underlying	Cash Flow	
1	10	-0.5		50	-500	Cash outflow for the initial purchase
2	11	-0.3	0.2	30	220	Cash inflow from selling 0.2 units of the intital long position
3	9.5	-0.55	-0.25	55	-237.5	Cash outflow from additional purchase of the underlying
4	9	-0.6	-0.05	60	-45	...
5	8.5	-0.75	-0.15	75	-127.5	
6	10	-0.5	0.25	50	250	
				Sum Total	-440	Convexity gains generated by delta hedging adjustments.
				Net	60	Unwinding the intitial short sale hedge yields net gains

#### Long Call Option

Time in days	Underlying Price	Delta	Change in Delta	Short Position in Underlying	Cash Flow
1	10	0.5		=100*C3	=E3*B3
2	11	0.7	=C4-C3	=100*C4	=100*D4*B4
3	9.5	0.45	=C5-C4	=100*C5	=100*D5*B5
4	9	0.4	=C6-C5	=100*C6	=100*D6*B6
5	8.5	0.25	=C7-C6	=100*C7	=100*D7*B7
6	10	0.5	=C8-C7	=100*C8	=100*D8*B8
				Sum Total	=SUM(F3:F8)
				<b>Net</b>	=F10-F3

#### Long Put Option

Time in days	Underlying Price	Delta	Change in Delta	Long Position in Underlying	Cash Flow
1	10	-0.5		=(-1)*100*C15	-500
2	11	-0.3	=C16-C15	=(-1)*100*C16	=100*D16*B16
3	9.5	-0.55	=C17-C16	=(-1)*100*C17	=100*D17*B17
4	9	-0.6	=C18-C17	=(-1)*100*C18	=100*D18*B18
5	8.5	-0.75	=C19-C18	=(-1)*100*C19	=100*D19*B19
6	10	-0.5	=C20-C19	=(-1)*100*C20	=100*D20*B20
				Sum Total	=SUM(F15:F20)
				<b>Net</b>	=F22-F15

## 4.4.2 Greek Neutrality

### Delta Neutrality

#### Case Study 4.4.2-1

Suppose we have written 30 calls. That is the call buyer has the right to buy 3000 shares of the underlying asset. In order to avoid issuing naked calls, we will want to hedge our position by purchasing a certain number of shares of the underlying. We have now covered our calls.

How many shares do we have to buy to cover our short call position? If we buy these shares will our portfolio be delta neutral? If so why?

Suppose that

$$\#_C = 30$$

$$\delta_C = 0.7$$

**Answer**

We'll need to buy  $\delta_C \times \#_C \times 100$  shares of the underlying asset.

$$0.7 \times 30 \times 100 = 2100$$

Delta of the underlying stock—long position—is equal to one.

$$\delta_S = 1$$

So, the delta of the stock position in the portfolio is

$$\delta_{StockPos} = 1 \times 2100 = 2100$$

And the delta of the option position in the portfolio is

$$\delta_{OpPos} = 0.7 \times (-3000) = -2100$$

(4.73)

Equation 4.73 : Portfolio delta

$$\delta_{Portfolio} = \delta_{OpPos} + \delta_{StockPos}$$

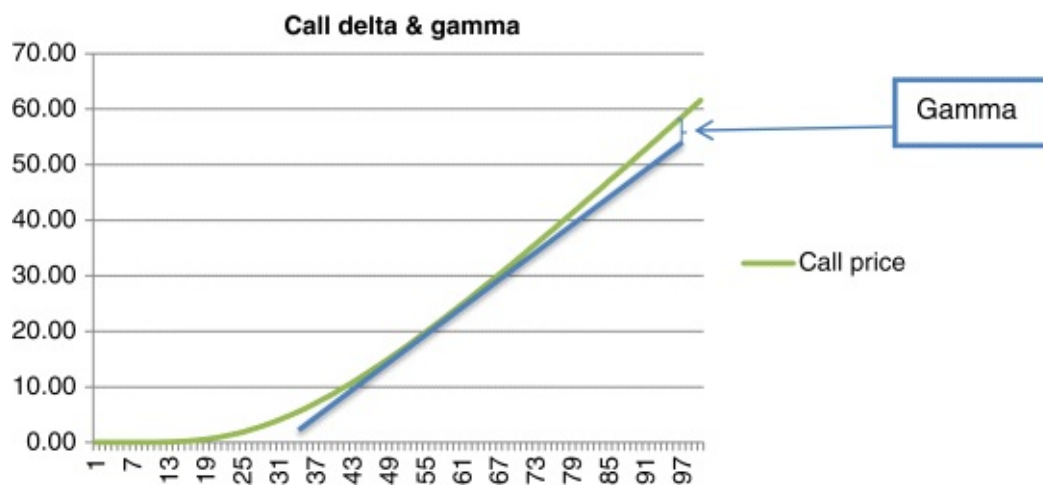
$$\delta_{Portfolio} = (-2100) + 2100 = 0$$

Yes, this portfolio is delta neutral, because the delta of the portfolio is equal to zero.

### Gamma Neutrality

Theta is sometimes used as a proxy for gamma. There is no theta neutrality, because the passage of time is completely predictable, thus creating zero ambiguity about its direction.

Gamma is the difference between the tangent line and the option price curve. Due to this, gamma is sometimes referred to as curvature.



How can we hedge gamma? In this case, it is impossible to use the underlying asset to hedge with. It is because the price profile of an

underlying has no curvature; therefore, its gamma is zero. We need an asset that has some curvature in order to hedge gamma—like options.

In order to make a portfolio gamma neutral, we need purchase the correct amount of options.

### *Notation*

$\#_C$	Number of liquid calls
$\Gamma_{Portfolio}$	Gamma of delta-neutral portfolio
$\Gamma_C$	Gamma of the liquid call

In order to get a portfolio to be gamma neutral, we need to add a certain number of the hedge asset—liquid call options in this case.

$$\begin{aligned}\#_C \times \Gamma_C + \Gamma_{Portfolio} &= 0 \\ \#_C \times \Gamma_C &= -\Gamma_{Portfolio}\end{aligned}$$

The number of call options added to the portfolio to make it gamma neutral is equal to

Equation 4.74 : Number of calls ensuring gamma neutrality

$$\boxed{\#_C = \frac{-\Gamma_{Portfolio}}{\Gamma_C}}$$

Making a delta-neutral portfolio gamma neutral can be regarded as a correction for the fact that we cannot delta-hedge continuously. Delta-hedging and the resulting delta neutrality is a perfectly good strategy for small price movements in the underlying (nonconvex) asset. For large price moves, a good protection requires gamma neutrality.

## Case Study 4.4.2-2

Suppose a call option has a  $\delta_C = 0.7$  and a  $\Gamma_C = 1.6$ . Further suppose that a delta-neutral portfolio has a gamma of  $-3200$ . What is the correct amount of call option needed to be added to the portfolio to make it gamma neutral? What will happen to the delta neutrality of the portfolio after the addition of calls?

**Answer**

$$\#_C = \frac{3200}{1.6} = 2000$$

We need to take a long position in 2000 options to make it gamma neutral.

This action will make the portfolio no longer delta neutral. In order to make it delta neutral, again we need to additionally sell a certain amount of the underlying nonconvex asset.

$$\delta_{OpPos} = 0.7 \times 2000 = 1400$$

To achieve delta neutrality, we need

$$\delta_{OpPos} + \delta_{StockPos} = 0$$

$$1400 + \delta_{StockPos} = 0$$

$$\delta_{StockPos} = -1400$$



Fourteen hundred units of the underlying asset have to be sold if we want the portfolio to be delta neutral. Now, the portfolio is both delta and gamma neutral.

### Vega Neutrality

Recall that the Black–Scholes model assumes a constant volatility parameter. This assumption is a simplification of reality. In fact, volatility should be modeled as a stochastic variable. Doing so would increase the formula's complexity substantially. It has been empirically verified, however, that vega calculated using stochastic volatility is very close to the Black–Scholes vega, and thus for simplicity the constant volatility parameter works well enough in practice.

Vega neutrality protects against changes in the volatility parameter  $\sigma$ . Vega-hedging resembles gamma hedging in that the hedge asset is once again an option. Note, however, that a gamma-neutral portfolio will not be vega neutral at the same time. If we need the portfolio to be both gamma and vega neutral, we would have to employ two types of options written on the same underlying.

#### Notation

$\#_C$	Number of liquid calls
$Vega_{Portfolio}$	Vega of delta-neutral portfolio
$Vega_C$	Vega of a liquid call

$$\#_C \times Vega_C + Vega_{Portfolio} = 0$$

Equation 4.75 : Number of calls ensuring vega neutrality

$$\#_C = \frac{-Vega_{Portfolio}}{Vega_C}$$

## Gamma and Vega Neutrality

### Case Study 4.4.2-3

Consider a portfolio that is delta neutral, with gamma of  $-5,000$  and a vega of  $-8,000$ . Suppose that a traded option has gamma of  $0.5$  and vega of  $2.0$  and a delta of  $0.6$ .

- How can we make this portfolio vega neutral?
- How can we make this portfolio both gamma and vega neutral?

#### **Answer**

The portfolio can be made vega neutral by including a long position in  $4000$  traded options.

$$\frac{-(-8000)}{2} = 4000$$

This would increase delta to  $2400$  and require that  $2400$  units of the asset be sold to maintain delta neutrality.

$$0.6 \times 4000 = 2400$$

In order to make the portfolio both gamma and vega neutral, we need to introduce a second traded option. Let's say that this option has a gamma of  $0.8$ , a vega of  $1.2$ , and a delta of  $0.5$ .

If  $w_1$  and  $w_2$  are the amounts of the two traded options included in the portfolio, we require gamma neutrality ensured by the first equation, and vega neutrality given in the second equation of this system.

$$-5000 + 0.5w_1 + 0.8w_2 = 0$$

$$-8000 + 2.0w_1 + 1.2w_2 = 0$$

The solution to this system of equations is  $w_1 = 400$  and  $w_2 = 6000$ .

This portfolio can be made gamma and vega neutral by including 400 of the first and 6000 of the second option. The delta of the portfolio after the addition of the positions in the two traded options will be 3240.

$$400 \times 0.6 + 6000 \times 0.5 = 3240$$

Hence, 3240 units of the underlying asset would have to be sold to maintain delta neutrality.

### 4.4.3 Implied Volatility

#### Options as Volatility Instruments

Options are instruments of volatility more than instrument for directional bets. For an end investor, directional motivation for options may be natural, for an institutional investor options represent much more frequently tools for trading volatility. Suppose you hold a net long position in options. If so, then you are expecting volatility to increase. Long-position investors are hopeful that the underlying price moves as much as possible and as often as possible. If you are short options then you expect volatility to decrease.

Market professionals are interested in the difference between actual daily volatilities of stock prices and the volatilities implied by option prices. For example, if the actual underlying price moves more than what the option price volatility implies, then we consider buying the option because it is

considered cheap by the actual vs. implied volatility measure.

In order to compare to an actual daily volatility of the underlying price, we must divide the percentage by the square root of the day count. This is known as the square root rule and holds due to the role played by Brownian motion in modeling stock-price dynamics. There the process increments have variance that is proportional to the time that has elapsed. Hence, the volatility is the standard deviation, and thus it is proportional to the square root of the time elapsed.

### Case Study 4.4.3-1

Suppose that we are quoted an implied volatility of 39%. This is always a number reflecting an annual implied volatility. We know that the stock has been known to move on average 3% a day over the last month. Let us determine whether we'd be getting these options relatively inexpensively.

Assume 250 trading days in a year.

**Answer**

$$39\% / \sqrt{250} = 2.47\%$$

$$2.47\% < 3\%$$

This means that we should buy these options because their prices imply volatility that's lower than the actual observed one. Thus, we would indeed be getting these options cheaply.

## Problems

**4.4.1-1.** Suppose a trader buys an ATM call option (long position) with current delta of 0.5. Once we he makes the purchase we will simultaneously want to hedge his exposure by selling (short position) 0.5 units of the underlying asset. Now, suppose that the ATM price was 15/share. The trader rehedges his exposure every day. The next day the price increased to 17/share. Suppose that on day 3 the price drops to 12/share. The corresponding delta is 0.43. Rehedging now consists of decreasing our short position in the underlying asset.

Please use Excel to calculate the net gains from hedging of this long call position.

Time in days	Underlying price	Delta	Change in delta	Short position in underlying	Cash flow
1	15	0.5			
2	17	0.7			
3	12	0.43			
4	10	0.4			
5	13	0.47			
6	15	0.5			
Sum Total					
Net					

**4.4.2-2.** Suppose, we have written 100 calls. That is the call buyer has the right to buy 10,000 shares of the underlying asset. We want to cover these calls. How many shares of the underlying asset do we have to buy to cover our short call position? If we buy these shares, will our portfolio be delta neutral?

Suppose that

$$\#_C = 100$$

$$\delta_C = 0.55$$

**4.4.2-3.** Suppose a call option has a  $\delta_C = 0.6$  and a  $\Gamma_C = 1.6$ . Further suppose that this delta-neutral portfolio has a gamma of

-4200. What is the correct amount of call option we need to add to the portfolio to make it gamma neutral? What will happen to the delta neutrality of the portfolio after the addition of calls?

**4.4.3-1.** Suppose that we are considering buying options with annual implied volatility of 23%. We know that the underlying stock has been known to move on average 1.5% a day over the last two months. Are these options cheap?

Please assume 263 trading days in a year.

## 4.5 Exotics

Exotic options exist essentially as an alternative to the complex option strategies introduced earlier. This alternative usually ends up being less costly, and thus often times preferable to clients. A number of the exotic options have become widely used and therefore are considered less and less “exotic”. Exotics can be hedged using plain vanilla puts and calls. Following are examples of some commonly used types of exotic options.

### 4.5.1 Asian Options

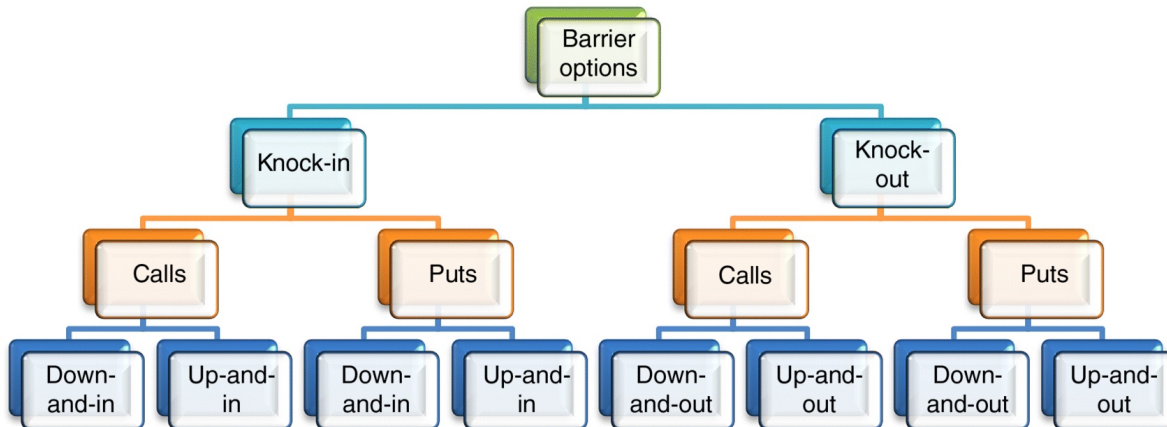
Asian options are options where the payoff depends on the average price of the underlying asset during a prespecified part of the life of the option. In the case of an average price call or put, the payoff at expiration is the difference between the average price of the underlying during the option life and the strike price.

Yet another type of an Asian option is an average strike price option. Here the payoff at expiration is given as the difference between the actual price of the underlying at that time and its average price over the life of the option.

### 4.5.2 Barrier Option

Barrier options are a combination of plain vanilla options and predefined thresholds. We recognize two basic types of barrier options: knock-in and knock-out option. With a knock-in option, the holder gains the right to exercise the option once an appropriate threshold has been crossed. In a knock-out option, the option ceases to exist once its threshold has been crossed. The threshold is typically the current/initial price of the

underlying asset (Fig. 4.40).



**FIGURE 4.40** Types of Binary Options

### 4.5.3 Basket Options

A basket option is an option where the payoff depends on the value of an underlying portfolio of assets. This portfolio may consist of any number of assets ranging from individual stocks to currencies, and so on.

Correct valuation of a portfolio of assets may be challenging. A suggested approach is a Monte Carlo simulation calibrated by correlated Brownian motion processes.

### 4.5.4 Binary Options

Binary options are quite popular and fairly easy to explain. A binary option offers a predetermined payoff at a prespecified date if a certain condition is met. Let us consider a European Put. It will generate gains if the price of the underlying asset at expiration time is less or equal to the option strike price.

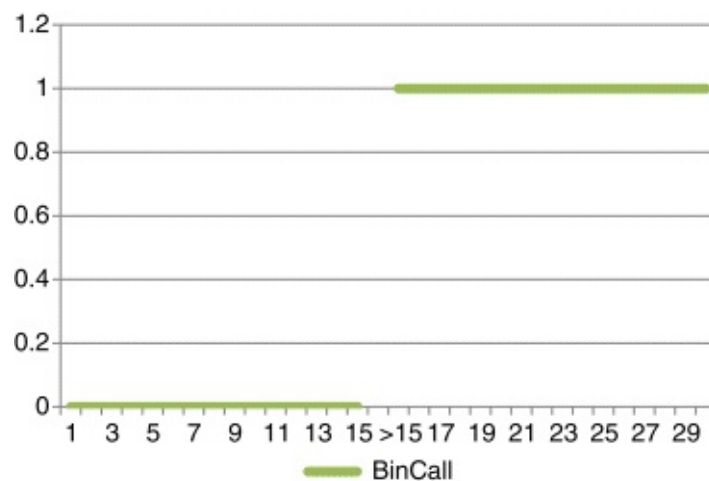
Equation 4.76 : Binary put payoff function

$$P_T^{Bin} = \begin{cases} \text{Payoff} \rightarrow S_T \leq X \\ 0 \rightarrow S_T > X \end{cases}$$

A European call works in a similar fashion, only the payoff occurs if the underlying price is greater or equal to the strike price ([Fig. 4.41](#)).

Equation 4.77 : Binary call payoff function

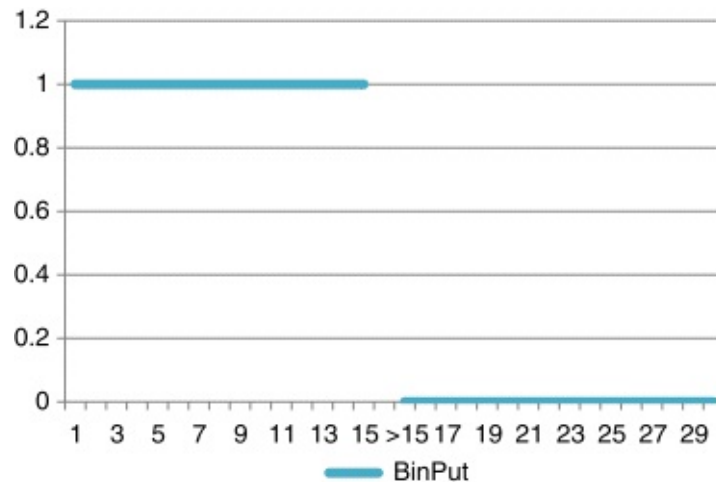
$$C_T^{Bin} = \begin{cases} \text{Payoff} \rightarrow S_T \geq X \\ 0 \rightarrow S_T < X \end{cases}$$



**[FIGURE 4.41](#) Binary Call Option Payoff**

A range option is constructed by using binary puts or calls with the same payoff. If the underlying price ends up being in the prespecified range, a certain payoff occurs. If it slips out of the range, the position will expire worthless and the trader will experience a loss equal to the net option premium he/she has paid ([Fig. 4.42](#)).





**FIGURE 4.42** Binary Put Option Payoff

### 4.5.5 Chooser Options

A chooser option is an option where the holder can, after some time of the life of the option has passed, choose whether the option should be a call or a put. If both of the vanilla options (i.e., a call and a put) underlying the chooser option are European and have the same strike price, we can value the chooser using the put–call parity.<sup>3</sup>

### 4.5.6 Forward Start Options

A forward start option is a vanilla option where the strike price is not determined. They start at some point in time in the future. The option will be ATM at the time it starts, which means that the exercise price is not set until the specified future time. However, please note that the option premium is paid in advance. Forward start options are quite common and mostly used as an incentive for employees.

### 4.5.7 Look-back Options

The payoff from look-back options depends on the price of the underlying during the life of the option. For calls, it is the difference between the minimum price and the price of the underlying asset at expiration time. In the case of puts, the option holders make positive profit if the difference between the maximum price and the price at expiration is positive.

## Problems

**4.5.1-1.** Explain why delta hedging is easier for Asian options than for regular options.

**4.5.2-1.** Does a down-and-out call become more valuable or less valuable as we increase the frequency with which we observe the asset price in determining whether the barrier has been crossed? What is the answer to the same question for a down-and-in call?

**4.5.2-2.** Explain why a regular European option is the sum of down-and-out European call and down-and-in European call. Is the same true for American call options?

**4.5.5-1.** Consider a chooser option where the holder has the right to choose between a European call and a European put at any time during a two-year period. The maturity dates and strike prices for the calls and puts are the same regardless of when the choice is made. Is it ever optimal to make the choice before the end of the two-year period? Explain your answer.

**4.5.6-1.** How can the value of a forward start put option on a nondividend paying stock be calculated if it is agreed that the strike price will be 10% greater than the stock price at the time the option starts?

**4.5.6-2.** Explain the difference between a forward start option and a chooser option.

## Notes

1. Remember that each options contract is for 100 shares of the underlying asset. That would make the transaction equal to  $0.5 * 100 = 50$  shares of the asset.
2.  $(0.7 - 0.5 = 0.2)$ .
3. See Section 4.2.2 for put–call parity.

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