The Mathematics of Interest Rate Derivatives, Markets, Risk and Valuation

ANALYTICAL FINANCE

VOLUME II

JAN R. M. RÖMAN

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The Mathematics of Interest Rate Derivatives, Markets, Risk and Valuation

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To my son and traveling partner –
Erik Håkansson

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Finally I will also give a special thanks to Thomas Gustafsson for all his comments, and great and deep discussion about financial mathematics.

Preface

This book is based upon my lecture notes for the course *Analytical Finance II* at Mälardalen University in Sweden. It's the second course in analytical finance in the program *Engineering Finance* given by the Mathematics department. The previous book, *Analytical Finance – The Mathematics of Equity Derivatives, Markets, Risk and Valuation*, covers the equity market, including some FX derivatives.

Both books are also a perfect choice for masters and graduate students in physics, astronomy, mathematics or engineering, who already know calculus and want to get into the business of finance. Most financial instruments are described succinctly in analytical terms so that the mathematically trained student can quickly get the expert knowledge she or he needs in order to become instantly productive in the business of derivatives and risk management.

The books are also useful for managers and economists who do not need to dwell on the mathematical details. All the latest market practices concerning risk evaluation, hedging and counterparty risks are described in separate sections.

This second volume covers the most central topics needed for the valuation of derivatives on interest rates and fixed income instruments. This also includes the mathematics needed to understand the theory behind the pricing of interest rate instruments, for example basic stochastic processes and how to bootstrap interest rate yield curves. The yield curves are used to generate and discount future cash-flows and value financial instruments. We include pricing with discrete time models as well as models in continuous time.

First we will give a short introduction to financial instruments in the interest rate markets. We also discuss the parameters needed to classify the instruments and how to perform day counting according to market conventions. Day counting is important when dealing with interest rate instruments since their notional amounts can be huge, millions or even billions of USD in one trade. One or a few missing days of discounting will change the total price with thousands of USD. We also discuss the most common types of interest rate quoting conventions used in the markets.

In Chapter 2 we present many of the different interest rates used in the market. We continue with swap interest rates in Chapter 3, where we also present details for several widely used interest rates such as LIBOR, EURIBOR and overnight rates in different currencies.

In Chapter 4, many of the common instruments are presented. This includes the basic instruments, such as bonds, notes and bills of different kinds, including some with embedded options. Then we introduce floating rate notes, forward rate agreements, forwards and futures, including cheapest to deliver clauses. We then discuss different kinds of interest rate swaps and the derivatives related to these swaps, like swaptions, caps and floors. This also includes some credit derivatives, such as credit default swaps. For swaptions, caps and floors we explicitly discuss recent changes in these models due to negative nominal interest rates and derive a quasi-analytical relationship between at-the-money lognormal and normal volatility.

In Chapters 5 and 6 we continue with yield curves and the term structure of interest rates. We show how to bootstrap interest rate curves from prices of financial instruments. We also present the Nelson-Siegel model and the extension by Svensson. A detailed analysis of interpolation methods follows and the pros and cons of each method is clearly outlined. Spreads in the interbank market are discussed in Chapter 7.

In Chapters 8 and 9, risk measures and some crucial features of modern risk management are discussed.

In Chapter 10, a new method for valuing instruments with an embedded optionality is presented. This method, the option-adjusted spread (OAS) method, can also be used to value callable and putable bonds, cancellable swaps etc. The call (put) structure can also be of Bermudan exercise type.

In Chapter 11 we begin to discuss the pricing theory and models based on stochastic processes. We continue with this, the continuous time models through Chapters 12–17. We derive and solve the partial differential equation for interest rate instruments based on arbitrage and relative pricing. Several stochastic models are presented. Some have an affine term structure, such as Vasicek, Ho-Lee, Cox-Ingersoll-Ross and Hull-White. Some models can be approximated by binomial or trinomial trees. These are Ho-Lee, Hull-White and Black-Derman-Toy. We also discuss the Heath-Jarrow-Morton framework and how to use forward measures in order to derive general option pricing formulas for interest rate instruments.

After a short presentation on how to handle some exotic instruments in Chapter 18, we discuss in Chapter 19 how to deal with some standard derivative instruments, such as swaptions, caps and floors. This also includes the recent case of negative interest rates.

In Chapter 20 is a brief introduction to convertible bonds.

Finally, there are some chapters on modern pricing. These chapters describes the dramatic changes in the markets after the financial crises in 2008 – 2009. Before the crises, credit risk was more or less ignored when valuing financial instruments. But, after the crises, collateral agreements have become a way to minimize counterparty risk. Also the funding of the deals were changed as well as the views on risk-free interest rates. During the crises even LIBOR rated banks did default. Also the LIBOR rates were manipulated by some of the panel banks. With collateral agreements in several currencies we need to use a multi-curve framework and bootstrap several curves to find the cheapest to deliver curve.

We also discuss credit value adjustment (CVA), debt value adjustment (DVA) and funding value adjustment (FVA). We also present the widely used LIBOR market model (LMM) and how to calibrate the LMM. Finally we present methods on how to manage exotic instruments by using linear Gaussian models (LGM). We also present something about the Stochastic Alpha Beta Rho (SABR) volatility model and how to convert between lognormal and normal distributed volatilities.

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Abbreviations

ABCDS	Asset-Backed Credit Default Swap
ALM	Asset and Liability Management
AML	Anti Money Laundering
ARCH	AutoRegressive Conditional Heteroskedasticity
ASW	Asset Swap
ATM	At-The-Money
BBA	British Bankers Association
B&S or BS	Black and Scholes
BDT	Black-Derman-Toy
BGM	Brace-Gatarek-Musiela
BIS	Bank for International Settlement
c-c	continuously compounded
CADF	Credit-Adjusted Discount Factor
CBOT	Chicago Board of Trade
CCBS	Cross-Currency Basis Swap
CCE	Current Credit Exposure
CCIRS	Cross-Currency Interest Rate Swap
CCS	Cross-Currency Swap
CCVN	Cross-Currency Variable Notional
CDI	Credit Default Index
CDO	Collateralised Debt Obligation
CDX	American Credit Default Indices
CET	Central European Time
CIRS	Cross-Currency Interest Rate Swap
CME	Chicago Mercantile Exchange

CMS	Constant Maturity Swap
COBE	Chicago Board of Option Exchange
CP	Commercial Paper
CPPI	Constant Proportion Portfolio Insurance
CSA	Credit Support Annex
CTD	Cheapest To Deliver
CVA	Credit Value Adjustments
C/W	Corporate-Week
DF	Discount Factor
DP	Dirty (purchase) Price
DVA	Debt Value Adjustment
ECB	European Central Bank
EIB	European Investment Bank
EONIA	Euro Over-Night Index Average
EVT	Extreme Value Theory
FRA	Forward Rate Agreement
FRN	Floating Rate Note
FSA	Financial Supervision Authority
FVA	Funding Value Adjustments
FX	Foreign Exchange
G-K	Garman-Kohlhagen
GARCH	Generalised Auto Regressive Conditional Heteroskedasticity
GBP	Great British Pound
HJM	Heath-Jarrow-Morton
IBOR	Inter Bank Offer Rate
IF	Implied Forward
IMM	International Monetary Market (based in CME)
IRS	Interest Rate Swap
ISDA	International Swap and Derivatives Association
ISMA	International Securities Market Association
ITM	Into-The-Money
KVA	Capital Value Adjustment (by regulations)
KWF	Kalotay-Williams-Fabozzy
LGD	Loss Given Default
LF	Likelihood Function
LGM	Linear Gaussian Models
LIBOR	London Inter-Bank Offered Rate
LIFFE	London International Financial Futures and Options Exchange
LMM	LIBOR Market Model

MC	Monte Carlo
NS	Nelson Siegel
NSS	Nelson Siegel Svensson
OAS	Option Adjusted Spread
ODE	Ordinary Differential Equation
OIS	Overnight Indexed Swap
O/N	Overnight
OTC	Over The Counter
OTM	Out of The Money
PCA	Principal Component Analysis
PDE	Partial Differential Equation
PDF	Probability Density Function
RVA	Replacement Value Adjustment.
PIP	Percentage In Point, sometimes also called a Price Interest Point
PRDC	Power Reverse Dual-Currency (Swaps)
PV	Present Value
PV01	Another name for PVBP
PVBP	Present Value of one Basis Point
QDS	Quanto Differential Swap
SARON	Swiss Average Rate Overnight
SABR	Stochastic Alpha Beta Rho (Volatility model)
SEK	Swedish Krona
SDE	Stochastic Differential Equation
S/N	Spot Text
SONIA	Sterling Over-Night Index Average
STIBOR	Stockholm Interbank Offered Rate
STINA	STIBOR T/N Average
STIR	Short Term Interest
TED	Treasury Euro Dollar
T/N	Tomorrow-Next
TRS	Total Return Swap
USD	United State Dollar
VaR	Value-at-Risk
VBA	Visual Basic
WB	World Bank
YTM	Yield To Maturity
ZAR	South African Rand
z-c	zero coupon

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Financial Instruments

1.1 Introduction

In the previous book, we studied derivatives in the equity markets and in this book, we will study the available instruments in the interest rate markets. First, we will shortly group the various instruments.

In order to group the wide variety of instruments that exist adequately, it is necessary to break the interest rate asset classes into two subdivisions: *long-term* and *short-term* debts. In addition, it is necessary to divide the derivatives into two groups: *standard derivatives* and *over-the-counter* (OTC) *derivatives.*

- Standard derivatives are traded on exchanges. In such trades, a clearing house act as a counterparty to both buyers and sellers. These trades have a daily settlement[1] to protect the clearing house for losses, if one of the counterparties cannot fulfil its obligations. The clearing house guarantees the delivery of payments or underlying securities to its counterparties.
- OTC derivatives are typically traded over telephone or via a broker firm. They are known as OTC instruments because each trade is an individual contract between the two counterparties making the trade. These contracts are *privately negotiated* which means that they are not negotiable, for example, if I lend you some money, I cannot trade that loan contract to someone else without your prior consent.

[1] Some exchanges use monthly settlement, for example, Nasdaq-OMX in Stockholm.

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Table 1.1 Instrument types and asset classes

Instrument-type asset class	Cash	Standard derivatives	OTC derivatives
Interest Rate (Long Term)	Bond, note Floating rate note	Bond futures Options Bond futures	Swaps, swaptions, caps & floors, IRG, Cross Currency swaps, Exotics
Interest Rate (Short Term)	Deposit/Loan, Bill, CD (Certificate of Deposit), CP (Commercial Paper)	Interest rate futures	Forward Rate Agreement FX-swap, Euro Dollar futures
Equity	Stock (Index)	Equity Options Equity futures	Equity Options Exotics
Foreign Exchange	Spot	FX futures	Options FX forwards

- The International Swaps and Derivatives Association (ISDA) provides standard contracts to facilitate the trading of OTC derivatives.
- Many clearinghouses also clear OTC instruments. In this case they are said to use central clearing. By using central clearing the counterparty risk can be minimized. Also the Capital requirements for buyers and sellers will be minimized by using central clearing

Further subdivisions of the categories give rise to the matrix as shown in Table 1.1.

1.1.1 Money

Money, in wholesale banking, exists only as an electronic entity in the banking systems. The reason is that paper money does not earn interest and is therefore not money in a financial view. Therefore, we consider paper money as an interest free loan to the government. An analogy is the old type of share certificates that was physical delivered between the counterparties who have made a deal. Nowadays, share certificates are no longer used, instead all ownership is registered electronically.

Also, dollars only exist in the US banking system, pound sterling only in the British banking system and Euro in European banks.

Every bank that accepts US dollar has a Nostro account in its correspondent bank in the US. Similar accounts exist in all currencies in banks in all countries. If for example Sanwa in London transfers 1 USD to Barclays in London, Sanwa instructs its correspondent bank in US to transfer the 1 USD to Barclays. The money therefore never leaves the US.

It is important to notice that payments can only be made or received when the banking system is up. Therefore, we have to consider when the banking holidays for all countries exist, because then, no money transactions can be made in that specific country.

1.1.2 Valuation of Interest Rate Instruments

We will start to study interest rate instruments and how to value them. The following instruments are examples of cash-flow instruments:

- Bonds, bills and notes
- Floating Rate Notes (FRN)
- Swaps, Currency swaps and FX swaps
- Swaptions
- Caps, floors, collars and Interest Rate Guarantees
- Forward Rate Agreements (FRA)
- Convertibles
- Deposits and Certificates of Deposits (CD)
- Repos and reverses
- Credit Default Swaps/Indices (CDS, CDI, CDX etc.).

Many of these instruments are treated only as cash-flow sequences. Some of them are treated as derivatives. That is, no assumption is made on the pattern of how the cash flows looks like in the valuation process. In this way, the description of how to value a single cash flow can be generalized for all cash-flow instruments.

The advantage of such method is its generality. It can be applied to any kind of cash-flow pattern, whether it is amortized, has non-consecutive interest rate periods or broken dates.

There are a number of different cash-flow types as well:

- Fixed amount
- Fixed rates
- Floating rates
- Caplets
- Floorlets
- Total return
- Credit default
- Return
- Redemption amount
- Call fixed rates
- Call float rates
- Zero-coupon fixed rates.

The different cash-flow types are described in terms of various parameters as shown in Table 1.2.

1.1.2.1 Parameters

Common parameters for all cash-flow types are the *Pay Date* – the calendar date when the cash flow is paid – and the *Currency* of the cash flow. All cash flows are discounted using a *zero-coupon curve* from the payout date to the valuation date.
The simplest cash-flow type is a single fixed payment, *Fixed Amount*. All other cash flows are related to interest rates payments in some way. They have the common attributes:

- day count – the day-count convention used for a certain period
- start day – the date on which the interest rate period starts
- end day – the date on which the interest rate period ends

The simplest interest payment is the fixed coupon rate, using the attribute, *Fixed Rate* – the fixed interest rate that applies for a specific period.

Table 1.2 Parameters for different cash flows

Parameter	Zero-coupon fixed	Return	Redemption amount	Fixed Amount	Fixed Rate	Float Rate	Caplet/ floorlet	Total Return	Credit Default	Call fixed rate	Call float rate
Pay Date	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Currency	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Fix Amount				Y							
Start Day	Y	Y			Y	Y	Y	Y	Y	Y	Y
End Day	Y	Y			Y	Y	Y	Y	Y	Y	Y
Day count	Y				Y	Y	Y			Y	Y
Fix Rate					Y	Y	Y			Y	Y
Float Rate						Y	Y				Y
Reset Day	Y	Y				Y	Y	Y			Y
Spread						Y	Y				Y
Strike							Y				

The different pay types are:

- Spot — An instant pay order (to pay in 2 days, the spot days)
- Future — "Mark to Market", daily
- Forward — Pay on expiration date
- IMM — On IMM days (International Monetary Market days that is the third Wednesday in March, June, September and December.
- Forward/Periodically — Make payments on certain days, for example, the 3rd Friday on each month

There are two delivery or exercise types for derivatives:

- Physical delivery — Typically a stock (equity) option.
- Cash settlement — Typically, an option with an index as underlying.

There are three types of option exercise:

- European — Exercise only at expiration date
- American — Exercise any time
- Bermudan — Exercise in pre-defined periods or days

There are two types of option underlying:

- The underlying asset itself
- A future or a forward (on the underlying asset)

We can arrange the types as in Table 1.3.

Table 1.3 Pay types, deliveries and underlying for different instruments

Instrument	Pay type	Delivery Eur./Am.	Underlying
Stocks	Spot		
Bonds	Spot		
Index forwards	Forward	Cash	
Index futures	Future	Cash	
Bond futures	Future	Physical	
Commodity	Future	Future	Physical
Stock options	Spot	Physical/American	Stock
Index options 1	Spot	Cash/European	Index
Index options 2	Future	Physical/American	Index futures
Bond options	Future	Physical/American	Bond futures
OTC derivatives	Spot/forward	European	...

1.1.2.2 Future Value and Present Value

When we value different financial instruments, we use different expressions for their rates of return. If we calculate the rate of return of an equity to find the payoff, we often use a simple period rate r over the holding period. This rate is the percentage return on annual basis of the invested amount P. To calculate the present value of this amount we use

$$F = P \cdot (1 + r)$$

where F is the value at the end of the period. It is also common to annualize the rate using some convention for counting the length of the holding period, that is, the number of days, as a fraction of a year. In the money market, we usually use the following measure for the yield

$$F = P \cdot \left(1 + r \cdot \frac{d}{360}\right)$$

where d is the number of days to maturity. Since no compounding was used above the rate is referred as the simple rate. If we use annual compounding with the same number of days, we can express this as a fraction of a 360-day year. We then use the compounded annual rate r_c

$$F = P \cdot (1 + r_c)^{\frac{d}{360}}$$

For money market instruments, such as treasury bills and CD, which have fixed dates of expiry, the quoting convention relating market prices to rates typically does not use compounding. Their values upon expiry equals their nominal amounts so we can solve for their current price

$$P = \frac{N}{1 + r \cdot \frac{d}{360}}$$

where N is the nominal amount, paid on expiry and r the simple annualized rate on a yearly basis. The simple rate r can then be expressed as

$$r = \frac{N - P}{P} \frac{360}{d}$$

1.1.3 Zero Coupon Pricing

The concept behind zero-coupon pricing is the evaluation of all individual cash flows as if they were zero-coupon bonds. The evaluation is made using a yield curve or, alternatively, a discount function, which accurately describes current market conditions.

The pricing of liquid, standardized instruments are quite simple – the current market price is used. The zero-coupon pricing methodology becomes important when pricing OTC instruments, for which no market prices are available. It is also needed for pricing standardized instruments, which do not have reliable market prices. In this case, zero-coupon pricing will be used to price these instruments consistently alongside the liquid instruments. This is a kind of relative pricing where user preferences only need to be taken account of to a small extent. Many risk management techniques also require the use of a yield curve to aggregate correctly the risk over several different instrument types.

1.1.3.1 The Discount Function

The discount function, $p(t_0, t)$, describes the present value at time t_0 of a unit cash flow at time t. This is a fundamental function that can be given, for each time in the future, as individual components, the discount factors. These factors are non-random and should be equal for all banks due to arbitrage conditions.[2]

In most cases, t_0 is the current time (equal zero) and is therefore dropped for notational convenience. The remaining variable $t(t - t_0)$ then refers to the time between $t_0(= 0)$ and t. The discount function is, as we will see, used as the base for all other interest rates. For any future date t this function also represent the value of a zero-coupon bond (also called a pure discount bond) at time $t_0(= 0)$ with maturity t. At maturity, a zero-coupon bond pays one cash unit (in USD, GBP, EUR SEK etc.). So therefore $p(t, t) = 1$. A discount function with rate $r = 2.0\%$ is shown in Fig. 1.1.

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[2] Since the financial crisis in 2008, this is not really true, since some currencies are more risky than others. Therefore, we have to add, a so-called cross currency basis spread to the discount function. This basis spread is set against the most liquid currency in a trade. Only USD will have a zero basis spread. We will discuss that later. But now we think about the discount function as generic.

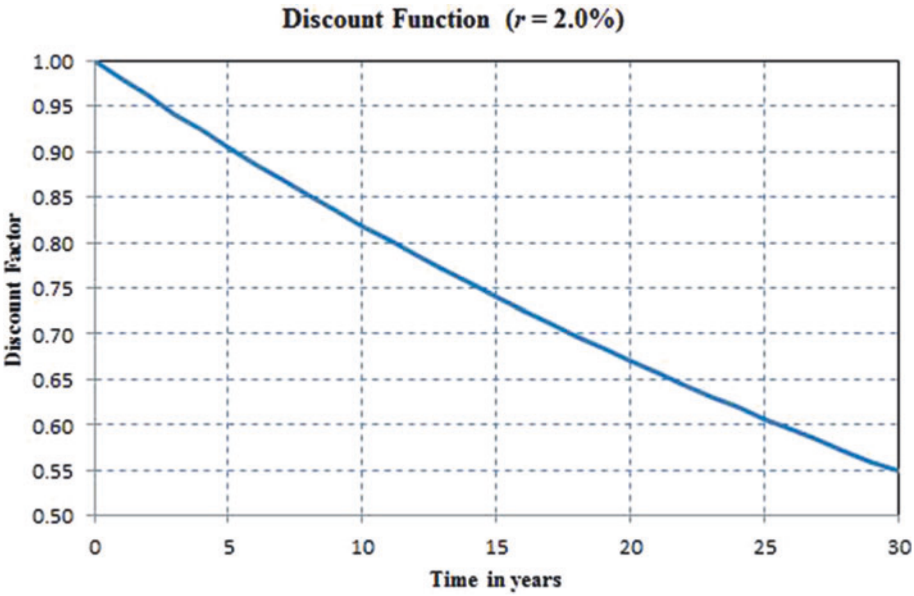

Fig. 1.1 The discount function for a constant interest rate at 2.0%

At $t = 0$, the discount function always has the value 1 ($p(0, 0) = 1$). One unit of cash today must have the value of one unit by definition. The discount function is monotonically decreasing, which corresponds to the assumption that interest rates are always positive. It never reaches zero since all cash flows, no matter how far in the future they are paid, should always be worth something.

The discount function has a mathematical relationship to the spot yield curve, although the "yield curve" is not a well-defined concept. The relationship between the discount function and the annually compounded yields of matching maturity, using a day-count convention that reflects the actual time between time t_0 and t measured in years, can be written as

$$p(t) \equiv p(0, t) = \frac{1}{(1 + r_1(t))^t}$$

This formula can be inverted to give

$$r_1(t) = P(t)^{-1/t} - 1.$$

Other used yields have a mathematical relationship to the discount function.

1.1.4 Day-Count Conventions

When using the discount function to express yield or interest rates, it is very important to known and consider the day-count convention used for each instrument and each market. The day-count convention is a user-defined, instrument-specific parameter and will typically have a substantial impact on the valuation of particular instruments.

1.1.4.1 Date Arithmetic

Dates are usually integers starting from 1900-01-01. Following the Excel convention, May 8, 2006 is day 38845, and May 9, 2006 as day 38846, etc. We therefore use an *Add-function* between different dates

$$t_{th} = \text{Add}\left(t, n, unit, EOMFlag, \ldots\right)$$

which adds n units (days, months, years or business days) to date t, where n can be positive, zero or negative.

We have the following End-Of-Months rules (*EOMFlag*)

1. If we add months or years and t_{th} ends up beyond the end of a particular month, we replace this date with the last day of month. Example:

   May 31 + 1 month = June 30
   December 31 + 2 month = February 28 (or 29 for a leap year)

2. If date t is the last day of a month, then

   If *EOMFlag* is *true*: adding months or years always gives the last day of the month:

   February 29 + 1 month = March 31
   April 30 – 1 month = March 31

   If *EOMFlag* is *false*: adding months or years always gives the same day of the month, provided that it exists:

   February 29 + 1 month = March 29
   April 30 – 1 month = March 31
   May 31 – 1 month = April 30

3. The *EOMFlag* is irrelevant if t is not the last day in a month.

We also must have a general add functionality

$$t_{act} = \text{Add}\left(t, n, \boldsymbol{unit}, EOMFlag, BDR, Hol1, Hol2, Hol3\ldots\right)$$

where *BDR* is the Business-Day-Rule. We first compute

$$t_{th} = \text{Add}\left(t, n, \boldsymbol{unit}, EOMFlag, \ldots\right)$$

If t_{th} is not a business day, we apply the *BDR* rule to resolve the date. t_{th} is a bad day if it is a bad day in any holiday *Holx*.

We have the five business day rules:

1. *none*: return t_{th} (banks can go into default also on non-banking days!)
2. *following* (*succeeding*): t_{act} is the first valid business day on or after t_{th}.
3. *proceeding*: t_{act} is the first valid business day on or before t_{th}.
4. *modified following*: t_{act} is the first valid business day on or after t_{th} if it is the same calendar month as t_{th}. Otherwise t_{act} is the first valid business day before t_{th}
5. *modified proceeding*: t_{act} is the first valid business day on or before t_{th} if it is the same calendar month as t_{th}. Otherwise t_{act} is the first valid business day after t_{th}

The *modified following* is the standard rule for payments. Typically, dates are generated backwards from the theoretical end date. Otherwise, it is difficult to do a rewind of a trade with a number of cash flows with another customer.

First, we get the theoretical end date. For an M month leg, starting at t_0 we have

$$t_n^{th} = \text{Add}\,(t_0,\ M,\ month,\ no,\ none,\ ccy1,\ ccy2\ldots)$$

For a leg with m months per period, we have

$$t_j^{th} = \text{Add}\,(t_n, -m\,(n-j)\,,\ month,\ no,\ none,\ ccy1,\ ccy2\ldots)$$
$$t_j^{act} = \text{Add}\,(t_n, -m\,(n-j)\,,month,\ no,\ modfol,\ ccy1,\ ccy2\ldots)$$

If an odd period is needed, the default is *short first* period, other possibilities are *long first, short last* and *long last*. The last two requires that we generate the dates from t_0, which we do not want. The holiday parameters *ccy1* and *ccy2* have to be used for the different currencies.

When dealing with interest rate payments, accrued from t_{j-1} to t_j and paid at t_j, we use the following rule:

1. If the swap leg is *adjusted* (which is the default situation), interest accrued from t_{j-1}^{act} to t_j^{act}.
2. If the swap leg is *unadjusted*, interest accrues from t_{j-1}^{th} to t_j^{th}.
3. Interest payments are $\alpha_j rN$ paid at t_j^{act} for $j = 1, 2, \ldots, n$ where N is the notional, r the interest rate and α_j the day-count fraction.

$$\alpha_j = DayCountFrac(t_{j-1}, t_j, basis)$$

Day-count basis are rules assigning official fractions of a year to any two dates. Some alternative day-count conventions are (there exist about 80 more day-count bases than those listed below):

- 30/360 corporate bonds, Eurobonds etc.
- 30E/360 money market Switzerland
- Act/360 US T-bills US, Euro and Switzerland money, etc.
- Act/365 US Treasury bonds/notes, UK gilts, German bunds etc.
- NL/365 Actual/365 with no leap year
- Act/Act New Euro bonds, LIFFE UK bond/bund futures etc.

The meaning of the abbreviations used in the naming of the above conventions is as follows:

- **Act**: Actual number of calendar days.
- **NL**: Actual number of calendar days, with no leap year.
  - Exception: If the year is a leap year then February is considered to have 28 days (instead of 29).
- **30**: Each month is considered to have 30 days.
  - Exception 1: If the later date is the last day of February, that month is considered to have its actual number of days.
  - Exception 2: When the later date of the period is the 31st and the first day is **not** the 30th or the 31st, the month that includes the later date is considered to have its actual number of days.
- **30E**: Each month has 30 days.
  - Exception: If the later date is the last day of the month of February, that month is considered to have its actual number of days.

Credit cards always use Act/360, which gives them five extra days of interest per year.

Interest rates are typically expressed for annual periods. The time period measured in years between two dates, t, is described as the fraction of the number of days between two dates, t_d, and the number of days in a year, t_y

$$t = \frac{t_d}{t_y}$$

t_d and t_y are determined according to the specified day-count convention.

Example 1.1

What is the time period between 11 January and 31 March?

30/360:	Number of days in January = 19 + 30 in February + 31 in March = 80: $t = 80/360$
30E/360:	Number of days in January = 19 + 30 in February + 30 in March = 79: $t = 79/360$
NL/365:	Number of days in January = 20 + 28 in February + 31 in March = 79: $t = 79/365$

Example 1.2

If we let $t = (d_1, m_1, y_1)$ (date, month and year) and $T = (d_2, m_2, y_2)$, then the 30/360 convention can be calculated as

$$\frac{\min(d_2, 30) + (30 - d_1)^+}{360} + \frac{(m_2 - m_1 - 1)^+}{12} + y_2 - y_1$$

where $(x)^+ = \max(x, 0)$. The time between t = January 4, 2005 and T = July 4 2007 is then

$$\frac{4 + (30 - 4)^+}{360} + \frac{(7 - 1 - 1)^+}{12} + 2007 - 2005 = 2.5.$$

1.1.4.2 International Monetary Market (IMM) Days

Many Fixed Income instruments have start days and maturities on International Monetary Market (IMM) days. IMM days are the third Wednesdays in Mars, June, September and December.

1.1.5 Quote Types

When pricing interest rate instruments, a number of different quote types are used. Quotes are the market prices traders do observe on screen from their trading system or from other price sources. We will now define some of them.

1.1.5.1 Per Cent of Nominal Amount

Quote is taken as a per cent of the nominal amount (also called the face value). This is used for bonds and can be given with or without accrued interests.

1.1.5.2 Clean Price

Quote is taken as a per cent of the nominal amount without the accrued interest. This is the normal quotes of bonds and other similar instruments.[3]

1.1.5.3 Price

Quote is taken as a per cent of the nominal amount included the accrued interest. This is also called the dirty price. Therefore the (dirty) price equals the (clean) price plus the accrued interest rate since the last coupon payment for a bond.

1.1.5.4 Coupon Rate

Quote given as the coupon rate. This can be used when comparing different bonds with similar maturities. With known coupon rate, the price can be calculated by discounting of the cash flows, included the nominal payout at redemption.

[3] Swedish bonds are quoted in yield (to maturity).

1.1.5.5 Yield/Yield-to-Maturity

Quote is given as a flat yield used to discount all future cash flows. This is how Swedish bills and bonds are quoted. Yield-to-maturity has a one-to-one relationship with the dirty price. It's based on that all the coupons can be reinvested at the same (flat) yield.

1.1.5.6 Volatility

This quote type is available for Options/Warrants, swaptions and caps/floors. The quote of an instrument with quote type "Volatility" is interpreted in terms of the implied volatility structure used for the instrument. Normally this is Black (lognormal) or Normal volatility. The reason for different volatilities like Black volatility and Normal volatility is that whatever the model used to value an instrument the price must be the same.

2
Interest Rate

2.1 Introduction to Interest Rates

As we will see, there exists many different definitions of interest rates in the markets. A repo trader talks about the simple rate, an option trader of the continuous compounding rate and a bond trader of yield-to-maturity (YTM). We will briefly 0name some of the rates and give a short description. Some of these rates will be discussed in detail in later sections.

2.1.1 Benchmark Rate, Base Rate (UK), Prime Rate (US)

This is the lowest interest rate an investor is willing to take to make an investment in a risk-less security. These rates are given as a yield curve of instruments with different maturities. Usually this yield curve is built from government securities or Over-Night Index Swaps (OIS) and is used to compare against other (risky) interest rates.

2.1.2 Deposit Rate

A typical deposit contract is a standardized agreement of a loan between two banks. It is a credit for the party who placed it, and it may be taken back, transferred to another party, or used for a purchase. Deposits are usually banks main source of funding. The rate for such loan is called deposit rate.

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_2

2.1.3 Discount Rate, Capitalization Rate

This is the rate used to discount a given cash flow in the future to a present value (*PV*). This rate reflects the time-value of money. This rate is not uniquely defined. For a certain deal it depends on how this deal is financed and your counterparty. If you have a collateral agreement with the counterparty you should discount with the collateral rate specified in the collateral agreement. A typical rate would be an OIS rate. Without a collateral agreement a typical rate would be the funding rate, like the inter-bank rate.

2.1.4 Simple Rate

The simple rate is the yield, expressed as a percentage per annum of an invested amount. If we receive all interest rates at the end of an investment period, we have the following relationship to the annual compounding rate

$$(1 + r_{annual})^t = (1 + r_{simple} \cdot t)$$

The relation to the discount function is then given by

$$p(t) = \frac{1}{1 + r_{simple}(t) \cdot t}$$

The difference between discount rate and simple rate is that the discount rate is applied to the nominal amount, while the simple rate is applied to the invested amount of a discount instrument. If we, for example, pay \$900 for a \$1,000 nominal amount maturing in 1 year (day-count Act/Act), the simple rate would be:

$$\left(\frac{1000}{900} - 1\right) \cdot 100 = 11.11\%$$

and the discount rate would be

$$\left(1 - \frac{900}{1000}\right) \cdot 100 = 10.00\%$$

2.1.5 Effective (Annual) Rate

The effective rate is the yield expressed as a percentage of the invested amount based on a year including the effect of compounding. If we receive interest, we have to ask us how often we get payments. If we let f be the period, for example, the number of annual payments we get

$$(1 + r_{annual})^t = \left(1 + \frac{r_f}{f}\right)^{f \cdot t}$$
$$(1 + r_{annual})^t = \left(1 + \frac{r_{quarterly}}{4}\right)^{4 \cdot t}$$

In continuous compounding this is expressed as

$$\left(1 + \frac{r_f}{f}\right)^{f \cdot t}, \quad f \to \infty \Rightarrow (1 + r_{annual})^t = e^{r_c \cdot t}$$

The annual rate is related to the discount function as

$$p\,(t) = \frac{1}{(1 + r_{annual}\,(t))^t}$$

The semi-annual rate is related to the discount function as

$$p\,(t) = \frac{1}{\left[1 + \frac{r_2(t)}{2}\right]^{2t}}$$

and the n-annual rate as

$$p\,(t) = \frac{1}{\left[1 + \frac{r_n(t)}{n}\right]^{nt}}$$

The continuous compounding rate is given as

$$p\,(t) = e^{-r_c(t) \cdot t}.$$

Each of these formulae can be inverted in the same way as for the annually compounded interest rate. The formulas also define the implicit relationship between the different interest rate types. Since there is a mathematical relationship between the concepts *discount function* and *yield curve*, both of these will be used in this text when we describe the information necessary to perform zero-coupon pricing.

When valuing options and other derivatives, like in Black-Scholes model, we use the continuous compounding. If r_c is the continuous compounded interest rate and r_m the same interest rate paid m times every year we have the relationship

$$r_c = m \cdot \ln\left(1 + \frac{r_m}{m}\right)$$
$$r_m = m \cdot \left\{\exp\left(\frac{r_c}{m}\right) - 1\right\}$$

In the interest rate markets it is very important how to discount cash flows, even for a short period as one day. So if you are given an overnight rate you have to know how this rate is quoted. Is it a simple rate, an effective annual rate or a continuous compounding rate? It is also important to know what day-count conversion that is being used. A one day discount rate can be expressed in many ways, such as

$$\frac{1}{(1+r)^{1/360}} \neq \frac{1}{(1+r)^{1/365}} \neq \frac{1}{1 + r \cdot \frac{1}{360}} \neq \frac{1}{1 + r \cdot \frac{1}{365}} \neq e^{-r/360} \neq e^{-r/365}$$

2.1.6 The Repo Rate

This is the interest rate for a repurchase agreement, that is, the rate you have to pay by selling a security and at the same time commit to buying it back after a short period. The period is usually one of

- O/N (Over-Night)
- T/N (Tomorrow-Next)
- C/W (Corporate-Week)
- S/N (Spot-Next)

The O/N rate is the rate for the period between now, sometime today until the closing time on the next business day. On a Friday, the O/N rate period will be 3 days (if the following Monday is a business day). The T/N will start on the next business day and end on the next following business day. All other rates usually begin two business days from today (if we use a spot lag of 2) and last for a given period time. We say that we are using two spot days.

A government repo rate is the rate at which the government buys their own bills, notes or bonds. Sometime this rate is used to calculate the carry cost for instruments with underlyings.

2.1.7 Interbank Rate

The interbank rate is the average rate at which XIBOR[1] rated banks can borrow from each other. We will discuss this in detail in a later section. XIBOR is a general name convention for the different interbank rates. LIBOR is the London Interbank Offered Rate, STIBOR is the Stockholm Interbank Offered Rate and EURIBOR is the Euro Interbank Offered Rate. Before the financial crises in 2008–2009 the interbank rate was considered as the risk-free interest rate. But at that time XIBOR rated banks such as the Lehman Brothers made default. After the crises more and more banks required collateral agreements for interbank loans. We will discuss this in a later section.

2.1.8 Coupon Rate

The coupon rate is given as the percentage of the nominal amount that is paid to the holder of a bond. These coupons are received with a certain frequency, usually one, two or four times per year. The coupons are paid by the issuer.

2.1.9 Zero Coupon Rate

The zero-coupon rate, or just zero rate, is the YTM on a zero-coupon bond, that is, a bond that pays no coupon. This rate can be bootstrapped from coupon bonds. The zero-coupon rates are often used for the discounting o future payments. Also risk managers use these rates to calculate the risk by making shifts of the curve.

2.1.10 Real Rate

The real rate is the interest rate adjusted for inflation. This rate can be found by bootstrapping Inflation linked bonds, sometimes referred to as Index linked bonds where the index is the Consumer Price Index, CPI.

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[1] XIBOR are used in general for Inter Bank Offer Rates in different currencies where X = L for London, ST for Stockholm EUR for EURO etc.

2.1.11 Nominal Rate

The interest rate including inflation. This means that the nominal rate is equal to the real rate plus inflation.

2.1.12 Yield – Yield to Maturity (YTM)

There are such a variety of fixed-income products, with different coupon structures, amortization, fixed and/or floating rates, that it is necessary to be able to consistently compare different products. One way to do this is through measures of how much each contract earns. There are several measures of this all coming under the same name, the yield.

2.1.13 Current Yield

The current yield have many other names such as interest yield, income yield, flat yield, market yield, mark to market yield or running yield: This yield is a financial term used in reference to bonds and other fixed-interest securities such as swaps. It is the ratio of the annual interest payment and the bond's current clean price of the bond.

The current yield therefor refers to the yield of the bond at the current moment. It does not reflect the total return over the life of the bond. In particular, it takes no account of reinvestment risk (the uncertainty about the rate at which future cashflows can be reinvested) or the fact that bonds usually mature at par value, which can be an important component of a bond's return.

For example, consider the 10-year bond that pays 2 cents every 6 months and $1 at maturity. This bond has a total income per year of 4 cents. Suppose that the quoted market price of this bond is 88 cents. The current yield is simply

$$0.04/0.88 = 4.5\%.$$

2.1.14 Par Rate and Par Yield

The par rate r_{par} is the (fixed) rate payments with the same value as a number of opposite floating rate payments so that their total values sum up to 0 as in Fig. 2.1. The typical instrument here is a plain vanilla interest rate swap.

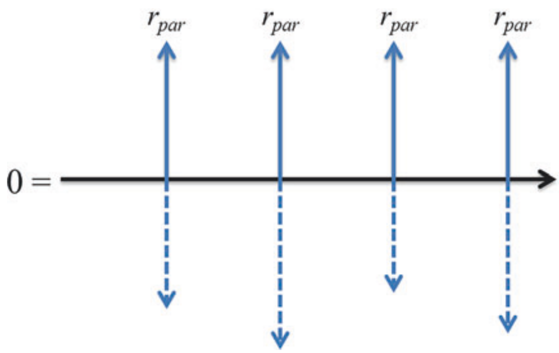

Fig. 2.1 The par rate r_{par} is the constant rate that equalizes the value of the floating leg (dotted arrows) to the fixed leg over the lifetime of the swap

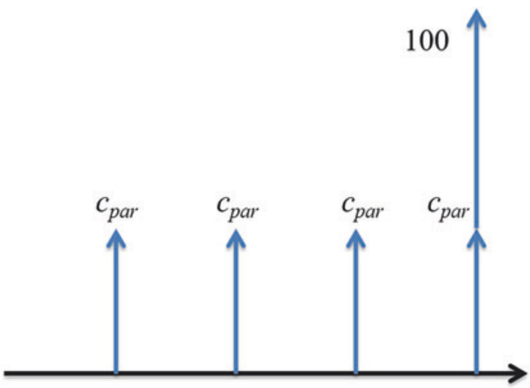

Fig. 2.2 The par yield is the yield that equals the coupon rate c_{par} so that the price of the bond is equal to its face value, nominal amount, here set to 100

The par yield is the fixed coupons of an instrument so that the total (discounted) value, included the nominal value equalize the nominal value itself as in Fig. 2.2

The par rate for a swap is calculated as

$$\sum_i \left(p(0,t_i) \cdot r_{par}\right) = \sum_i \left(p(0,t_i) \cdot r_{forward}^{t_i - t_{i-1}}\right) \Rightarrow r_{par} = \frac{\sum_i \left(p(0,t_i) \cdot r_{forward}^{t_i - t_{i-1}}\right)}{\sum_i p(0,t_i)}$$

Where $p(0, t_i)$ is the discount factor at time t_i, that is, the price of a zero-coupon bond with maturity at t_i. The rate $r_{forward}^{t_i - t_{i-1}}$ represent the floating rate, given by the forward rate (see next) between time t_i and (with maturity) t_{i-1}.

Similarly, the par rate of a bond is calculated as

$$100 = \sum_{i=1}^{n} \left(p(0,t_i) \cdot c_{par}\right) + \left(p(0,t_n) \cdot 100\right) \quad \Rightarrow \quad c_{par} = \frac{100 \cdot (1 - p(0,t_n))}{\sum_{i=1}^{n} p(0,t_i)}$$

2.1.15 Prime Rate

The prime rate or prime lending rate is a term applied in many countries to reference an interest rate used by banks. The term originally indicated the interest rate at which banks lent to favoured customers, that is, those with good credit, but this is no longer always the case.

2.1.16 Risk Free Rate

This is defined as the rate you can earn by taking a risk-less position. Many time, this rate is based on treasury bonds with the same time to maturity as the period used (see benchmark rate). If any rate really is risk-free can be discussed, and is discussed a lot in the literature. Some uses the swap rate, or the OIS rate as risk free and other says that also government zero-rate is not risk-free, since also a governments too can default.

2.1.17 Spot Rate

The spot rate or short rate is defined as the theoretical profit given by a zero-coupon bond. We use this rate when we calculate the amount we will get at time t_1 (in the future) if we invest X today (i.e. at time t_0)

$$X_{t_1} = \left(1 + r_{spot}\right)^{t_1} X_{t_0}$$
$$PV\left(X_{t_1}\right) = \frac{1}{\left(1 + r_{spot}\right)^{t_1}} X_{t_1}$$

where $PV(X_t)$ is the present value of X_t. The relation between the spot rate and the discount function is

$$p(t) = \frac{1}{\left(1 + r_{spot}(t)\right)^t}$$

The spot rate is calculated by bootstrapping, by fitting the yield curve. We also see that this rate is the same as the annual effective rate.

2.1.18 Forward Rate

From a yield curve describing the interest rates that apply between the current date and the set of future dates ordered by maturity, it is possible to calculate an implied forward rate curve, that is, the rate that "should" apply between two future dates. The formula for implied forward rates is based on an arbitrage argument, where the rate for a specific nominal amount between two future dates can be locked in by borrowing and lending at the current rates to the future dates.

A projection of the future interest rate, from one time to another, calculated from the spot rate (as shown earlier) or a yield curve is given by

$$(1+r_{t_1}^{spot})^{t_1} \cdot (1+r_{t_2-t_1}^{forward})^{t_2-t_1} = (1+r_{t_2}^{spot})^{t_2} \Rightarrow r_{t_2-t_1}^{forward} = \left(\frac{(1+r_{t_2}^{spot})^{t_2}}{(1+r_{t_1}^{spot})^{t_1}}\right)^{\frac{1}{t_2-t_1}} - 1$$

An easy way to represent the forward rate is via the discount function. We then have

$$p(0,t_1) \cdot p(t_1,t_2) = p(0,t_2) \quad \Rightarrow \quad p(t_1,t_2) = \frac{p(0,t_2)}{p(0,t_1)} \equiv \frac{p(t_2)}{p(t_1)}$$

In terms of continuous compounding we then have

$$e^{-r(t_1)\cdot t_1} \cdot e^{-f(t_2,t_1)\cdot(t_2-t_1)} = e^{-r(t_2)\cdot t_2} \quad \Rightarrow \quad f(t_2,t_1)\cdot(t_2-t_1) = r(t_2)\cdot t_2 - r(t_1)\cdot t_1$$

or

$$f_{t_2,t_1} = \frac{r_2 \cdot t_2 - r_1 \cdot t_1}{t_2 - t_1} = \frac{r_2 \cdot t_2 - r_2 \cdot t_1 + r_2 \cdot t_1 - r_1 \cdot t_1}{t_2 - t_1} = r_2 + (r_2 - r_1)\frac{t_1}{t_2 - t_1}$$

where $p(t, T)$ represent a pure discount bond or zero-coupon bonds at time t with maturity T. We have the boundary condition $p(T, T) = 1$, that is, the zero-coupon bond pays 1 cash unit (CU) at maturity.

2.1.19 Swap Rate

The fixed rate used to price a swap to zero value. A swap is a contract where the buyer and the seller exchange their cash flows, typical floating interest rate cash flows against fixed rate cash flows. Sometimes such a rate is used as the risk-free interest rate. We will discuss swaps in a later section.

2.1.20 Term Structure of Interest Rates

The term structure of interest rates is a set of market interest rates ordered by maturity, that is, the rates on for example, treasury bonds with different times to maturity. An instant term structure is shown in Fig. 2.3. This yield curve is used to discount cash flows to a present value.

2.1.21 Treasury Rate

Treasury rate is the rate you get if you lend money to a government in their own currency. Sometimes, this is used as the risk-free interest rate.

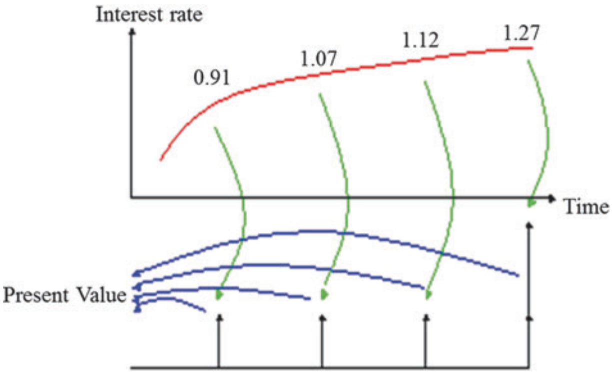

Fig. 2.3 Here we use a yield curve to discount a number of cash-flows PV

2.1.22 Accrued Interest

Accrued interest is calculated for the holder of a coupon bond every day as part of the market convention for the sharing of the annual or semi-annual coupon payment when the bond is bought and sold. For example, from a bond since the last coupon payment.

2.1.23 Dividend Rate

Dividend rate is the fixed or floating rate paid by a preferred stock in Great Britain.

2.1.24 Yield to Maturity (YTM)

The rate an investor will earn if he keeps an interest paying security, typically a bond, until maturity. This only holds true if the received cash flows during the lifetime of the security can be reinvested at the same interest rate. The YTM depends on the coupon rate, time to maturity and the market price of the instrument. Some instruments are quoted in yield to maturity since there is a one-to-one relationship between YTM and the price. This means that you can express the price in YTM.

2.1.25 Credit Rate

The credit rate is an interest rate depending on the ranking of a company. This is in many cases defined as a spread on some benchmark rate.

2.1.26 Hazard Rate

The Hazard rate is the rate based on the risk that the lender might default. If we model the probability that the counterparty will default and therefore cannot pay all of the obligations we use

$$r_{t_1}^{discount} = \frac{1}{\left(1 + r_{t_1}^{spot}\right)^{t_1}} \cdot [(1 - P(t_1)) + R \cdot P(t_1)]$$

where $P(t)$ is the probability that the counterparty will default between the time 0 (today) and the time t, and R the amount we will receive if default occur. R is given in per cent and is called recovery rate.

2.1.27 Rates and Discounting Summary

We summarize the most important discounting methods as follows:

Simple annualized rate: $p(t) = \frac{1}{1+r_{simple}(t)\cdot t}$

Annual compounding rate: $p(t) = \frac{1}{(1+r_{annual}(t))^t}$

Periodically compounding rate: $p(t) = \frac{1}{\left(1+\frac{r_f(t)}{f}\right)^{f\cdot t}}$

Continuous compounding rate: $p(t) = e^{-rt}$

To be able to compare two different yields or interest rates, they have to be in the same day-count basis and method. The "golden rule" for converting yields is: *The discount factor must be equal before and after the conversion.* This means, that we are able to solve the equation by setting the discount factors before and after the conversion equal, and then solve for the unknown yield. For example, to convert from a simple interest rate, Actual/360 basis to simple interest rate, Actual/365 basis, we can do it in the following way

$$1 + r_1 \cdot days/360 = 1 + r_2 \cdot days/365$$

After some simple calculations, we find

$$r_2 = r_1 \cdot 365/360$$

Therefore, for example simple interest rate 5% in actual/365 basis is equivalent to 5% · 365/360 = 5.069444% in actual/365 basis.

The following equations summarize the conversion formulas between periodically compounded and simple interest rates. For annually compounded rates, the f is 1 and the formulas take a simpler form. The formulas assume that year fractions for original and destination basis have been calculated.

From compounding yield basis to simple interest basis and vice versa

$$\left(1+\frac{r_f}{f}\right)^{t_1 f} = 1 + r_{simple} \cdot t_2$$

giving

$$r_f = f \cdot \left(1 + r_{simple} \cdot t_2\right)^{\frac{1}{t_1 \cdot f}} - 1 \qquad \Leftrightarrow \qquad r_{simple} = \frac{1}{t_2}\left(\left(1+\frac{r_f}{f}\right)^{t_1 \cdot f} - 1\right)$$

From compounding yield basis to another compounding yield basis we have

$$\left(1+\frac{r_{f_1}}{f_1}\right)^{t_1 \cdot f_1} = \left(1+\frac{r_{f_2}}{f_2}\right)^{t_2 \cdot f_2} \qquad \Leftrightarrow \qquad r_{f_1} = \left(\left(1+\frac{r_{f_2}}{f_2}\right)^{\frac{t_2 \cdot f_2}{t_1 \cdot f_1}} - 1\right) \cdot f_1$$

2.1.28 Black-Scholes Formula

In almost all literature in option theory, Black-Scholes formula (without dividends) for a call option is given by

$$C = S \cdot N(d_1) - e^{-rT} X \cdot N(d_2)$$

Here r is the risk free interest rate. We can rewrite the Black-Scholes equation as

$$C = e^{-r \cdot T}\left[S e^{r \cdot T} N(d_1) - X \cdot N(d_2)\right]$$

where we have moved the discount factor outside the bracket. The first term inside the bracket is recognized as the forward price of the underlying security, times the probability that the option will be at-the-money at maturity. In practical situations it is favourable to rewrite the equation as

$$C = e^{-r_{discount} \cdot T} \cdot \left[S \cdot e^{r_{repo} \cdot T} \cdot N(d_1) - X \cdot N(d_2)\right]$$

As we can see, we use two different interest rates. The discount rate $r_{discount}$ is used for discounting to a PV, and the repo rate r_{repo} as the risk-free rate in the valuation of the forward.

3

Market Interest Rates and Quotes

3.1 The Complexity of Interest Rates

In many, if not in all discussion about valuing financial instruments, especially interest rate derivatives, the risk-free interest rate is an important topic. The risk-free interest rate are used to discount projected or expected cash-flows to a present value. But, what rate should be used? A short answer should be that this depends on what instrument to value, the counterparty and the agreements made. A better answer might be that the rate should be chosen to reflect the funding cost of buying the instrument. In this section we will discuss how the market situations in the near future have changed the view about the risk-free interest rate.

Before 2007, the London Inter-Bank Offered Rate (LIBOR) rate was frequently used as the risk-free interest rate. Today, we know that this is not correct. To understand why, we have to go back to the definition.

3.1.1 The LIBOR Rates

On 1 February 2014 the administration of LIBOR was transferred from the British Banker's Association (BBA) to the Intercontinental Exchange (ICE), and BBA LIBOR is now known by the name ICE LIBOR. The need for a new administrator of LIBOR was highlighted in the Wheatley Review[1] due to the findings by various authorities in regard to the attempted manipulation of LIBOR.

[1] https://www.gov.uk/government/uploads/system/uploads/attachment_\penalty\z@data/file/191762/wheatley_review_libor_finalreport_280912.pdf

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_3

LIBOR reflects the average rate at which banks can obtain unsecured funding in the London inter-bank market for a particular currency and a particular time period. It is used globally as a benchmark to calculate payments made under all manner of finance documents – for example, derivatives, syndicated and bilateral loan agreements and floating rate notes.

The appointment of ICE as the new administrator will need to be reflected in the LIBOR definition in finance documents entered into after 1 February 2014. With regard to pre 1 February 2014 finance documents, they will typically define LIBOR by reference to BBA LIBOR. On the basis that ICE LIBOR retains substantially the same attributes as BBA LIBOR and the transfer of the administration function does not involve a fundamental change in the way in which the relevant data is collected and the calculation made, the widely held view in the market is that a reference to BBA LIBOR will operate to reference ICE LIBOR.

In 2015, the ICE Benchmark Administration (IBA) has a reference panel of 11–17 banks, see Table 3.1 for five different currencies,[2] which includes CHF (Swiss Franc), EUR (Euro), GBP (Pound Sterling),

Table 3.1 ICE Benchmark Administration panel banks

BANK/CCY	USD	GBP	EUR	CHF	JPY
Lloyds TSB Bank plc	O	O	O	O	O
Bank of Tokyo-Mitsubishi UFJ Ltd	O	O	O	O	O
Barclays Bank plc	O	O	O	O	O
Mizuho Bank, Ltd.		O	O		O
Citibank N.A. (London Branch)	O	O	O	O	
Cooperative Rabobank U.A.	O	O	O		
Credit Suisse AG (London Branch)	O		O	O	
Royal Bank of Canada	O	O	O		
HSBC Bank plc	O	O	O	O	O
Santander UK Plc		O	O		
Bank of America N.A (London Branch)	O				
BNP Paribas SA, London Branch		O			
Credit Agricole Corporate & Investment Bank	O	O			O
Deutsche Bank AG (London Branch)	O	O	O	O	O
JPMorganChase Bank, N.A. (London Branch)	O	O	O	O	O
Societe Generale (London Branch)	O	O	O	O	O
Sumitomo Mitsui Banking Corporation Europe limited	O				O
The Norinchukin_Bank	O				O
The Royal Bank of Scotland plc	O	O	O	O	O
UBS AG	O	O	O	O	O

[2] Before May 2013 there were 11 currencies. The following currencies have been removed; NZD, DKK, SEK, AUD and CAD. At the same time the tenors 2W, 4M, 5M, 7M, 8M, 9M, 10M and 11 M were removed for CHF, EUR, GBP, JPY and USD.

the JPY (Japanese Yen), and USD (US Dollar). It is a polled rate from the panel of banks.

The LIBOR rate is determined by every contributor bank, which are determined yearly by the IBA and regulated by the Financial Conduct Authority. Only banks that have a significant presence in the London market are considered to be placed on the ICE LIBOR panel. All of the panel banks are asked the following question: “At what rate could you borrow funds, were you to do so, by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 a.m.?” The banks are obligated to submit a rate at which they would borrow cash from another bank.

Once the banks submit their rates in response to the question, ICE calculates the LIBOR rate using a trimmed mean excluding both the highest and lowest quartiles of the submissions to exclude outliers, while the rest are averaged. The average rate is published to the market daily at approximately 11:45 a.m. Greenwich Mean Time.

ICE are using an ICE LIBOR HOLIDAY CALENDAR that can be found at https://www.theice.com/publicdocs/Fixing_Calendar_2016.pdf

Quoted LIBOR rates are given as in Table 3.2 and Table 3.3.

3.1.1.1 Calculation

All ICE LIBOR rates are quoted as an annualized interest rate. This is a market convention. For example, if an overnight Pound Sterling rate from a contributor bank is given as 2.00000%, this does not indicate that a contributing bank would expect to pay 2% interest on the value of an overnight loan. Instead, it means that it would expect to pay 2% divided by 365.

Every ICE LIBOR rate is calculated using a trimmed arithmetic mean. Once each submission is received, they are ranked in descending order and then the highest and lowest 25% of submissions are excluded. This trimming of the top and bottom quartiles allows for the exclusion of outliers from the final calculation. The number of rates for different numbers of contributors are shown in Table 3.4

3.1.2 The EURIBOR Rates

The Euro Interbank Offered Rate (EURIBOR) is a daily reference rate, published by the European Money Markets Institute (EMMI), based on

Table 3.2 Euro LIBOR quotes

EUR	08-15-2016	08-12-2016	08-11-2016	08-10-2016	08-9-2016
Euro LIBOR – overnight	–0.40000%	–0.39929%	–0.40000%	–0.40000%	–0.40000%
Euro LIBOR – 1 week	–0.38714%	–0.38714%	–0.38714%	–0.38714%	–0.38714%
Euro LIBOR – 2 weeks	–0.34129%	–	–	–	–
Euro LIBOR – 1 month	–0.37143%	–0.37071%	–0.37214%	–0.37214%	–0.37143%
Euro LIBOR – 2 months	–0.32143%	–0.33971%	–0.33971%	–0.33900%	–0.33829%
Euro LIBOR – 3 months	–	–0.31929%	–0.31929%	–0.31857%	–0.31857%
Euro LIBOR – 4 months	–	–	–	–	–
Euro LIBOR – 5 months	–	–	–	–	–
Euro LIBOR – 6 months	–0.20214%	–0.20219%	–0.19843%	–0.19829%	–0.19729%
Euro LIBOR – 7 months	–	–	–	–	–
Euro LIBOR – 8 months	–	–	–	–	–
Euro LIBOR – 9 months	–	–	–	–	–
Euro LIBOR – 10 months	–	–	–	–	–
Euro LIBOR – 11 months	–	–	–	–	–
Euro LIBOR – 12 months	–0.07271%	–0.07257%	–0.07257%	–0.07171%	–0.07143%

the averaged interest rates at which Eurozone banks offer to lend and borrow *unsecured* funds from each in the euro interbank market. EURIBOR was first published on December 30, 1998. Prior to 2015, the rate was published by the European Banking Federation and calculated by Tomson Reuters.

At present there are eight EURIBOR maturities – 1 week, 2 weeks, 1 month, 2 months, 3 months, 6 months, 9 months and 12 months (until October 2013 there were 15 maturities). The rates are used as a reference rate for euro-denominated forward rate agreements, short-term interest rate futures contracts and interest rate swaps. EURIBOR are used in the same way as LIBOR rates are commonly used for Sterling and US dollar-denominated instruments.

Table 3.3 USD LIBOR quotes

USD	08-15-2016	08-12-2016	08-11-2016	08-10-2016	08-09-2016
USD LIBOR – overnight	0.41889%	0.41910%	0.42020%	0.41970%	0.41880%
USD LIBOR – 1 week	0.44078%	0.44270%	0.44345%	0.44370%	0.44245%
USD LIBOR – 2 weeks	–	–	–	–	–
USD LIBOR – 1 month	0.50744%	0.50665%	0.50765%	0.51765%	0.51315%
USD LIBOR – 2 months	0.63206%	0.63255%	0.62880%	0.63280%	0.62955%
USD LIBOR – 3 months	0.80411%	0.81825%	0.81700%	0.81760%	0.81600%
USD LIBOR – 4 months	–	–	–	–	–
USD LIBOR – 5 months	–	–	–	–	–
USD LIBOR – 6months	1.19744%	1.20670%	1.20395%	1.20370%	1.19620%
USD LIBOR – 7 months	–	–	–	–	–
USD LIBOR – 8 months	–	–	–	–	–
USD LIBOR – 9 months	–	–	–	–	–
USDLIBOR – 10 months	–	–	–	–	–
USD LIBOR – 11 months	–	–	–	–	–
USD LIBOR – 12 months	1.50661%	1.52570%	1.51950%	1.52450%	1.52250%

Table 3.4 Number of used rates for given numbers of contributors

#CONTRIBUTORS	METHODOLOGY	#OF RATES
18 Contributors	Top 4 highest rates, tail 4 lowest rates	10
17 Contributors	Top 4 highest rates, tail 4 lowest rates	9
16 Contributors	Top 4 highest rates, tail 4 lowest rates	8
15 Contributors	Top 4 highest rates, tail 4 lowest rates	7
14 Contributors	Top 3 highest rates, tail 3 lowest rates	8
13 Contributors	Top 3 highest rates, tail 3 lowest rates	7
12 Contributors	Top 3 highest rates, tail 3 lowest rates	6
11 Contributors	Top 3 highest rates, tail 3 lowest rates	5

As at February 2016 the panel of banks contributing to EURIBOR consists of 24 banks: Whereas in September 2012, the panel of banks contributing to EURIBOR consisted of 44 banks, see Table 3.5.

Table 3.5 The LIBOR panel banks at 2012-09-01 and 2014-09-01

Country	Banks 2012-09-01	Banks 2014-09-01
Austria	Erste Group Bank AG	
	RZB Raiffeisen Zentralbank Österreich AG	
Belgium	Belfius	Belfius
	KBC	
Finland	Nordea	Nordea
	Pohjola	Pohjola
France	Banque Postale	
	BNP-Paribas	BNP-Paribas
	HSBCFrance	HSBC France
	Société Général	Société Général
	Natixis	Natixis
	Credit Agricole s.a.	Credit Agricole s.a.
	Credit Industriel et Commercial CIC	
Germany	Landesbank Berlin	
	Bayerische Landesbank Girozentrale	
	Deutsche Bank	Deutsche Bank
	Commerzbank	Commerzbank
	DZ Bank Deutsche Genossenschaftsbank	DZ Bank Deutsche
	Norddeutsche Landesbank Girozentrale	
	Landesbank Baden-Wurttemberg Girozentrale	
	Landesbank Hessen-Thuringen Girozentrale	
Greece	National Bank of Greece	National Bank of Greece
Italy	Intesa Sanpaolo	Intesa Sanpaolo
	Banca Monte dei Paschi di Siena	Banca Monte dei Paschi di Siena
	UniCredit	UniCredit
	UBI Banca	
Ireland	Bank of Ireland	
	AIB	
Luxembourg	Banque et Caisse d'Epargne de l'Etat	Banque et Caisse d'Epargne de l'Etat
Netherlands	ING Bank	INGBank
	Rabobank	
Portugal	Caixa Geral de Depósitos (CGD	Caixa Geral de Depósitos (CGD)
Spain	Banco Bilbao Vizcaya Argentaria	Banco Bilbao Vizcaya Argentaria
	Banco Santander Central Hispano	Banco Santander
	Confederacion Espaòola de Cajas de Ahorros	CECABANK
	CaixaBank S.A.	CaixaBank S.A.
Great Brittain	Barclays	Barclays
Denmark	Den Danske Bank	Den Danske Bank
Sweden	Svenska Handelsbanken	
Non-EU banks	UBS (Luxembourg) S.A.	
	Citibank	
	J.P.Morgan Chase & Co	London Branch of JP Morgan Chase
	The Bank of Tokyo-Mitsubishi UFJ	The Bank of Tokyo-Mitsubishi

The panel of banks provide daily quotes of the rate, rounded to two decimal places, that each panel bank believes one prime bank is quoting to another prime bank for interbank term deposits within the Euro zone. The maturities are ranging from 1 week to 1 year.

Every Panel Bank is required to directly input its data no later than 11:00 a.m. (CET) on each day that the Trans-European Automated Real-Time Gross-Settlement Express Transfer system (TARGET) is open. At 11:02 a.m. (CET), GRSS (Global Rate Set Systems) will instantaneously publish the reference rate on Reuters, Bloomberg and a number of other information providers which will then be made available to all their subscribers. The published rate is a rounded, truncated mean of the quoted rates. The highest and lowest 15% of quotes are eliminated and the remainder are averaged and the result is rounded to three decimal places. EURIBOR rates are spot rates, that is, for a start two working days after measurement day. Like US money-market rates, they are Actual/360, that is, calculated with an exact day count over a 360-day year. (Fig. 3.1)

3.1.2.1 EURIBOR+

At a SIFMA[3] Roundtable on December 2, 2015, representatives of the European Money Market Institute (EMMI) explained their plan to change the EURIBOR rate from the current quotation-based system to a rate based on actual transactions. The new rate will be called EURIBOR+. According to EMMI, one of the goals is to achieve a "seamless transition" in which no current EURIBOR-based contracts would be disrupted. At the end of the transition, EURIBOR+ will continue to be published on the same data vendor pages, such as Reuters page EURIBOR01. EMMI administers the EURIBOR and Euro Overnight Index Average (EONIA) rates.

Currently, EURIBOR is defined as "the rate at which euro interbank term deposits are being offered within the EU and EFTA countries by one Prime Bank to another at 11:00 a.m. Brussels time." The definition of EURIBOR+ would be "the rate at which banks of sound financial standing could borrow funds in the EU and EFTA countries in the wholesale, unsecured money markets in euro."

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[3] Securities Industry and Financial Markets Association

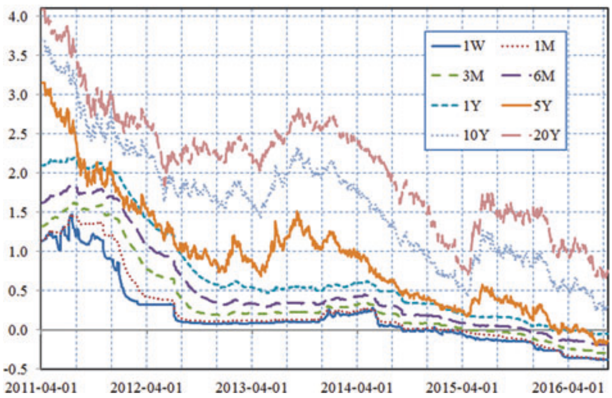

Fig. 3.1 EURIBOR rate quotes between 2011-04-01 and 2016-08-15[4]

The key difference between current EURIBOR and EURIBO+ is that EURIBOR relies on quotes and member bank estimates of prime bank activity, while EURIBOR+ will rely on actual wholesale borrowing transactions executed by the member bank. EURIBOR's current estimate of bank funding rates as a point-in-time average will be replaced by the EURIBOR+ backward- looking period average.

The transition to EURIBOR+ is targeted to take effect on July 4, 2016.

The first element to consider is that the number of entities that provide data is increased, in addition to changing the methodology, since it will take into account not only the deposits that banks make to each other (interbank lending) but also those of big companies and financial institutions, non-financial small and medium entrepreneurs, insurance companies, pension funds, etc. The current EURIBOR measures the average interest rate at which banks lend money in Europe and currently only 24 institutions are providing information. The problem is that banks do not provide accurate information on operations with real interest but on estimates of the interest that would be charged between them.

The various manipulations of EURIBOR (also from other indices such as Libor or Tibor) between 2005 and 2009 made the European Commission in 2013 to fine several entities. The new EURIBOR Plus

[4] Source Swedbank AB (publ)

calculation would be based on a more realistic rate which would mean that an application of a rate more realistic, although not until the test period to see if this is so.

3.1.3 The EONIA Rates

The other widely used reference rate in the euro-zone is EONIA, also published by the European Banking Federation, which is the daily weighted average of overnight rates for unsecured interbank lending in the euro-zone, that is, like the federal funds rate in the US. The banks contributing to EONIA were the same as the Panel Banks contributing to EURIBOR. However, "On 1st June 2013 the Eonia® and Euribor® respective panels of contributing banks have been differentiated."(EMMI website)

The reference rate referred to as EONIA is computed as a weighted average of **all overnight unsecured lending transactions** in the interbank market, initiated within the euro area by the Panel Banks. Note that this is an average of **actual** transactions that has taken place between banks – not any indicative quote as used in the calculations of LIBOR or EURIBOR rates. It is reported on an act/360 day count convention with three decimal places.

"Overnight" means from 1 day to the next business day, until the interbank payment system TARGET, The Trans-European Automated Real-time Gross settlement Express Transfer system closes. The panel of reporting banks is the same as for EURIBOR, so that only the most active banks located in the euro area are represented on the panel and the geographical diversity of banks in the panel is maintained.

All specified transactions initiated during the business day shall be reported by the Panel Banks in aggregate, that is, the sum of all lending transactions carried out before the closing of real-time gross settlement (RTGS) systems at 6:00 p.m. (London time). Each Panel Bank shall, on each day that the TARGET system is open and no later than 6:30 p.m., report to the ECB the total volume of unsecured lending transactions that day and the weighted average lending rate for these transactions. Thus, the calculation of the weighted average for the overnight transactions for each bank is made by the respective Panel bank itself.

The amount of lending transactions shall be reported by Panel Banks in millions of euro, and the individual average rates shall be reported

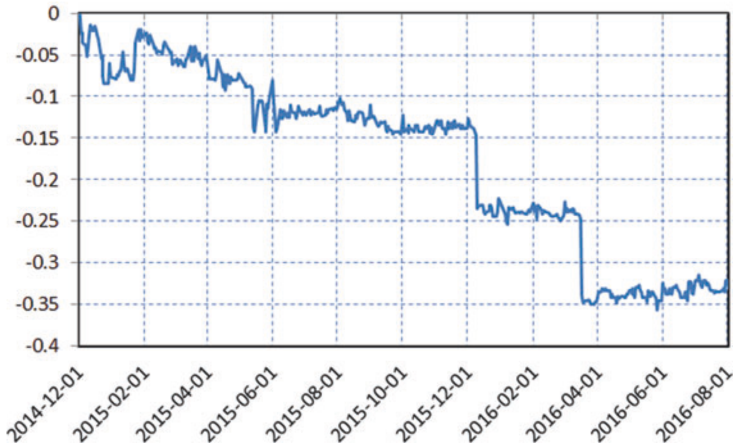

Fig. 3.2 The Over-Night rate EONIA[5]

with three decimals. Rounding shall be carried out following established rounding rules in the market. In Fig. 3.2, the evolution of the EORNIA rate between 2014-12-01 and 2016-08-01 are shown.

3.1.3.1 Calculation and Publication of EONIA

Based on the reported volumes and average rates from each Panel Bank the European Central Bank (ECB) calculates EONIA, the weighted average for all the Panel banks. ECB shall aim to make the computed rate available for publication as soon as possible so that EONIA can be published between 6:45 p.m. and 7:00 p.m. on the same evening.

The ECB will undertake control measures to assess the quality of EONIA and may report to the Steering Committee on the performance of individual Panel Banks.

3.1.4 The Euro Repurchase Agreement Rate – Eurepo[6]

For the reference rate Eurepo,[7] a representative panel of prime banks provide daily quotes of the rate, rounded to three decimal places. Each Panel Bank reports its believes on what one prime bank is bidding another prime bank (and offering money) for term repo with

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[5] Source, Swedbank AB (publ)
[6] Eurepo was discontinued on 2 January 2015.
[7] http://www.emmi-benchmarks.eu/assets/files/Eurepo_tech_features.pdf

generalized collateral (Eurepo GC). Eurepo is quoted for spot delivery (T +2) using the act/360 day-count convention. Eurepo is quoted for the following maturities: T/N, 1, 2 and 3 weeks and 1, 2, 3, 6, 9 and 12 months.

Contribution of data

- Every Panel Bank will be required to directly input their data to the Calculation Agent platform no later than 10:45 a.m. (CET) on each day that the Trans-European Automated Real-Time Gross-Settlement Express Transfer system (TARGET) is open.
- Each Panel Bank will be allocated a private page by the Calculation Agent on which to contribute its data. Each contribution can only be viewed by the contributing Panel Bank and by the Calculation Agent staff involved in the calculation process.
- From 10:45 to 11:00 a.m. at the latest, the Panel Banks can correct, if necessary, their quotations.

3.1.4.1 Calculation

At 11:00 a.m. (CET), the Calculation Agent will process the Eurepo calculation. The Calculation Agent shall, for each maturity, eliminate the highest and lowest 15% of all the quotes collected. The remaining rates will be averaged and the result will be rounded to three decimal places.

3.1.4.2 Fall-Back Rules

Before calculating at 11:00 a.m. (CET) on each Target Day the Eurepo for that day, the Calculation Agent shall verify if all the Panel Banks have made their data available for that day in accordance with the established procedures.

If one or more Panel Banks have failed to do so, the Calculation Agent shall use reasonable efforts to remind such Panel Banks by telephone or any other means of communication of their obligation to provide the data and shall invite them to submit the data immediately.

Should any Panel Bank after such a reminder still not provide its data until 11:00 a.m. (CET), the Calculation Agent shall calculate the Eurepo for that day without the missing data and promptly notify EMMI in writing.

At 11.00 am:

- if eight or more Panel Banks from three or more countries have provided data, calculate and display the Eurepo based on this data; or
- if fewer than eight Panel Banks have provided data or if the Panel Banks which have provided data are from fewer than three countries, the Calculation Agent shall delay the calculation of the Eurepo for that day until eight or more Panel Banks from three or more countries have provided data. The Calculation Agent shall, at 11:15 a.m. (CET), indicate the delay to all Authorized Vendors and promptly notify EMMI.
- If fewer than eight Panel Banks have provided data by 12:30 p.m. (CET), Eurepo rates of the previous business day will be republished at 12:30 p.m. (CET) and will be used as the Eurepo rates for that day. Any republished rates from the previous business day will be identified as such by the Calculation Agent.

In this event, the Eurepo Steering Committee shall be convened in special session as soon as practicable on notification of a contingency event, in order to devise a resolution strategy preserving the continuity of Eurepo. This strategy should be implemented within a period no longer than three fixing days of the prior fixing established under the regular process. The prior fixing may be re-published as the fixing for the days in this period.

3.1.4.3 Publication of Eurepo

After the calculation has been processed at 11:00 a.m. (CET), the calculation agent will publish the Eurepo reference rate which will be made available simultaneously to all Authorised Vendors.

At the same time, the underlying Panel Bank rates will be published on a series of composite pages which will display all the rates by maturity.

Historical data and individual submissions for Eurepo are also published on a delayed basis on the EMMI official website.

3.1.5 Sterling Overnight Index Average (SONIA)

Sterling Overnight Index Average (SONIA) was introduced by the Wholesale Markets Brokers' Association (WMBA) in March 1997 as a benchmark for the cost of overnight funds in sterling. It was London's first Overnight Index and it stimulated the development of Overnight Index Swaps (OIS) in the Sterling Money Market. SONIA provides a methodology for the fixing of Overnight Indexed Swap rates. Although some central banks calculate and publish daily fixing rates for overnight funds in their respective currency, the Bank of England did not. So if no appropriate rate existed in Sterling, the WMBA, with the BBA's backing, decided to create the SONIA calculations. Historical data are available on the WMBA website.

The Bank of England and the WMBA announced in April 13 2016 that the Bank of England will become the administrator of the SONIA interest rate benchmark on 25 April 2016.[8]

SONIA tracks actual Sterling overnight funding rates experienced by market participants during the day. SONIA is the weighted average rate to four decimal places of all **unsecured sterling overnight** cash transactions brokered in London by WMBA member firms between midnight and 4:15 p.m. with all counterparties in a minimum deal size of £25 million.

The creation of SONIA led to new derivative products, which have been used to reduce the risk and increase the transparency for overnight funding. The foremost example of this is the OIS. In such a swap a fixed rate interest rate is swapped against a floating rate index, for example, SONIA or EONIA. OIS contracts replicate a mismatched interbank deposit position through either:

- a short-term loan funded by an overnight deposit; or
- an overnight loan funded by a short-term deposit.

OISs allow banks to manage their liquidity requirements more effectively although part of the overnight risk still remains.

A typical OIS contract looks like this: Two parties agree to exchange the difference between the interest accrued at a pre-specified fixed interest rate on a given notional amount for a fixed period – say 3 months – and the compounded interest obtained from rolling-over the

[8] http://www.bankofengland.co.uk/publications/Pages/news/2016/046.\penalty\z@aspx

daily SONIA rates over the term of the swap. At the end of the period settlement of the contract is made and payments are netted so the principal never changes hands.

3.1.6 Federal Funds

Federal funds, or fed funds, are unsecured loans of reserve balances at Federal Reserve Banks that depository institutions make to one another. The rate at which these transactions occur is called the fed funds rate.

The most common duration or term for fed funds transaction is overnight, though longer-term deals are arranged. The Federal Open Market Committee (FOMC) sets a target level for the fed funds rate, which is its primary tool for implementing monetary policy. Fed Funds Transactions Redistribute Bank Reserves

Fed funds are unsecured loans of reserve balances at Federal Reserve Banks between depository institutions. Banks keep reserve balances at the Federal Reserve Banks to meet their reserve requirements and to clear financial transactions. Transactions in the fed funds market enable depository institutions with reserve balances in excess of reserve requirements to lend them, or "sell" as it is called by market participants, to institutions with reserve deficiencies. Fed funds transactions neither increase nor decrease total bank reserves. Instead, they redistribute bank reserves and enable otherwise idle funds to yield a return.

3.1.7 Summary

In Table 3.6, we show a summary of the most common interest rates.

Table 3.6 A summary of some interest rates

	Libor	Euribor	Eonia	Europe
Definition	London Interbank Offered Rate	Euro Interbank Offered Rate	Euro Overnight Index Average	Euro Repurchase Agreement rate
Market	London Interbank	Euro Interbank	Euro Interbank	Euro Interbank
Side	Offer	Offer	Offer	Offer
Rate Quotation Specs	EURLibor=Euribor, Other currencies: minor differences(e.g. act/365, T+0, London calendar for GBP Libor).	TARGET calendar settlement T+2, act/360, three decimal places, modified following, end of month, tenor variable.	TARGET calendar settlement T+1, act/360, three decimal places, tenor Id.	TARGET calendar settlement T+2, act/360, three decimal places, modified following, end of month, tenor variable.
Maturities	1d–12m	1w,2w,3w,1m . . . 12m	1d	T/N-12m
Publication Time	12.30 CET	11:00am CET	6:45–7:00pm CET	11:00am CET
Panel Banks	11–17 banks per currency	24 banks from 11 EU countries+2 international banks	24 banks from 11 EU countries+2 international banks	24 EU banks from plus some large international banks from Non-EU countries
Calculation agent	ICE	EMMI	ECB	EMMI
Transaction based	No	No	Yes	No
Collateral	No (unsecured)	No (unsecured)	No (unsecured)	Yes (secured)
Counterparty risk	Yes	Yes	Low	Negligible
Liquidity Risk	Yes	Yes	Low	Negligible
Tenor based	Yes	Yes	No	No

4
Interest Rate Instruments

4.1 Introduction to Interest Rate Instruments

We will now describe some instruments in the interest rate markets, where there exist a huge number of different instrument types. To mention all variants is far out of the scope in this book, if possible at all. Some of these instruments are referred as Fixed Income instruments. The name refer to the fact that all income, that is, all cash flows, are known prior to the actual trade. Bonds are typical fixed income instruments since the coupon rate and the nominal amount are known.

4.1.1 Bonds, Bills and Notes

Bond, bills and notes are the most common debt instruments and the starting point in most financial theory. In a later section we will consider the bond prices (zero-coupon bonds) as stochastic processes related to similar processes of the short rates and forward rates. The value of such an instrument is found by studying the individual cash flows and discount them to present values. The total value is then given as the sum of its components.

4.1.1.1 Bills

Bills are the simplest debt instruments. It is a promissory note with a fixed time of expiry when it promises to pay a fixed nominal amount. There are no intermediate payments. Bills are negotiable securities that

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_4

are issued by central or local governments, private corporations and banks and can be resold. The lifetime of a bill is normally a year or less. If the time value of money is positive the bills are traded at a price that is lower than the nominal amount ahead of expiry. This is known as a “discount”. The discount is the nominal value of the bill minus its current price. Before the recent financial crisis and negative central bank rates bills typically traded at a “discount”.

Bills issued by the central government are known as T-bills (Swedish SSVX). If issued by local government municipalities or private firms the bills are known as CDs or Commercial Paper.

4.1.1.2 Bonds and Notes

Bonds too, have a fixed time of expiry but in addition to bills typically pay interest during their lifetimes. The interest payments are known as coupons. In addition bonds typically have several years to expiry upon issue. These are long-term debt instruments that can be resold. When a bond is issued, the buyers are essentially lending money to the issuer in return for the promise of regular interest payments and the promise of repayment of the principal at a future date. Most bonds also trade in the secondary market, like stocks and other securities. Understanding the pricing of a bond and, in particular, the relative valuation of different bond issues is very important for investors and traders.

A note is similar to a bond, but with a shorter lifetime. Notes have normally a lifetime between 1 and 10 years, while bonds have a lifetime of more than 10 years. In UK there are bonds, war loans (from the Second World War) that never expires. In UK bonds are referred to as gilts.

The main considerations that enter the pricing of a bond are:

- The principal (or notional amount of the bond)
- Time to maturity
- The interest payments (coupons)
- Call provisions and other features such as conversion to shares, etc.
- Credit quality of the issuer.

The first three points define the structure of cash flows that the investor expects to receive if the bond was held to maturity.

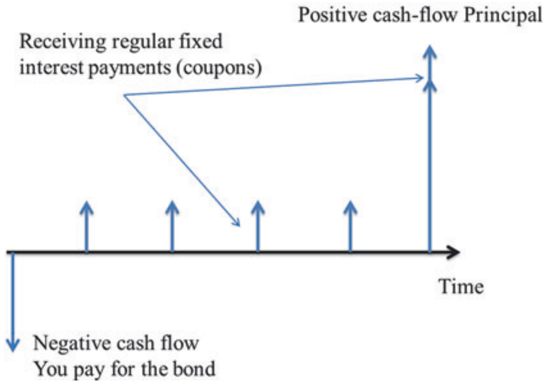


Fig. 4.1 The bond cash flows consist of an initial payment , the fixed coupon payments and the payback of the principal (the nominal amount)

Callable bonds are bonds where the issuer has the right, but not the obligation, to call back/repurchase the bond at one or more specified points over the bond's lifetime. If called, the issuer pays the investor the pre-specified call price, the strike. The call price is usually higher than the bond's par value. The difference between the call price and par value is called the call premium. For the investor, this means that there is uncertainty as to the true maturity of the bond.

Putable bonds are bonds where the holder has the right, but not the obligation, to put back the bond to the issuer at one or more specified points over the bond's lifetime. If putted, the investor pays the issuer pre-specified put price. The difference between the put price and the par value of the bond is called the put premium.

Convertible bonds (usually issued by corporations) can be converted into stocks at a given price (the "conversion ratio"). Conversion events complicate the pricing because the investor is uncertain about what cash flows will be received.

Zero-coupon bonds (also called a "zeroes" or a "pure discount bonds") are bonds that have a single payment of the principal at maturity, without any intermediate interest payments.

A **Bullet bond** is a conventional bond paying a fixed periodic coupon and having no embedded optionality. Such bonds are non-amortizing, that is, the principal remains the same throughout the lifetime of the bond and is repaid in its entirety at maturity. Bullet bonds are also called straight bonds. In the United States such bonds usually pay semi-annual coupons. The coupon rate (CR) is stated as a simple annualized rate (usually with semi-annual coupon payments) and paid on the bond's par value (par). Thus, a single coupon payment is equal to $\frac{1}{2} \times \text{CR} \times \text{Par}$.

A **benchmark bullet bond** is a bullet bond issued by a sovereign government and assumed to have no credit risk. These are also called Treasury bonds.

A **non-benchmark bullet bond** is a bullet bond issued by an entity other than the sovereign and which, therefore, has some credit risk.

Eurobonds are bonds in the Eurobond market. The name can be confusing because of its name. Although the Euro is the currency used by participating European Union countries, Eurobonds refer neither to the European currency nor to some European bond market. A Eurobond instead refers to any bond that is denominated in a currency other than that of the country in which it is issued. Bonds in the Eurobond market are categorized according to the currency in which they are denominated. As an example, a Eurobond denominated in Japanese yen but issued in the US would be classified as a Euro Yen bond.

Foreign bonds are denominated in the currency of the country in which a foreign entity issues the bond. One example is the samurai bond, which is a yen-denominated bond issued in Japan by an American company. Other popular foreign bonds include Bulldogs and Yankee bonds.

The credit quality is a very important variable because it represents our beliefs about the issuer's capacity to repay the principal and interest. US Federal Government bonds are considered to be the most creditworthy, since they are backed by the "full faith and power" of the government.

Bonds issued by corporations have a certain probability of defaulting in case the corporation can no longer meet its obligations. Thus, corporate bonds have a lower credit quality. Low-credit bonds have higher coupons than high-grade bonds trading at the same price. Investors demand a premium for taking on the default risk. In most cases we will ignore credit quality considerations.

Arbitrage Pricing Theory (APT), as we will see, gives a way of expressing the value of a zero-coupon bonds in terms of a risk-neutral measure on the paths of the short-term interest rates. The yield of the zero-coupon bond is, by definition, the constant interest rate that would make the bond price equal to the discounted value of the final cash flow. In other words, the yield is the continuously compounded (constant) rate of return that the investor would receive if the zero-coupon bond was bought and held to maturity. Notice that the price and the yield varies inversely to each other: an increase in price corresponds to a decrease in yield and vice-versa. Moreover, the price is a convex function of yield.

There is an important practical consideration regarding the calculation of yields. In fact, expressing the time-to-maturity, $T - t$, as a fraction of a year, a decimal number, requires using a specific day-count convention, in order to convert days and months into fractions of a year.

Most bonds have intermediate interest payments, typically called coupons, as well as repayment of the principal. The "generic" bond must therefore specify a maturity date in which the principal payment is made, as well as a schedule of intermediate interest payments.

- Maturity date
- Principal (notional, nominal amount or the face value)
- Coupon rate
- Frequency of the coupons and payment dates

The coupon rate is the annualized intermediate payment of the bond. The frequency represents how many payments are made per year (1, 2, 4 or 12). Most bonds have annual or semi-annual coupon payments. Thus, a 10-year bond with face value of \$1000 and a semi-annual coupon of 6.25 will pay the investor an interest of $0.5 \times 0.0625 \times 1000 = \$31.25$ every six months (totally 20 payments) and the principal will be paid at the 20th payment date.

4.1.2 Bonds, Market Quoting Conventions and Pricing

When a trader buys a bond, the price is normally quoted as the clean price. The price actually being paid, however, is equal to the clean price plus the accrued interest since the last coupon. This is known as the "dirty price". The price of a bond as function of the yield-to-maturity (YTM) can be written as

$$P = \frac{N}{(1+ytm)^T} + \sum_{i=1}^{n} \frac{C}{(1+ytm)^{t_i}}$$

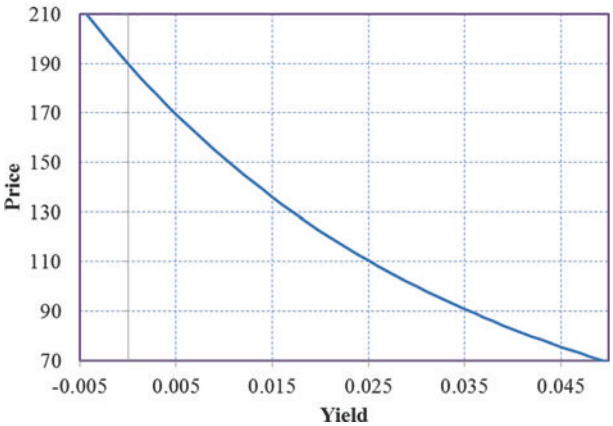


Fig. 4.2 A 30 year to maturity bond price as function of ytm. The coupon rate = 3%.

Remark! This is the relationship between the quoted yield and the dirty price is known as the present value formula although in the markets one does not use it to value the bonds. The price P versus ytm is shown in Fig. 4.2.

The formula for the price, as a function of the yield can be simplified as we will show below. Denote T as the time to maturity, N the nominal amount, C the size of the coupon ($C = c \cdot N$ where c is the coupon rate) ant t_i the times for the individual coupons. If the market rate is the same as the coupon rate, we say that the bond is traded at par. For simplicity we denote yield-to-maturity with the single letter y from now on.

We start by studying the present value formula for only the coupons, that is,

$$PV_c = \sum_i \frac{C}{(1+y)^{t_i}} = \frac{C}{(1+y)^{\{T\}}} \cdot \sum_{i=0}^{[T]} \frac{1}{(1+y)^i}$$

where $[T]$ is the (integer) number of years to maturity and $\{T\} = T - [T]$ the time (part of a year) to the next coupon. The last sum is then

$$x = \sum_{i=0}^{[T]} \frac{1}{(1+y)^i} = 1 + \sum_{i=1}^{[T]} \frac{1}{(1+y)^i}$$

If we multiply this sum with $(1 + y)$ we get

$$(1+y)\,x = (1+y) + \sum_{i=0}^{[T]-1} \frac{1}{(1+y)^i}$$

We then have

$$(1+y)\,x - x = x \cdot y = (1+y) - \frac{1}{(1+y)^{[T]}}$$

Giving

$$x = \frac{1+y}{y} - \frac{1}{(1+y)^{[T]} \cdot y}.$$

Then

$$PV_c = \frac{C}{(1+y)^{\{T\}}} \cdot \left(\frac{1+y}{y} - \frac{1}{(1+y)^{[T]} \cdot y} \right) = \frac{C/y}{(1+y)^{\{T\}}} \cdot \left(\frac{(1+y)^{[T]+1} - 1}{(1+y)^{[T]}} \right)$$

And we finally get

$$PV(y) = \frac{N + \frac{C}{y} \cdot \left[(1+y)^{[T]+1} - 1\right]}{(1+y)^T}$$

If there are several coupons per year (with the frequency f) we get

$$PV(y,f) = \left(1 + \frac{y}{f}\right)^{-T \cdot f} \cdot \left\{ N + \frac{C}{y} \left[\left(1 + \frac{y}{f}\right)^M - 1 \right] \right\}$$

where M is the number of the remaining cash flows. In C/C++ or Excel, M is calculated using the function `ceil` as `ceil` $(Tf)/f$. The formula can be used to calculate the interest rate risk, where we shift the yield-to-maturity with one basis point (bp)

$$R = PV(y) - PV(y + 1\ bp)$$

We can also use this to calculate an implied coupon rate for floating rate notes (FRNs) if we have the market price and yield-to-maturity

$$C = \frac{y \cdot \left(PV \cdot \left(1 + \frac{y}{f}\right)^{T \cdot f} - N \right)}{\left(1 + \frac{y}{f}\right)^M - 1}$$

Using the yield-to-maturity, the price of a bond can be calculated in C/C++ as

```
double Price(double Coupon, double YTM, double T, int f)
{
  double price;

  price = Coupon*(pow(1.0 + YTM, ceil(T*f)/f) - 1.0)/
          (pow(1.0 + YTM, 1.0/f) - 1.0);
  price = (100.0 + price)/pow(1.0 + YTM, T);
  return price;
}
```

The present value, that is, the market (dirty) price of a bond, is given by

$$P = \frac{N}{(1 + r(T))^T} + \sum_{i=1}^{n} \frac{C}{(1 + r(t_i))^{t_i}}.$$

With a known constant spread above the interest rate we calculate this as

```
double BondPrice(double Coupon, double *SpotRate, int N,
                 int *PayDay, double spread)
{
  double Price = 0.0;
  int    i;
  for (i = 0; i < N; i++) {
    Price += Coupon/pow(1.0 + (SpotRate[i] + spread),
                        PayDay[i]/365.0);
  }
  Price += 100.0/pow(1.0 + (SpotRate[N-1] + spread),
                     PayDay[N-1]/365.0);
  return Price;
}
```

4.1.3 Accrued Interest

The market prices of bonds, when published in newspapers, are quoted as *clean prices*.[1] That is, they are quoted without any accrued interest. The accrued interest is the amount of interest that has built up since the last coupon payment. In contrast to stock markets, in fixed income the convention is to share the dividend payments, that is, the coupon payments, in a special way between buyers and sellers if the bond is bought at a time between coupon dates.

[1] In some countries, such as Sweden, bond prices are quoted as yield (to maturity).

The accrued interest is equal to the upcoming coupon payment times the number of days since the last coupon date divided by the number of days in the period between coupon payments.

The actual payment is called the *dirty price* and is the sum of the quoted clean price and the accrued interest.

How the dirty price is related to the clean price is shown in Fig. 4.3. The owner of the bond wants to get his share of the upcoming coupon payment if he sells the bond between the cash flows.

The clean price is given as

$$Clean = P - Coupon \cdot \frac{365 - d}{365}$$

where d is the number of days until next coupon payment. The clean price in the Fig. 4.3 is constant if the market rate and the coupon rate are the same.

If the market rate is higher than the coupon rate, the clean price will have a positive slope. This is illustrated in Fig. 4.4.

On the other hand, if the market rate is below the coupon rate, the clean price will have a negative slope as in Fig. 4.5.

As we have seen, when we calculate the price of a bond and consider the bond as a number of cash flows that we use the present value formula. In Table 4.1, we show the coupon frequencies and the day-count convention for typical bonds in some countries.

Bills in US with maturity less than 1 year are called T-bills. They are usually zero-coupon bonds. Bonds with maturity 1 to 10 years

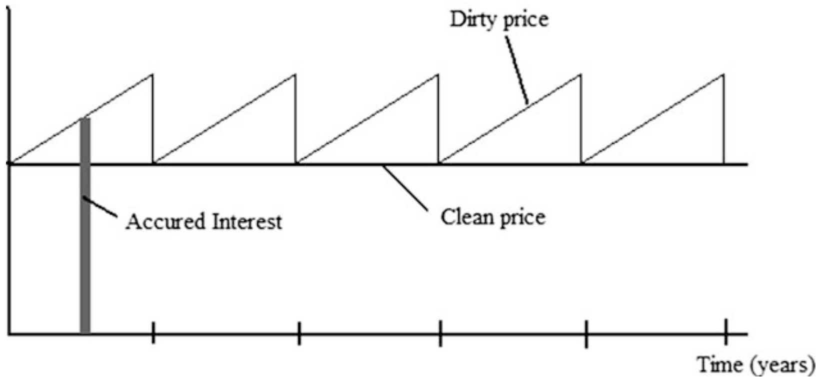


Fig. 4.3 The clean- and dirty price of a bond as function of a constant yield over time

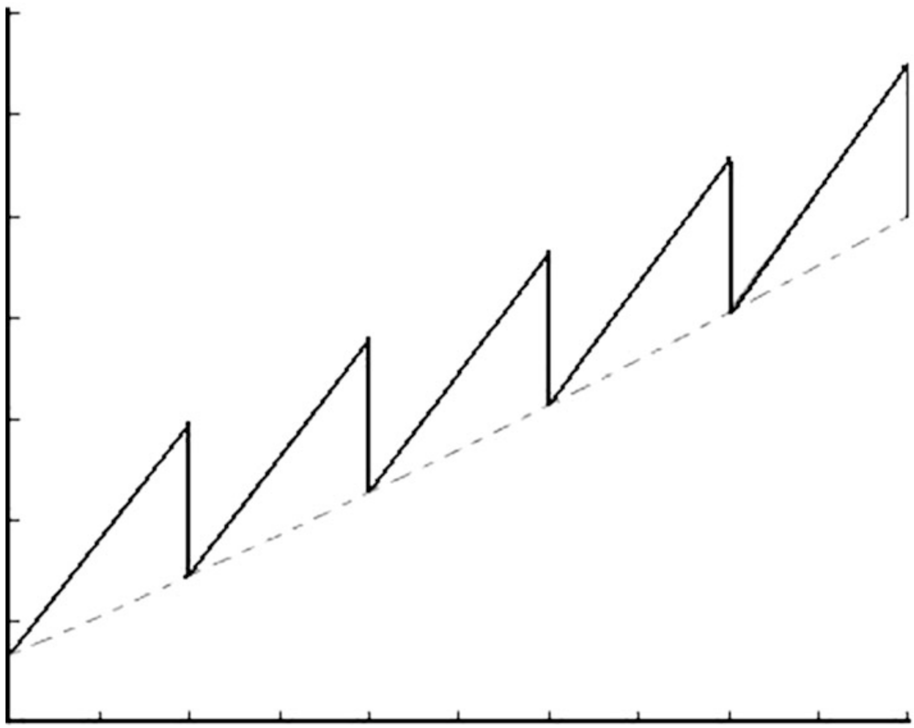


Fig. 4.4 The bond dirty price as function of a constant upward sloping yield

are called T-notes and they with maturity above 10 years are called T-bonds. They are coupon bearing and quoted in USD and 32nds of a USD for a USD 100 face value. Thus, a quote of 99–16 means a decimal price of USD 99.5 for a USD 100 face value.

Bonds traded in the United States foreign bond market, which are issued by non-US institutions, are called *Yankee bonds*. Since the beginning of 1997, the US Federal Government has also issued bonds linked to the rate of inflation.

Bonds issued by the UK Government are called *gilts*. Some of these bonds are callable; some are irredeemable, meaning that they are perpetual bonds (we will discuss such a bond in more detail in a later section) having a coupon but no repayment of principal. The government also issues convertible bonds, which may be converted into another bond issue, typically of longer maturity. Finally, there are index-linked (inflation-linked) bonds having the amount of the coupon and principal payments linked to a measure of inflation, the Retail Price Index (RPI).

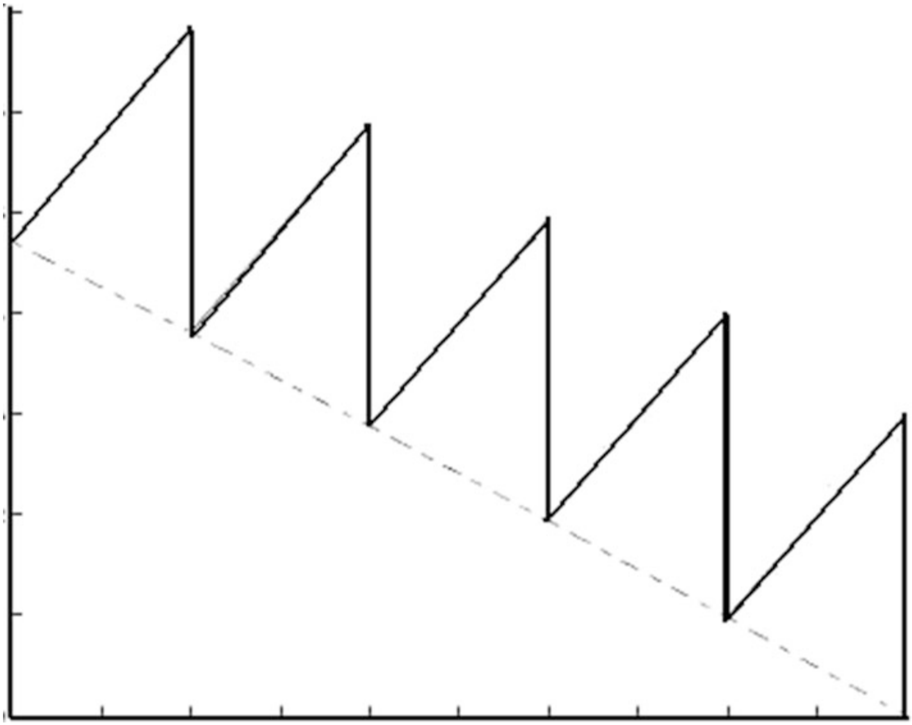


Fig. 4.5 The bond dirty price as function of a constant downward sloping yield

Table 4.1 Coupon frequency and day count for bonds

Treasury bonds	Coupon frequency	Day-count convention
Sweden	Annual	30!360
US	Semi annual	Actual/Actual
Japan	Semi annual	Actual/365
UK	Semi annual	Actual/365,Actual/Actual
Germany	Annual	30E/360 or Actual/Actual
Italy	Semi annual	Actual/Actual
Corporate bonds		
US	Semi-annual /Annual	30!360
UK	Semi annual	Actuall365 or Actual/Actual
Eurobonds	Annual (Semi-annual)	30E/360

Japanese government bonds (JGBs) come as short-term, medium-term, long-term (10 year maturity) and super long-term (20 year maturity). The long- and super long-term bonds have coupons every six months. The short-term bonds have no coupons and the medium-term bonds can be either coupon-bearing or zero-coupon bonds. Yen-denominated bonds issued by non-Japanese institutions are called **Samurai bonds**.

Example 4.1.3.1

The future value, FV of a quarterly paying bond with 10% coupon rate and a nominal of 100 is given by:

$$FV = 100\left(1 + \frac{0.10}{4}\right)^4 = 110.38$$

This corresponds to an effective rate (EAR, Equivalent Annual Rate) of 10.38%.

Example 4.1.3.2

The present value formula for three annual coupons of 5000 with 5% discount rate

$$\begin{aligned} P &= \frac{5000}{1+0.05} + \frac{5000}{(1+0.05)^2} + \frac{5000}{(1+0.05)^3} \\ &= 4761.90 + 4535.15 + 4319.19 \\ &= 13616.24 \end{aligned}$$

The discounted present value can be expressed as

$$P = \frac{C}{y}\left[1 - \frac{1}{(1+y)^n}\right]$$

Remark! The bond price is proportional to the inverse of the discount rate.

The *Yield-To-Maturity* is sometimes called *Redemption Yield* and denoted as *Red* in *Financial Times*. In *Financial Times* another rate is also given, denoted *Int* which is the coupon divided by the dirty price. This is the flat yield, sometimes called *current-, interest-, running-* or *income yield*. Yield-to-maturity is interpreted as the yield an investor gets, by holding a bond to maturity and if he/she can re-invest all the coupons at the same rate, YTM. When trading bonds, the price, if quoted in yield-to-maturity, ytm, can be related to the bonds coupon rate. If the quote is equal to the coupon rate, we say that the bond is traded at par. If the quote is lower than the coupon rate, we say that the bond is traded above par. Similarly, if the quote is higher than the coupon rate, the bond is traded below par, see Table 4.2.

A common way to value risky bonds is via a spread s, over the yield to maturity y as

$$P = \sum_{n=1}^{N} \frac{CFn}{(1+y+s)^n}$$

Table 4.2 Bond par versus yield

If the bond is traded	The Yield is
On par	Same as coupon rate
Above par	Below coupon rate
Below par	Above coupon rate

where CF_n is the cash flow at time n. This means that due to a higher risk we want to get a better payoff compared with the less risky bond. A positive spread gives a lower price of the bond. This spread is called risk a premium.

Definition 4.1.3.1. *Current- and adjusted current yield* are defined as:

$$Current\ yield = \frac{Coupon\ rate}{Clean\ price} \cdot 100$$

$$Adjusted\ current\ yield = \frac{[Coupon\ rate + (100 \cdot Clean\ price)\,/\mathrm{n}]}{Clean\ price} \cdot 100$$

where n is the number of years to maturity with use of day-count convention.

4.1.4 Floating Rate Notes

An FRN is a hybrid between a short- and a long-term debt security

- Its original maturity typically exceeds 12 months (indeed most FRNs have longer maturities than straight corporate bonds), and its price is quoted as a percentage of int face value, like a bond.
- Its coupon rate is reset at each coupon date in line with a money market reference rate such as LIBOR, plus a fixed spread s.

Unlike a straight bond, the price of an FRN is not very sensitive to changes in market rates, because its coupons are reset periodically in line with the market. However, its price is sensitive to changes in the credit quality of the issuer: a note rated single-A paying LIBOR +45 bps and issued at par will trade at a discount to par in the secondary market if the debt of the issuer wee to be downgraded to BBB/Baa.

The full price of a floating-rate note on a coupon date is given by discounting the implied future coupons using the issuer's discount curve as follows

$$P = \sum_{i=1}^{N} [L(i-1, i) + s]\, p(0, i) + p(0, N)$$

where $p(0, i)$ is the discount factor from today to the subsequent coupon dates i, and $L(i-1, i)$ represents the forward LIBOR rate, which sets at time $i-1$ and pays at time i. For simplicity, we have assumed that the bond pays coupons annually.

If at time t the issuer has a T-maturity **par floater spread** of F, then the discount factors are given by the following iterative scheme

$$p(0, i) = \frac{p(0, i-1)}{1 + L(i-1, i) + F}$$

where $p(0, 0) = 1$. Clearly, $F$is a measure of the credit quality of the issuer since it is the fixed spread to LIBOR used to discount all cash flows. Note also that F changes over time as the credit quality of the issuer changes. If we substitute $F = S$ above, then we find that $P =$ 100%; that is, if the par floater spread, F, equals the fixed spread, S, on a coupon date, the floating rate note prices at par.

Dealers and investors assess the investment value of the FRN by reference to its **discount margin**.

Definition 4.1.4.2. *Discount Margin*

- The risk premium which, when added to the risk-free rate, makes the PV of the FRN equal to its market price.
- The spread over LIBOR which should be paid on the FRN in order to make its market price equal to par.

Discount margin is also known as: **Effective LIBOR Spread**.

The discount margin, M, is defined by the following relationship:

$$P = \frac{\left[\left(L^{next} + s\right) + \sum_{i=1}^{N} \frac{L+s}{(1+L+M)^i} + \frac{1}{(1+L+M)^N}\right]}{(1 + L^* + M)}$$

where for simplicity we have ignored day-count fractions and assumed that coupon dates are integers. The symbols are:

P	= full bond price
L^*	= stub LIBOR coupon to next coupon date
L	= current LIBOR fixing
L^{next}	= the next LIBOR payment (which was fixed on previous coupon date)
M	= discount margin for which we solve
s	= quoted margin

This calculation assumes that all future LIBOR cash flows are equal to the previous fixing. As a result, no account is taken of the shape of the LIBOR forward curve as in the par floater calculation.

Like a yield to maturity, M is calculated by a process; **iteration** – trying different values of M until you arrive at the one that equates the note's PV with its market price.

If the note trades at par:	Discount margin = LIBOR spread
If the note trades at a discount to par:	Discount margin > LIBOR spread
If the note trades at a premium to par:	Discount margin < LIBOR spread

Example 4.1.4.1

What is the discount margin on the following security?

Security:	GBP FRN maturing 29 October 2026
Rating:	Single A
Coupon rate:	6 month LIBOR + 0.15%
Day count:	Actual/365
Settlement:	14-apr-16
Current LIBOR fix:	1.25%
Clean price:	98.75

What is the discount margin on this security?

Analysis:

Using method 1, first we "fix" the coupon rate on the note at the current LIBOR

$$\text{Coupon} = 1.25 + 0.15 = 1.40\%$$

Then, we compute the yield to maturity on this "bond", given its current market price.

$$\text{Calculated yield (semi – annual, actual/365)} = 1.558\%$$

Finally, we calculate the discount margin as the difference between this yield and LIBOR

$$\text{Discount Margin} = 1.558\% - 1.250\% = \mathbf{0.308\%}$$

In other words, if the note paid LIBOR + 30.8 bps, instead of LIBOR + 15, then it would trade at par. An investor expecting to earn LIBOR + 35 on single-A-rated paper would consider this note to be trading rich.

This approach is very straightforward, but you must be careful to use the appropriate day-count conventions when moving between the bond markets and the money markets. Thus, if the coupons on this note were calculated on an Actual/360 basis, the bond-equivalent coupon rate would be closer to $365/360 \times 1.40\% = 1.419\%$. The calculated bond yield would be 1.579%, which is a money market equivalent yield of $360/365 \times 1.579\% = 1.557\%$. The discount margin would then be: $1.557 - 1.250 =$ **0.307%**.

In this example, the difference is very small, but in different conditions, it could be significant.

If we simplify the general pricing formula for an FRN by setting $s = M = 0$, we can discount with the LIBOR forward rate L_i with day-count period Δ_i

$$PV = \frac{\Delta_0 \cdot L_0 \cdot N}{1 + \Delta_0 \cdot L_0} + \frac{\Delta_1 \cdot L_1 \cdot N}{(1 + \Delta_0 \cdot L_0) \cdot (1 + \Delta_1 \cdot L_1)} + \dots + \frac{\Delta_n \cdot L_n \cdot N + N}{(1 + \Delta_0 \cdot L_0) \cdot (1 + \Delta_1 \cdot L_1) \cdots (1 + \Delta_n \cdot L_n)}$$

But

$$\frac{\Delta_n \cdot L_n \cdot N + N}{(1 + \Delta_0 \cdot L_0) \cdot (1 + \Delta_1 \cdot L_1) \cdots (1 + \Delta_n \cdot L_n)} = \frac{(1 + \Delta_n \cdot L_n) \cdot N}{(1 + \Delta_0 \cdot L_0) \cdot (1 + \Delta_1 \cdot L_1) \cdots (1 + \Delta_n \cdot L_n)} = \frac{N}{(1 + \Delta_0 \cdot L_0) \cdot (1 + \Delta_1 \cdot L_1) \cdots (1 + \Delta_{n-1} \cdot L_{n-1})}$$

which when combined with the previous term can be written as

$$\frac{(1 + \Delta_{n-1} \cdot L_{n-1}) \cdot N}{(1 + \Delta_0 \cdot L_0) \cdot (1 + \Delta_1 \cdot L_1) \cdots (1 + \Delta_{n-1} \cdot L_{n-1})} = \frac{N}{(1 + \Delta_0 \cdot L_0) \cdot (1 + \Delta_1 \cdot L_1) \cdots (1 + \Delta_{n-2} \cdot L_{n-2})}$$

Continuing to combine the last term with the previous one we finally get

$$PV = \frac{\Delta_0 \cdot L_0 \cdot N}{1 + \Delta_0 \cdot L_0} + \frac{\Delta_1 \cdot L_1 \cdot N + N}{(1 + \Delta_0 \cdot L_0) \cdot (1 + \Delta_1 \cdot L_1)}$$
$$= \frac{\Delta_0 \cdot L_0 \cdot N + N}{1 + \Delta_0 \cdot L_0} = \frac{(1 + \Delta_0 \cdot L_0)N}{1 + \Delta_0 \cdot L_0} = N$$

By induction, we have then proved that $PV = N$ at each reset day.

Therefore, we have a market risk with the duration to the next reset day. The duration of the credit risk (the risk in the change of discount margin, spread risk or basis risk) will on the other hand persist for the whole duration of the FRN, that is be similar to that of a bond of matching maturity.

The value between two resets days is given by:

$$PV = \frac{1 + \Delta \cdot L^{last}}{1 + L \cdot \frac{d}{D}} \cdot N$$

where d/D is the time to next reset with day-count, L the next reset, Δ the length of the current time period the and L^{last} the last fixing rate.

A **floating rate note (FRN)** initiated at time t_0 involves:

1. Buying the FRN at time t_0 for a fixed price N;
2. a series of floating interest payments: L_{t_0} at time t_1, L_{t_1} at time $t_2, \ldots, L_{t_{n-2}}$ at time $t_{n-1}, L_{t_{n-1}} + N$ at time t_n.

We will consider the most common floating rate note, which is a bullet note, where the coupon rate effective for the payment at the end of one period is set at the beginning of the period at the current market interest rate for that period (Fig. 4.6).

Suppose that the payment dates of the bond are $t_1 < \cdots < t_n$, where $t_i - t_{i-1} = \delta$ for all i.

In practice, δ will typically equal 0.25, 0.5 or 1 year, corresponding to quarterly, semi-annual or annual payments. The annualized coupon rate valid for the period $[t_{i-1}, t_i]$ is the δ-period market rate at date t_{i-1} computed with a compounding frequency of δ. We will denote this interest rate by $l(t_i, t_{i-1})$, although the rate is not necessarily a LIBOR rate (plus a number of bps), but can also be a Treasury rate. If the face value of the bond is N, the payment at time ti equals

$$N\delta l\,(t_i, t_i - \delta)\,, \quad \text{for } i = 1,\ 2, \ldots, n-1,$$

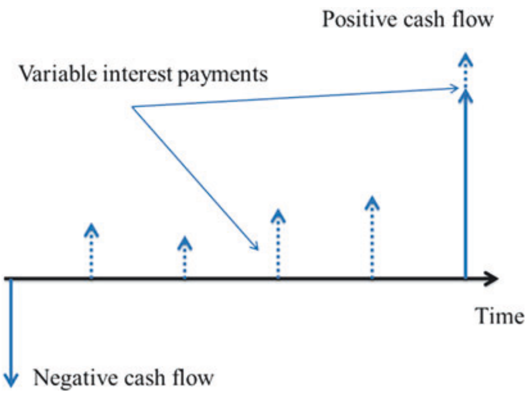


Fig. 4.6 The cash flows for a floating rate note (FRN).

and the final payment at time t_n equals

$$N\,(1 + \delta l\,(t_i,\ t_i - \delta))$$

If we define $t_0 = t_1 - \delta$, the dates $t_0, t_1, \ldots, t_{n-1}$ are often referred to as the reset dates of the note. Let us look at the valuation of a floating rate note. We will argue that immediately after each reset date, the value of the bond will equal its face value.

To see this, first note that immediately after the last reset date t_{n-1}, the bond is equivalent to a zero-coupon bond with a coupon rate equal to the market interest rate for the last coupon period.

By definition of that market interest rate, the time t_{n-1} value of the bond will be exactly equal to the face value N. In mathematical terms, the market discount factor to apply for the discounting of time t_n payments back to time t_{n-1} is $(1 + \delta l(t_n, t_{n-1}))^{-1}$.

The time t_{n-1} value of a payment $N(1 + \delta l(t_n, t_{n-1}))$ at time t_n is precisely N.

Immediately after the next-to-last reset date t_{n-2}, we know that we will receive a payment of $N\delta l(t_{n-1}, t_{n-2})$ at time t_{n-1} and that the time t_{n-1} value of the following payment (received at t_n) equals N. We therefore have to discount the sum

$$N\delta l(t_{n-1}, t_{n-2}) + N = N(1 + \delta l(t_{n-1}, t_{n-2}))$$

from t_{n-1} back to t_{n-2}. The discounted value is exactly N. Continuing this procedure, we get that immediately after a reset of the coupon rate, and the floating rate note is valued at par.

We can also derive the value of the floating rate bond between two payment dates. Suppose we are interested in the value at some time t between t_0 and t_n. Introduce the notation

$$i(t) = min\{i\{1,\ 2,\ \ldots, n\} \,|\, t_i > t\},$$

so that $t_i(t)$ is the nearest following payment date after time t. We know that the following payment at time $t_{i(t)}$ equals $N\delta l(t_{i(t)},\ t_{i(t)-1})$ and that the value at time $t_{i(t)}$ of all the remaining payments will equal N. The value of the bond at time t will then be

$$B^{fl}(t) = N\left(1 + \delta l(t_{i(t)}, t_{i(t)-1}\right) p(t, t_{i(t)}), \quad t_0 \leq t < t_n$$

where $p(t, t_{i(t)})$ is the market price of a zero-coupon bond with nominal amount N at time t with maturity at time $t(i)$. This expression also holds at payment dates $t = t_i$, where it results in N, which is the value excluding the payment at that date.

Inverse floaters pay a variable coupon rate that changes in direction opposite to that of short-term interest rates. An inverse floater subtracts the benchmark from a set coupon rate. For example, an inverse floater that uses LIBOR as the underlying benchmark might pay a coupon rate of a certain percentage, say 6%, minus LIBOR.

4.1.5 FRA – Forward Rate Agreements

A **forward rate agreement (FRA)** is an OTC derivative that trades as part of the money markets. It is essentially a forward-starting loan, but with no exchanges of principal, so that only the difference in interest rates is traded. So FRAs are *off-balance sheet* instruments. By trading today at an interest rate that is effective at some point in the future, FRAs can be used to hedge future interest rate exposure. They may also be used to speculate on the level of future interest rates.

An FRA is therefore an agreement to borrow or lend a notional cash sum for a period of time, at a pre-specified fixed rate of interest (the FRA rate). The "buyer" of an FRA is borrowing a notional sum of money and paying the agreed fixed rate while the "seller" is lending this cash sum. Note that, in the FRA market, to "buy" is to "borrow". The notional sum is the amount on which interest payment is calculated.

Many banks and large corporations will use FRAs to hedge future interest or exchange rate exposure. The *buyer* of an FRA (the borrower of the notional) will be protected (hedged) against rising interest rates

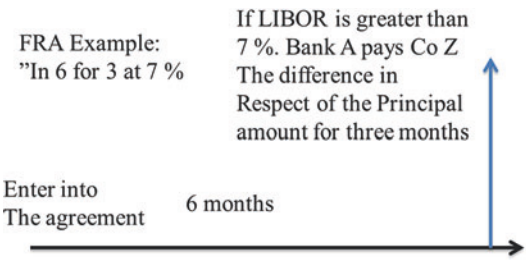


Fig. 4.7 An FRA "In 6 for 3 at 7 %".

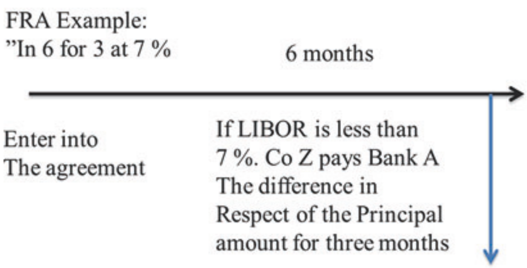


Fig. 4.8 An FRA "In 6 for 3 at 7 %".

between the date that the FRA is traded and the date that the FRA comes into effect. If there is a fall in interest rates, the buyer must pay the difference between the rate at which the FRA was traded and the actual rate, as a percentage of the notional. The seller hedges against the risk of falling interest rates. Other parties that use forward rate agreements are speculators purely looking to make bets on future directional changes in interest rates.

Since there isn't any delivery of the underlying loan amount, the contract can be considered as a CFD, contract for difference. One may also say that FRA is an interest rate swap with only one payment at maturity.

For example Bank A may agree to fix the PIBOR (The Paris Interbank Offer Rate) "in 6 for 3 at 7 %" for Company Z. This means that, if in six months' time the PIBOR exceeds 7%, A will pay Z the difference (Fig. 4.7).

On the other hand if PIBOR is less than 7% Z will pay A the difference (Fig. 4.8).

The following standard terms are used in the market.

- **Notional**: The amount for which the FRA is traded.
- **Trade date**: The date on which the FRA is dealt.

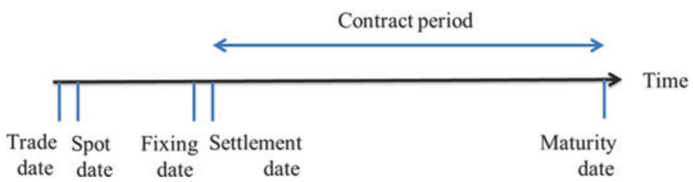


Fig. 4.9 The FRA contract period definition.

- **Settlement date**: The date on which the notional loan or deposit of funds becomes effective, that is, is said to begin. This is also called the **effective date**.
- **Fixing date**: This is the date on which the *reference rate* is determined, that is, the rate to which the FRA dealing rate is compared.
- **Maturity date**: The date on which the notional loan or deposit expires.
- **Contract period**: The time between the settlement date and maturity date.
- **FRA rate**: The pre-specified fixed interest rate at which the FRA is traded.
- **Reference rate**: This is the rate used as part of the calculation of the settlement amount, usually the LIBOR rate on the fixing date for the contract period in question.
- **Settlement sum**: The amount calculated as the difference between the FRA rate and the reference rate as a percentage of the notional sum, paid by one party to the other on the settlement date (Fig. 4.9).

As we have seen, a forward rate agreement (FRA) is a type of forward contract on short-term deposits, determined on the basis of a short-term interest rate, referred to as the Reference rate, over a predetermined time period at a future date. Typical, the reference is the forward LIBOR rate for the period $[T_1, T_2]$ contracted time $t, L(t, T_1, T_2)$.

The netted payment made at the effective date is

$$P = N \cdot \tau \cdot \frac{L(t, T_1, T_2) - X}{1 + \tau \cdot L(t, T_1, T_2)}$$

where X is the pre-specified contracted rate (strike) to be paid at maturity T_2, and $\tau = T_2 - T_1$, (the actual number of days in the interval divided by 360). In the new *Multiple Curve Framework* (see below) the formula is better expressed as

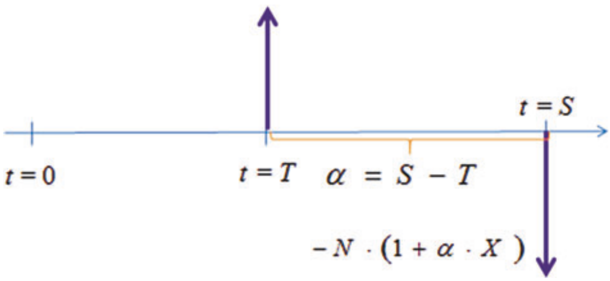


Fig. 4.10 An FRA with both cash flows.

$$P = N \cdot \tau \cdot [L_\tau(t, T_1, T_2) - X] \cdot p_D(t, T_2)$$
$$L_\tau(t, T_1, T_2) = \frac{1}{\tau}\left(\frac{p_\tau(t, T_1)}{p_\tau(t, T_2)} - 1\right)$$

so

$$P = N \cdot \left[\frac{p_\tau(t, T_1)}{p_\tau(t, T_2)} - 1 - \tau \cdot X\right] \cdot p_D(t, T_2)$$

where we discount with another curve, D. In single curve framework, the last formulae would be expressed as

$$P = N \cdot \left[p_\tau(t, T_1) - p_\tau(t, T_2)\right] \cdot (1 + \tau \cdot X)$$

Standardized (exchange traded) contracts at par ($X = L(t, T_1, T_2)$) are quoted in some markets. Typically such contracts are written as three-month contracts between IMM-days (International Money-Market days), that is, the third Wednesdays in March, June, September and December.

As we have seen, an FRA is a contract consisting of a synthetic forward-starting loan. The cash flows of the loan can be described as in Fig. 4.10.

Here N is the notional amount, T is the settlement time, S the maturity and X the forward strike rate. Since we only have one payment, the difference between the forward interest rate F, fixed at time $t = T$ and the strike rate X, we can illustrate the payment in Fig. 4.11.

However, since the payment is known at $t = T$ it is also paid at this time, as illustrated in Fig. 4.12.

Here we must discount the cash flow with the forward rate between the maturity date and the pay date.

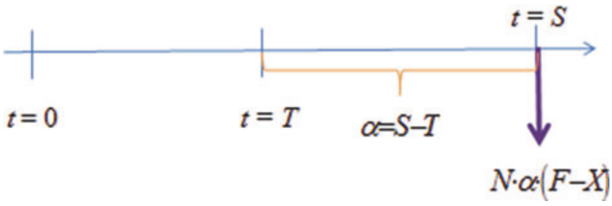


Fig. 4.11 An FRA with the maturity cash flow.

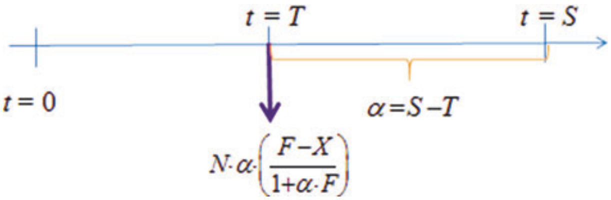


Fig. 4.12 An FRA with the initial cash flow.

To illustrate that we still have risk at $t = S$ we do the following calculation. The cash flow at $t = T$ is

$$CF(T) = N \cdot \alpha \cdot \left(\frac{F - X}{1 + \alpha \cdot F} \right)$$

where F is the forward rate between T and S observed at time t, that is,

$$F = F(t, T, S) = -\frac{\ln p(t, T, S)}{S - T} \equiv -\frac{\ln \left[p(t, S)/p(t, T) \right]}{S - T} = \{t = 0\}$$

$$= -\frac{\ln \left[e^{-r(S) \cdot S + r(T) \cdot T} \right]}{S - T} = \frac{1}{\alpha} (r(S) \cdot S - r(T) \cdot T)$$

Here we have used continuous compounding of the interest rate. $p(t, T, S)$ is the forward discount function between T and S observed at t. $p(t, S)$ and $p(t, T)$ are the discount factors time t to S and T respectively. $r(S)$ and $r(T)$ represent the zero-coupon rates at time S and T observed at time t.

Using the forward rate above, the cash flow can be expressed as

$$CF(T) = N \cdot \left(\frac{r(S) \cdot S - r(T) \cdot T - \alpha \cdot X}{1 + r(S) \cdot S - r(T) \cdot T} \right)$$

To find the net present value of the cash flow we need to discount the cash flow as

Table 4.3 FRA contract notation

Notation	Date from now	Maturity from now	Underlying Rate
1 × 4	1 month	4 months	4-1 = 3months LIBOR
1 × 7	1 month	7 months	7-1 = 6months LIBOR
3 × 6	3 months	6 months	6-3 = 3months LIBOR
3 × 9	3 months	9 months	9-3 = 6months LIBOR
6 × 12	6 months	12 months	12-6 = 6months LIBOR
12 × 18	12 months	18 months	18-12 = 6months LIBOR

$$PV_{CF(T)} = N \cdot e^{-r(T)\cdot T} \cdot \left(\frac{r(S)\cdot S - r(T)\cdot T - \alpha \cdot X}{1 + r(S)\cdot S - r(T)\cdot T} \right)$$

Similarly, we can study the equivalent cash flow at time S

$$CF(S) = N \cdot \alpha \cdot (F - X) = N \cdot (r(S)\cdot S - r(T)\cdot T - \alpha \cdot X)$$

with the present value (discounted from time S)

$$PV_{CF(S)} = N \cdot e^{-r(S)\cdot S}\,(r(S)\cdot S - r(T)\cdot T - \alpha \cdot X)$$

We can easily see that $PV_{CF(S)} \equiv PV_{CF(T)}$ from

$$e^{-r(T)\cdot T} = \frac{e^{-r(S)\cdot S}}{1 + r(S)\cdot S - r(T)\cdot T} = \frac{e^{-r(S)\cdot S}}{1 + \alpha \cdot F(T,S)}$$

From this analysis we see that we have zero-coupon risk in two nodes ($t = T$ and $t = S$) in an FRA contract until we reach time T.

OTC FRA deposit contracts sometimes use the notation as seen in Table 4.3.

FRA are also used when bootstrapping the interest rate curve (see below)

Example 4.1.5.1

Hedging an FRA with a future. Below, we will study a hedging situation where we will hedge an FRA with a future contract.

An FRA market-maker sells a EUR 100 million 3-v-6 FRA, that is, an agreement to make a notional deposit (without exchange of principal) for three months in three months' time, at a rate of 7.52%. He is exposed to the risk that interest rates will have risen by the FRA settlement date in three months' time. He wants to hedge this with a number of matching futures contracts. The question is, how many should he buy?

Date	14 December
3-v-6 FRA rate	7.52%
March futures price	92.50%
Current spot rate	6.85%

Action: The dealer first needs to calculate a precise hedge ratio. This is a three-stage process:
In the first stage we calculate the nominal value of a bp move in LIBOR on the FRA settlement payment

$$BPV = FRA_{nom} \cdot 0.01\% \cdot \frac{n}{360}$$

Therefore: $N_{bpv} = €100,000,000 \times 0.01\% \times 90/360 = €2500.$

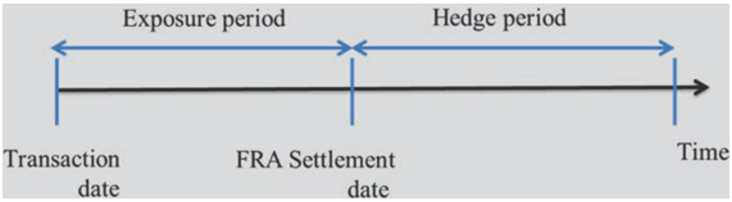
In the next stage we calculate the present value of 1. By discounting it back to the transaction date using the FRA and spot rates. Present value of a bp move is given by

$$PV01 = \frac{N_{bpv}}{\left(1 + r_{spot} \cdot \frac{d}{360}\right) \cdot \left(1 + r_{FRA} \cdot \frac{d}{360}\right)}$$
$$= \frac{2500}{\left(1 + 0.0685 \cdot \frac{90}{360}\right) \cdot \left(1 + 0.0752 \cdot \frac{90}{360}\right)} = 2412.55$$

The final stage consist of determine the correct hedge ratio by dividing *PV*01 by the futures tick value.

Hedge Ratio = 2413/25 = 96.52.

The appropriate number of contracts for the hedge of a EUR 100,000,000 3-v-6 FRA would therefore be 97. To hedge the risk of an increase in interest rates, the trader sells 97 EUR three months' futures contracts at 92.50. Any increase in rates during the hedge period should be offset by the gain realized on the futures contracts through daily variation margin receipts.



Outcome	
Date	15 March
Three month LIBOR	7.625%
March EDSP	92.38

The hedge is lifted upon expiry of the March futures contracts. Three-month LIBOR on the FRA settlement date has risen to 7.625% so the trader incurs a loss of EUR 25,759 on his FRA position (i.e. EUR 26,250 discounted back over the three month FRA period at current LIBOR rate), calculated as follows:

$$N \cdot \frac{LIBOR_{FRA} \cdot (d/360)}{1 + LIBOR \cdot \frac{d}{360}} = 100{,}000{,}000 \cdot \frac{0.00105 \cdot (90/360)}{1 + 0.07625 \cdot \frac{90}{360}} = 25{,}759$$

Futures P/L: 12 ticks $(92.50 - 92.38) \times €25 \times 97$ contracts $=$ EUR29, 100. The EUR 25,759 loss on the FRA position is more than offset by the EUR 29,100 profit on the futures position when the hedge is lifted. If the dealer has sold 100 contracts his futures profit would have been EUR 30,000, and, accordingly, a less accurate hedge. The excess profit in the hedge position can mostly be attributed to the arbitrage profit realised by the market maker (i.e. the market maker has sold the FRA for 7.52% and in effect bought it back in the futures market by selling futures at 92.50 or 7.50% for a two tick profit.)

4.1.6 Interest Rate Futures

An interest rate future is a *futures contract* with an interest-bearing instrument as the underlying asset. Buying an interest rate futures contract allows the buyer of the contract to lock in a future investment rate; not a borrowing rate as many believe. Being long an IR future means you have agreed to receive a rate at a certain period in the future.

Interest rate futures are based on an *underlying security* which is a debt obligation and moves in value as interest rates change. Typical contracts are EuroDollar futures and Euribor futures.

EuroDollars are USD deposited in banks outside the United States, and thus are not under the jurisdiction of the Federal Reserve. Futures on Euribor are similar contracts in Euro.

A single future is similar to a *forward rate agreement* to borrow or lend a nominal amount for a time (typical three months) starting on the contract settlement date. Buying the contract is equivalent to lending money, and selling equivalent to borrowing money.

The futures contracts are traded with delivery at IMM. The interest rate underlying the contract is the interest rate typically applicable to a 91-day period. The contracts are settled in cash on the second London business day before the IMM day.

The relation between Q, the quoted price, and P, the contract price (in points of 100%), is given by

$$P = 100 - \alpha \cdot (100 - Q)$$

where α is the day-count fraction (Act/360).

As the *futures contract* refers to cash settled financial futures contract based upon the LIBOR rate on expiry, they can be used to hedge future interest rate exposures and are quoted in price. A quoted price of 95.00 implies an interest rate of 100.00–95.00, or 5%. The settlement price of a contract is defined to be 100.00 minus the official British Bankers Association (BBA) fixing of 3-month LIBOR/Euribor on the day the contract is settled.

On IMM, the actual interest rate for the period is known and the contract is settled in cash. The final marking to market sets the futures price equal to $100 - R$, where R is the interest rate expressed with quarterly compounding and an actual/360 day-count convention.

1. When interest rates move higher, the buyer of the futures contract will pay the seller an amount equal to that of the benefit received by investing at a higher rate versus that of the rate specified in the futures contract. Conversely, when interest rates move lower, the seller of the futures contract will compensate the buyer for the lower interest rate at the time of expiration.
2. To accurately determine the gain or loss of an interest rate futures contract, an interest rate futures price index was created. When buying, the index can be calculated by subtracting the futures interest rate from 100, or (100 – futures interest rate). As rates fluctuate, so does this price index. You can see that as rates increase, the index moves lower and vice versa.
3. Typically, the interest rate futures contract has a base price move (tick) of 0.01, or 1 bp. However, some contracts have a tick value of 0.005 or half of 1 bp. For example, for Eurodollar contracts, a tick is worth $12.50 and a move from 94 to 94.50 would result in a $1250 gain per contract for someone who is long the futures.
4. Interest rate futures contracts, when used in conjunction with the duration measure of fixed income instruments, can be used to hedge a company's risk exposure to interest rate movements.

Common IR futures contracts include Treasury bond and Eurodollar futures contracts that trade in the United States. Interest rate futures in the US markets are traded on the CME (Chicago Mercantile Exchange). Euribor futures are typically traded at LIFFE (London International Financial future and Option Exchange).

The face value of an IR-future is calculated as follows

$$\mathit{Face\ Value} = (1 - r \cdot \alpha) \cdot \mathit{Contract\ Size}$$

where r is the annualized forward interest rate and α the length of the deposit period in years. The quote converted to a price is given by

$$\mathit{Price} = 100(1 - r)$$

For example, if a EuroDollar future is quoted at 94.25, this corresponds to an interest rate of 5.75%.

Interest rate futures are priced as

$$PV(t) = N \cdot \alpha \cdot \left(\frac{P(t) - X}{100} \right) \cdot D(T_M)$$

where

$$P(t) = 100 \cdot (1 - F(t, T_S, T_M))$$

and

- N the notional amount,
- α the tenor (time between the maturity and settlement): $T_M - T_S$,
- X the strike rate in the agreed time period,
- $D(T_M)$ the discount-factor to maturity T_M and
- $F(t, T_S, T_M)$ the market reference forward rate for the time between Settlement and Maturity.

IR-futures are quoted in price, based on the expectation in the forward rate. Hence, the sensitivity in the rate is calculated as an increase in the zero-rate by one bp. The relationship between the forward rate and the zero rates (using continuous compounding) is given by

$$F(t, T_S, T_M) = r(t, T_M) + \{r(t, T_M) - r(t, T_S)\} \cdot \frac{T_S}{\alpha}$$

An interest rate future is a contract between a party such as a bank or investor and an exchange. It is a way to fix an interest rate for a nominal amount for a period in the future.

In Fig. 4.13, we illustrate an interest future with a fixed rate of 4% that is agreed for a nominal of \$100 on a period starting in six months

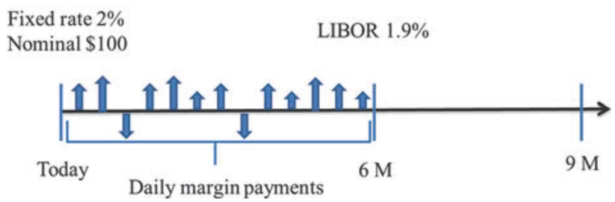


Fig. 4.13 An example of interest rate future.

and ending in nine months. The reference rate that the fixed rate will be compared against is the LIBOR.

At the start date (six months), the agreed fixed rate (4%) is higher than the reference rate (3.9%). This means money must be paid to the exchange. The amount is given by

$$\text{Interest rate differential} \times \text{Nominal amount} \times \text{Time}$$
$$= (4 - 3.9) \times 100 \times 3/12 = £10.$$

As is typical for futures, daily margin payments (marking to the market) must be paid to or received from the exchange to effectively close out the position each day (so that no credit risk arises for either party).

On most exchanges (e.g. Eurex, LIFFE, CBOT) interest rate futures are quoted using type 100 – Rate. If you trade a future at price 96, you actually have locked in an interest rate of 4%.

Interest rate futures are traded with standardized expiry dates, the IMM dates. Expiry is the Monday preceding the third Wednesday every last month every quarter, that is, March, June, September and December.

4.1.7 Interest Rate Bond Futures and CTD

As we have seen, financial future contracts are contracts to either sell or buy a certain underlying financial asset on a specified future date at a fixed price or rate. Furthermore, they are exchange traded with daily settlement.

We will here study the so-called bond futures. Usually the underlyings are one (or more) specific government bonds. Usually the underlying asset is an index calculated from the prices of one (or more) specific government bonds. Since different futures on the different markets have different names (EUR-Bund future, US treasury bond future, etc.) we will use bond future as a synonym for a future on a medium- or a long-term bond.

In a bond futures contract, the underlying asset is a synthetic bond with a defined term and defined coupon. The advantage of this synthetic bond over an actual bond is that the futures price can be better compared over time.

Example 4.1.7.1

The underlying asset of a EUR-Bund future is a synthetic bond with a 10-year term and a 6% coupon. The T-bond (note) futures underlying specifications are 30 and 10 years respectively, both with a 6% coupon.

The buyer of a bond future contract is obliged to buy the underlying bond at a fixed price on an agreed date. Because the prices of bonds rise when interest rates fall, a purchased future contract can be used to speculate on falling interest rates.

The seller of a bond future is obliged to deliver the underlying bond at a fixed price on an agreed date. Because the prices of bonds fall when interest rates rise, a sold future can be used to speculate on rising interest rates or to secure existing short positions against rising interest rates.

As with Money-Market futures, a tick is the minimum price movement of a futures contract. In contrast to Money-Market futures where a tick is typically one hundredth of 1%, long-term futures like T-bond futures, sometimes move in 1/32 of 1% (i.e. 0,0003125 or 3.125 bps) or 1/64 of 1% (i.e. 0,00015625 or 1.5265 bps). The tick size is typically defined according to the quoting conventions of the underlying bond. For example, EUR-Bunds are quoted in decimals on 1 bp, thus the tick value of the Bund-future is 1 bp. A tick has always an exactly defined value in relation to the contract; the tick value is the product of the contract value times the bps of a tick. The tick value of a EUR-Bund future and a 10-y T-note future respectively are

EUR-Bund future:	$100000 \times 0.0001 = \text{EUR } 10$
10-year T-note future:	$100000 \times 0.00015625 = \text{USD } 15.625$

Contrary to Money-Market futures, bond futures are delivered physically if they have not been closed out prior to delivery date. The delivery of the futures contract must tackle the problem that the underlying bond is a synthetic instrument. Therefore, the seller can deliver from a basket of bonds. The settlement price is determined by means of a conversion factor (or price factor) that makes the price of the synthetic bond comparable to the price of the deliverable bond.

Since the deliveries consist of a basket of underlying bonds, it is important to calculate which of the deliverable bonds is the cheapest to deliver. This is called Cheapest-To-Deliver (CTD). Bond futures are quoted as clean prices, exchange traded and any gains or losses during the lifetime of the contract are settled daily via each participant's variation margin account.

4.1.7.1 Spot Based Forwards

First, we study a traditional spot price based forward. Let us look at a picture of the cash flow for a couple of transactions (see Fig. 4.14).

t_m : Market date for the future/forward, that is, today.
t_c : Date of a coupon, if any, before delivery date.
t_e : Delivery date for the future/forward contract.
t_{ck} : The time for the first coupon after the delivery date.
C_{me} : The coupon, if any that, occurs between t_m and t_e, if no coupon occurs $C_{me} = 0$.
R_{kl} : The market interest rate for the interval between the time t_k and t_l.
Δt_{kl} : The time interval between t_k and t_l expressed in actual days.
A_m : Accrued interest for the bond at time t_m.
P_m : Spot bond price expressed as the clean price at delivery, t_m.
F_{me} : Forward prices contracted at t_m and valid on t_e.

When setting the price of the forward contract, we start by looking at the cash flows. First, we have the underlying bond, where the value consists of the clean spot price P_m and the accrued interest rate A_m. The sum $P_m + A_m$ is the price. We use the clean price and not the price

Fig. 4.14 A time view of a spot price based future.

since bonds are quoted as clean. To get the forward value we multiply with the inverse discount factor:

$$[P_m + A_m] \cdot \left(1 + \frac{\Delta t_{me}}{360} R_{me}\right)$$

Here we use the money-market discount factor since Δt_{me} is less than a year. The sum of P_m and A_m, is exactly the amount needed to purchase a bond in the spot market and sell a forward position at price F_{me} in the future.

Next, we have to consider the coupon, if any. The present value of a coupon at time t_c is

$$-\frac{C_{me}}{\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)}$$

Since this coupon should be returned at t_c the value is negative. At the maturity this corresponds to

$$-\frac{C_{me}}{\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)} \cdot \left(1 + \frac{\Delta t_{me}}{360} R_{me}\right)$$

Finally, we have accrued interest rate at delivery, A_e. Therefore, the total price of the forward must be

$$F_{me} = [P_m + A_m] \cdot \left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) - C_{me} \cdot \left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right) - A_e$$

or

$$F_{me} = \left\{ P_m + A_m - \frac{C_{me}}{\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)} \right\} \cdot \left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) - A_e$$

In exchange for this we deliver the bond at maturity.

Above, we have used the fundamental connections between interest rates

$$\left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)\left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right) = \left(1 + \frac{\Delta t_{me}}{360} R_{me}\right)$$

The forward rates above are calculated from the known term structure.

Example 4.1.7.2

Bond Forward

Future data:		
Market date:	2016-01-03	
Delivery date:	2016-03-18	
Interest rate to delivery:	5.55%	
Interest rate of coupon:	5.80%	
Bond maturity:	2017-01-21	
Bond coupon:	11.0%	
Yield-To-Maturity	6.0%	
Bond dirty price:	115.380 (Calculated, $P_m + A_m$)	
Days to next coupon:	18	(Calculated)
Days to delivery:	75	(Calculated)

The accrued interest at delivery date is:

$$A_e = 11 \cdot \frac{75 - 18}{360} = 1.742.$$

Then

$$\begin{aligned} F_{me} &= \left[P_m + A_m - C_{me} \left(1 + \frac{\Delta t_{mc}}{360} R_{mc} \right)^{-1} \right] \cdot \left(1 + \frac{\Delta t_{me}}{360} R_{me} \right) - A_e \\ &= \left[115.380 - 11 \left(1 + \frac{18}{360} 0.0580 \right)^{-1} \right] \cdot \left(1 + \frac{75}{360} 0.0555 \right) - 1.742 \\ &= 103.877 \end{aligned}$$

103.877 would be the clean forward price.

4.1.7.2 Implied Repo Rate for Forwards

As a complement to this way of calculating forward price we also want to calculate the "implied repo rate" for a forward contract. This means that we already have the forward price and want to know the rate R_{me} in an effort to deduce arbitrage opportunities. From above, we have

$$F_{me} + A_e - \left\{ P_m + A_m - \frac{C_{me}}{\left(1 + \frac{\Delta t_{mc}}{360} R_{mc} \right)} \right\} \cdot \left(1 + \frac{\Delta t_{me}}{360} R_{me} \right) = 0$$

or

$$\begin{aligned} &\left\{ P_m + A_m - C_{me} \cdot \left(1 + \frac{\Delta t_{mc}}{360} R_{mc} \right)^{-1} \right\} \cdot \frac{\Delta t_{me}}{360} R_{me} \\ &\quad = F_{me} + A_e - P_m - A_m + C_{me} \cdot \left(1 + \frac{\Delta t_{mc}}{360} R_{mc} \right)^{-1} \end{aligned}$$

so we get the implied repo rate as

$$R_{me} = \frac{360}{\Delta t_{me}} \cdot \frac{F_{me} + A_e - P_m - A_m + C_{me} \cdot \left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}}{P_m + A_m - C_{me} \cdot \left(1 + \frac{\Delta t_{mc}}{360} R_{mc}\right)^{-1}}$$

We can also do the following calculations

$$F_{me} - [P_m + A_m] \cdot \left(1 + \frac{\Delta t_{me}}{360} R_{me}\right) + C_{me} \cdot \left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right) + A_e = 0$$

or

$$[P_m + A_m] \cdot \frac{\Delta t_{me}}{360} R_{me} = F_{me} + A_e + C_{me} \cdot \left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right) - [P_m + A_m]$$

and express the implied repo rate as

$$R_{me} = \frac{360}{\Delta t_{me}} \cdot \frac{F_{me} + A_e - P_m - A_m + C_{me} \cdot \left(1 + \frac{\Delta t_{ce}}{360} R_{ce}\right)}{P_m + A_m}$$

Observe that the two equations above are equivalent, and the choice of which one to use is a question about how we easiest handle interest rates.

Example 4.1.7.3

Future data:	
Market date:	2016-01-03
Delivery date:	2016-03-18
Interest rate to delivery:	5.55%
Interest rate of coupon:	5.80%
Bond maturity:	2017-01-21
Bond coupon:	11. 0%
Bond dirty price:	115.380
Days to next coupon:	18
Days to delivery:	75
Forward price:	103.877 (calculated in previous example)

$$R_{me} = \frac{360}{75} \cdot \frac{103.877 + 1.742 - 115.380 + 11 \cdot \left(1 + \frac{18}{360} 0.058\right)^{-1}}{115.380 - 11 \cdot \left(1 + \frac{18}{360} 0.058\right)^{-1}} = 5.5481$$

4.1.7.3 Futures

For the forward case we have a known deliverable bond. In most futures we have a collection of deliverable bonds in which the seller is free to pick anyone. Except for this collection we have the daily mark-to-market that gives us a daily settlement amount that is to be administrated for each trading day. This administrative task makes Futures exchange traded instruments. Except for these differences the actual construction is very similar to that of forwards.

In Table 4.4 we are given the most common bond future contracts

Table 4.4 Some of the most common bond future contracts

Contract	UST-Bond	B1md	UK Gilt	Fr Notionel
Delivery months	Mar (H), Jun (M), Sep (U), Dec (Z)	Mar (H), Jun (M), Sep (U), Dec (Z)	Mar (H), Jun (M), Sep (U), Dec (Z)	Mar (H), Jun (M), Sep (U), Dec (Z)
Quotation	Percentage	Percentage	Percentage	Percentage
Contract size	$100000	€ 100000	£ 100000	€ 100000
Coupon	8%	6%	7%	3.50%
Tick size	1/32 = $31.25	0.01 = €10	0.01 = £10	0.01 = €10
Last trading-day	7 business day prior to the delivery day	2 business day prior to the delivery day	2 business day prior to the delivery day	2 business day prior to the delivery day

Futures on Eurex

On Eurex they use the following contract standards (Table 4.5). They are short-, medium- or long-term debt instruments issued by the Federal Republic of Germany, the Republic of Italy, the Republic of France or the Swiss Confederation.

The Contract Values are EUR 100,000 or CHF 100,000.

Table 4.5 Standard future contracts on Eurex

Contract	Product ID	Remaining Term in Years	Coupon Percent	Currency
Euro-Schatz futures	FGBS	1.75 to 2.25	6	EUR
Euro-Bobi futures	FGBM	4.5 to 5.5	6	EUR
Euro-Bund futures	FGBL	8.5 to 10.5	6	EUR
Euro-Buxl futures	FGBX	24.0 to 35.0	4	EUR
Short-Term Euro-BTP futures	FBTS	2 to 3.25	6	EUR
Mid-Term Euro-BTP futures	FBTM	4.5 to 6	6	EUR
Long-Term Euro-BTP futures	FBTP	8.5 to 11	6	EUR
Mid-Term Euro-OAT futures	FOAM	4.5 to 5.5	6	EUR
Euro-OAT futures	FOAT	8.5 to 10.5	6	EUR
CONF futures	CONF	8.0 to 13.0	6	CHF

On settlement there is a delivery obligation arising out of a short position that may only be fulfilled by the delivery of certain debt securities issued by the Federal Republic of Germany, the Republic of Italy, the Republic of France or the Swiss Confederation with a remaining term on the Delivery Day within the remaining term of the underlying. Settlement of debt securities issued by the Republic of France in case of physical delivery will be done via Clearstream Banking Luxemburg.

Debt securities issued by the Federal Republic of Germany must have an original term of no longer than 11 years.

Debt securities issued by the Republic of Italy must have an original term of no longer than 16 years (only for Long-Term Euro-BTP futures).

Debt securities issued by the Republic of France must have an original term of no longer than 17 years.

In the case of callable bonds issued by the Swiss Confederation, the first and the last call dates must be between eight and 13 years.

Debt securities must have a minimum issue amount of EUR 5 billion, such issued by the Republic of Italy no later than 10 exchange days prior to the Last Trading Day of the current maturity month, otherwise, they shall not be deliverable until the delivery day of the current maturity month.

Debt securities issued by the Swiss Confederation must have a minimum issue amount of CHF 500 million.

The Price Quotation, the Minimum Price Change and The Price Quotation in percent of the par value are shown in Table 4.6.

Contract Months

Up to 9 months: The three nearest quarterly months of the March, June, September and December cycle.

Table 4.6 Quotation of future contracts

Contract	Minimum Price Change Percent	Minimum Price Change Value
Euro-Schatz futures	0.005	EUR 5
Euro-Bobl futures	0.01	EUR 10
Euro-Bund futures	0.01	EUR 10
Euro-Buxl®futures	0.02	EUR 20
Short-Term Euro-BTP futures	0.01	EUR 10
Mid-Term Euro-BTP futures	0.01	EUR l0
Long-Term Euro-BTP futures	0.01	EUR 10
Mid-Term Euro-OAT futures	0.01	EUR 10
Euro-OAT futures	0.01	EUR 10
CONF futures	0.01	CHF 10

Delivery Day

The tenth calendar day of the respective quarterly month, if this day is an exchange day; otherwise, the exchange day immediately succeeding that day.

Notification

Clearing members with open short positions must notify Eurex on the Last Trading Day of the maturing futures which debt instrument they will deliver. Such notification must be given by the end of the Post-Trading Full Period.

Last Trading Day

Two exchange days prior to the Delivery Day of the relevant maturity month. Close of trading in the maturing futures on the Last Trading Day is at 12:30 CET.

Daily Settlement Price

The Daily Settlement Prices for the current maturity month of CONF futures are determined during the closing auction of the respective futures contract.

For all other fixed income futures, the Daily Settlement Price for the current maturity month is derived from the volume-weighted average of the prices of all transactions during the minute before 17:15 CET (reference point), provided that more than five trades transacted within this period.

For the remaining maturity months the Daily Settlement Price for a contract is determined based on the average bid/ask spread of the combination order book.

Final Settlement Price

The Final Settlement Price is established by Eurex on the Final Settlement Day at 12:30 CET based on the volume-weighted average price of all trades during the final minute of trading provided that more than 10 trades occurred during this minute; otherwise the volume-weighted average price of the last 10 trades of the day, provided that these are not older than 30 minutes. If such a price cannot be determined, or does not reasonably reflect the prevailing market conditions, Eurex will establish the Final Settlement Price.

Example 4.1.7.4

Bond Future

We follow a case to get the feeling for the calculations. In this set-up we have sold 100 future contracts. We position us at expiration of the future.

Market date:	2016-03-13	
Delivery date:	2016-03-18	
Contract price:	97.454	
Last fix:	97.465	
Nominal amount:	1 000 000 SEK	

Deliverable bond A

Bond expire date:	2024-10-25	
Bond coupon:	6.500%	
Days to coupon:	217	(Calculated)
Bond YTM :	6.833%	
Clean price:	96.548	(Calculated)
Accrued interest:	2.492	(Calculated)

Deliverable bond B

Bond expire date:	2026-05-05	
Bond coupon:	6.500%	
Days to coupon:	48	(Calculated)
Bond YTM:	6.352%	
Clean price:	96.307	(Calculated)
Accrued interest:	5.525	(Calculated)

Deliverable bond C

Bond expire date:	2027-04-20	
Bond coupon:	9.000%	
Days to coupon:	34	(Calculated)
Bond YTM:	6.386%	
Clean price:	114.843	(Calculated)
Accrued interest:	8.075	(Calculated)

First of all the future should be marked-to-market as usual.

$$MM = (97.465 - 97.454) \times 1\,000 \times 100 = 1100$$

This amount should be exchanged before delivery procedure takes over. After this point the future price for all positions is 97.465. Assume that we have to deliver a bond among the deliverable and hand over without adjusting for accrued interest. Therefore we want to buy a bond at the spot market and deliver. The actual profit from receiving the futures fix and deliver the newly bought bond is

$$\left.\begin{array}{ll} A: & (97.465 - (96.548\text{ă} + 2.492\text{ă})) \cdot 1000 \cdot 100 = -157500 \\ B: & (97.465 - (96.307 + 5.525)) \cdot 1000 \cdot 100 \quad = -436700 \\ C: & (97.465 - (114.843\text{ă} + 8.075)) \cdot 1000 \cdot 100 = -2545300 \end{array}\right\}$$

Therefore we would buy and deliver bond A. But, if I am a very fast customer I could have bought that future at the fix price and since we have a net loss for all the deliverable bonds, I have actually made an arbitrage. This should be impossible in a developed market! In practical terms this means that the future price should clearly converge to the dirty spot price for the cheapest deliverable bond. We also see that since different bonds have their coupon at different times the accrued interest is a problem. If we were to use the dirty price directly, this would be taken care of, but since the spot market in most countries is denoted in clean price we chose to explicitly include the accrued interest.

4.1.7.4 Introducing Price Factors

If we only had one deliverable bond then the buyer of the future would be **forced** to buy this bond. This could lead to a situation with acute shortage of supply and increasing prices. The market solution is to permit delivery of several bonds.

Therefore we have to find a method to choose the bond to deliver, that is, a factor that we can multiply with the futures fix price, to narrow the profit/loss from delivering different bonds. These factors are calculated prior to the start of trading the future. If we could multiply the fix with something that is near the bond price with a nominal amount of one, we would be quite satisfied.

To calculate the CDT we therefore introduce a conversion factor, on LIFFE[2] also called price factor P_f. This price factor times the price of the future is equal to the clean price of the bond on the delivery day.

Recall the clean price of a bond with one coupon, C per year with yield to maturity y and face value 100

$$PV = \frac{1}{(1+y)^{d/360}} \cdot \left\{ \frac{100}{(1+y)^T} + \sum_{t=0}^{T} \frac{C}{(1+y)^t} \right\}$$

---

2 The London International Financial futures and Option Exchange.

Here T is time to maturity in whole years and d is the number of days until the next coupon. The price factor is then given by $P_f = PV/100$ – accrued interest rate.

The formula of the price factor (used in Sweden) can be written as

$$P_{f_i} = \frac{1}{(1 + r_c)^{n_i + m_i/12}} \left[\frac{C_i}{r_c} \left((1 + r_c)^{n_i + 1} - 1 \right) + 1 \right] - C_i \left(1 - \frac{m_i}{12} \right)$$

where

r_c = the coupon rate of the constructed bond (usually 6.00%).

C_i = the coupon rate of the deliverable bond.

n_i = the number of whole years to maturity of the deliverable bond measured from the next receivable coupon payment for bond i.

m_i = the number of whole months to the next receivable coupon payment from the futures delivery date for bond i.

If $C_i > r_c \Rightarrow P_{fi} > 1$ the bond is traded to a high price
$C_i < r_c \Rightarrow P_{fi} < 1$ the bond is traded at par
$C_i = r_c \Rightarrow P_{fi} = 1$ the bond is traded to a low price

As one can see, the price factors are just estimates of the clean price for individual bonds with the approximation to round time to whole years and month of expiry. Not being as accurate as possible the formula would lead to a situation where all the deliverable bonds were equally profitable to deliver from start. We would then have a situation where CTD could change very rapidly. Since this could inflict negatively on the trading we use some approximations to widen the distance between them.

The formula above might be slightly different in different markets, like on CBoT, EUREX and for Japanese treasury bonds. One of the bonds will be cheaper than the others since the delivery price is the clean futures price time the price factor. Therefore, the contract will be priced upon that bond. The "built in" imperfection in the formula above is used to give a favour for one of the bonds. The reason for this is that we do not want any changes of the CTD bond when there are only small changes in the market rate.

On EUREX the conversion factor is calculated as

$$Cf = \frac{1}{(1 + r_c)^{1 + \delta_e / act_1}} \times \left[C \frac{\delta_i}{act_2} + \frac{C}{r_c} \left((1 + r_c) - \frac{1}{(1 + r_c)^n} \right) + \frac{1}{(1 + r_c)^n} \right] - C \left(\frac{\delta_i}{act_2} - \frac{\delta_e}{act_1} \right)$$

where

DD	is the Delivery Date
NCD	the Next Coupon after the Delivery date
NCD1y	1 year before the NCD
NCD2y	2 years before the NCD
LCD	the Last Coupon Date before the delivery date. Start interest period if the last coupon date is not available.
δ_e	NCD1y – DD
δ_i	NCD1y – LCD
act_1	NCD – NCD1y if $\delta_e < 0$, else NCD – NCD2y
act_2	NCD – NCD1y if $\delta_i < 0$, else NCD – NCD2y
n	Integer number of years from the NCD until the maturity date of the bond.

Using the price factors above and the individual bond data as the ongoing example give us

$$Pf_A = 1.032337$$
$$Pf_B = 1.036880$$
$$Pf_C = 1.237580$$

With these price factors we are prepared to take a second look at delivery. (Standard for these factors is to round them to six decimals.) Let us be a little bit stricter about what we receive from the futures contract.

$$(F_{Fix} \cdot P_{fi} + U_i) \cdot N \cdot n$$

where

F_{Fix}:	The futures fix price
P_{fi}:	The individual price factor for each deliverable bond.
N:	Nominal amount
U_i:	Accrued interest for each deliverable bond.
n:	Number of contracts
P_i:	Clean price for each deliverable bond.

The actual profit we get from delivering a special bond is given by

$$(F_{Fix} \cdot P_{fi} + U_i - \{P_i - U_i\}) \cdot N \cdot n$$

With this approach we get

$$\left.\begin{array}{l} A\text{: } (97.465 \cdot 1.032337 + 2.492 - (96.548 + 2.492)) \cdot 1000 \cdot 100 = 406872 \\ B\text{: } (97.465 \cdot 1.036880 + 5.525 - (96.307 + 5.525)) \cdot 1000 \cdot 100 = 475250 \\ C\text{: } (97.465 \cdot 1.237680 + 8.075 - (114.843 + 8.075)) \cdot 1000 \cdot 100 = 578748 \end{array}\right\}$$

We see that the differences between the deliverable bonds are much less now with the price factors than without.

4.1.7.5 Cheapest to Deliver

Now when we know the basic construction of the futures we are standing with all the deliverable bonds and wondering which one of them one should buy at the same time as one is selling the future. The reason for doing this investigation could be to decide whether to deliver the bond that already is in our possession or do we benefit from buying new ones and deliver those. The process of selecting the optimal bond is called determining which one is "cheapest to deliver (CTD)".

One thing that we will not discuss in the following sections is the fact that futures are marked-to-market on a daily basis. This means that we cannot perform true arbitrages with bonds and futures due to these daily cash flows. Because of this we cannot, without explanation, use today's futures price in equations describing future events. If however the futures price and the forward price are equal, we can make positions that **momentarily** can be regarded as an "arbitrage". With a more strict formulation we have that the futures price equals the forward price when we add the expected value of all the mark-to-markets for the period. For simplicity we assume that the mark-to-market is "approximately" equal to zero.

To calculate which of the deliverable bonds in the basket that is the CDT bond, we can calculate the implied repo rate for each of them, or simpler, the maximum net basis

$$CTD = \max_i \{P_{future}^{\cdot} P_f^i - P_{bond}^i\}$$

On delivery, we calculate the delivery price as

$$P = Fix \cdot P_f \cdot N \cdot n + U$$

where *Fix* is the fixing, *N* the face value, *n* number of contracts and *U* the accrued interest.

Example 4.1.7.5

Future data

Coupon:	6.0%	
Market date:	2016-01-03	
Delivery date:	2016-03-18	
Future price:	98.00	
Nominal amount:	1 000 000 SEK	
Interest rate to delivery:	4.5%	
Days to deliver:	75	(Calculated)

Deliverable bond A

Bond maturity:	2024-10-25	
Bond coupon:	6.500%	
Bond YTM:	6.750%	
Days next coupon:	292	(Calculated)
Clean price:	98.343	(Calculated)
Market date accrued interest:	1.2278	(Calculated)
Delivery date accrued interest:	2.5819	(Calculated)
Price factor:	1.0329	(Calculated)
Forward price:	97.926	(Calculated)

Deliverable bond B

Bond maturity:	2026-05-05	
Bond coupon:	6.500%	
Bond YTM:	6.697%	
Days next coupon:	123	(Calculated)
Clean price:	98.548	(Calculated)
Market date accrued interest:	4.2611	(Calculated)
Delivery date accrued interest:	5.6153	(Calculated)
Price factor:	1.0373	(Calculated)
Forward price:	98.1172	(Calculated)

Deliverable bond C

Bond maturity:	2025-04-20	
Bond coupon:	9.000%	
Bond YTM:	6.632%	
Days next coupon:	109	(Calculated)
Clean price:	118.331	(Calculated)
Market date accrued interest:	6.3250	(Calculated)
Delivery date accrued interest:	8.200	(Calculated)
Price factor:	1.2399	(Calculated)
Forward price:	97.926	(Calculated)

Deliverable bond A: $F_{fut}P_f - F_{me}$: 3.30478
Deliverable bond B: $F_{fut}P_f - F_{me}$: 3.49555
Deliverable bond C: $F_{fut}P_f - F_{me}$: 3.88736

From this data we see that the bond C is CTD.

One common way is to calculate the implied futures price (IFP) for the deliverable bonds.

Definition 4.1.7.3. The *Implied futures price* for a bond is the price that provides zero profit on the purchase, carry and delivery of a specific bond.

$$IFP = \mathrm{F_{me}}/\mathrm{P_f}$$

Here one argues that the bond with the smallest *IFP* will be CTD, for at that futures price, any other bond will, upon delivery, provide a negative profit.

Use the same data as before and calculate IFP: s.

Deliverable bond A: IFP_{A}:	94.80
Deliverable bond B: IFP_{B}:	94.63
Deliverable bond C: IFP_{C}:	94.86

This means that bond B should be CTD! Let us plot the individual profit that these bonds produce against futures price, under the assumption that forward prices are constant. (Fig. 4.15)

We see that bond C crosses the other two bonds in the plot interval. This means that we have three intervals, which we must inspect separately:

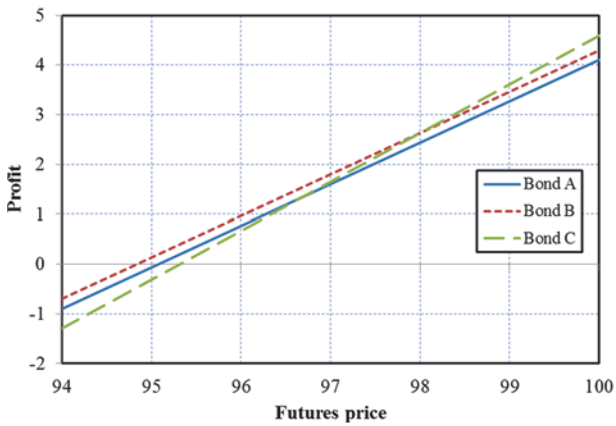


Fig. 4.15 Profit of the bonds in a CTD contract.

$F_{fut} \leq 96.069$	CTD = bond B
$96.069 < F_{fut} \leq 97.246$	CTD = bond B
$97.246 \leq F_{fut}$	CTD = bond C

We get problems with the last approach if the different IFPs are located in different intervals. If we are unlucky the calculated CTD could be the wrong one.

4.1.7.6 Duration

The modified duration for an interest rate bond futures contract is defined as the duration of the CTD bond divided by the conversion factor.

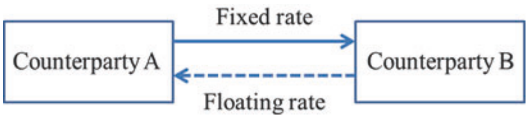
4.1.8 Swaps

The swap market are huge and swaps are the most popular fixed-income derivatives at this writing. The total notional principal amount is, in US dollars, currently comfortably in 14 digits. The market really began in 1981 although there were a small number of swap-like structures arranged in the 1970s. Initially the most popular contracts were currency swaps, but they were quickly overtaken by interest rate swaps.

Many different market participants use swaps for several purposes. One of the most common applications is the hedging of interest-rate risk by financial institutions, corporations and large institutional investors. An often-cited motivation for using swaps is the theory of comparative advantages counterparties often has different abilities for borrowing in Capital markets. This could be due to differences in credit rating or tax treatment, or for accounting reasons. Swaps can serve, for instance, to change the fixed- coupon debt into floating rate debt and vice-versa. Due to the liquidity of the swap market, they can be used to hedge the interest rate risk of fixed-income positions.

A **swap** is shortly explained as a contractual agreement between two parties in which they agree to make periodic payments to each other according to two different indices.

A **plain vanilla interest-rate swap** specifies the notional amount or face value of the swap, the payment frequency (quarterly, semi-annually, etc.) the tenor, maturity, the coupon, that is the fixed rate, and the floating rate. One party (Counterparty A) makes fixed rate payments on the stipulated notional. The other party (Counterparty B) makes floating rate payments to Counterparty A based on the same notional. Most swaps are arranged so that their net value is zero at the starting date.



For US dollar swaps, floating rates are typically the 3-month or 6-month LIBOR rates prevailing over the period before the interest payment is made. The interest rates are determined in advance or equivalently, the payments are made in arrears. In practice, there are many variants of this basic structure. For instance, swaps can be such that the notional is different for the two counterparties, or the notional(s) amortized, or where the floating leg is LIBOR plus or minus a fixed coupon, etc.

4.1.8.1 Swap Valuation

The present value of a plain vanilla swap can easily be computed using standard methods for finding the discounted value of the components.

The swap requires from one party a series of payments based on variable rates, which are determined at the agreed dates of each payment. At the time the swap is entered into, the actual payment rates are known only in the future, but the market provides a yield curve from bonds with various maturity dates stretching from the short term to the long term. Each variable rate payment is calculated based on the forward rate for each respective payment date.

Using these interest rates leads to a series of cash flows. Each cash flow is discounted by the zero-coupon rate for the date of the payment; this is also sourced from the yield curve data available from the market. Zero-coupon rates are used because these rates are for bonds, which pay only one cash flow. The interest rate swap is therefore treated like a series of zero-coupon bonds.

This calculation leads to a present value (PV). The fixed rate offered in the swap is the rate, which values the fixed rate payments to the same PV as the variable rate payments using today's forward rates.

Therefore, at the time the contract is entered into, there is no advantage to either party, and therefore the swap requires no upfront payment. During the life of the swap, the same valuation technique is used, but since, over time, the forward rates change, the PV of the variable-rate part of the swap will deviate from the unchangeable fixed-rate side of the swap.

The rates of interest in the fixed leg of a swap are quoted for various maturities. These rates make up the **swap curve**.

Single Currency Swap Valuation

Denote by $D(T)$ the discount factor (when using pure discount bonds we denote discounting by $p(t,T)$) from the swap curve for a cash flow at time T. Consider a fixed-floating standard interest rate swap with reference dates $0 = \overline{T}_0, \overline{T}_1, \dots, \overline{T}_n$ on the fixed leg and reference dates $0 = T_0, T_1, \dots, T_m$ for the floating leg. Also let the end dates be the same. Denote by $\overline{\Delta}_i$ and Δ_i the length (day-count fraction) of the time periods according to the specified fixed and floating leg day-count convention. For the period $[T_{i-1}, T_i]$ the Libor rate L_i is set (fixed) in the market at time T_{i-1} and the amount $\Delta_i \cdot L_i$ is paid at time T_i. The **forward rate** F_i for the period $[T_{i-1}, T_i]$ is defined as

$$D(T_i) = D(T_{i-1})D(T_{i-1}, T_i) = \frac{D(T_{i-1})}{1 + \Delta_i \cdot F_i}$$

giving

Here $D(T_{i-1}, T_i)$ is a forward discount factor between T_{i-1} and T_i resetted at time T_{i-1}. One says, the forward rate F_i is projected or forecasted from the discount curve. The value today of the floating cash flow $\Delta_i \cdot L_i$ for period $[T_{i-1}, T_i]$ is from above equal to its discounted forward rate

$$\Delta_i \cdot F_i \cdot D(T_i) = \Delta_i \cdot \frac{D(T_{i-1}) - D(T_i)}{\Delta_i \cdot D(T_i)} \cdot D(T_i) = D(T_{i-1}) - D(T_i)$$

Therefore, the value of the whole floating leg is simply

$$\sum_{i=1}^{m}\Delta_i \cdot F_i \cdot D(T_i) = \sum_{i=1}^{m}\Delta_i \cdot \frac{D(T_{i-1}) - D(T_i)}{\Delta_i D(T_i)} \cdot D(T_i) = \sum_{i=1}^{m}\{D(T_{i-1}) - D(T_i)\}$$
$$= D(T_0) - D(T_m) = 1 - D(T_m) \equiv 1 - D(\overline{T}_n)$$

Consequently, the value of a floating rate bond is always at par on reset days:

$$\sum_{i=1}^{m}\Delta_i \cdot F_i \cdot D(T_i) + D(T_m) = 1 - D(T_m) + D(T_m) = 1$$

The value of the fixed leg with rate C can then be solved for as

$$\sum_{i=1}^{n} C \cdot \overline{\Delta}_i \cdot D(\overline{T}_i) = C \cdot \sum_{i=1}^{n}\overline{\Delta}_i \cdot D(\overline{T}_i)$$

If C_n is the fair swap rate at maturity, $\overline{T}_n$ we get the following equation

$$\sum_{i=1}^{n} C_n \cdot \overline{\Delta}_i \cdot D(\overline{T}_i) + D(\overline{T}_n) = 1$$

This is the basis for the recursive bootstrapping relationship for the discount factors

$$D(\overline{T}_n) = \frac{1 - C_n \cdot \sum_{i=1}^{n-1}\overline{\Delta}_i \cdot D(\overline{T}_i)}{1 + \overline{\Delta}_n \cdot C_n}, \qquad n = 1, \ldots$$

From market quoted fair swap rates C_n.

Example 4.1.8.1

In the examples below, we show a number of exotic swaps.

1. First we start with a plain vanilla swap. Suppose that we want to calculate the 3-year swap rate when the spot rates are as follows:

Maturity (years)	Discount factor, D	Spot rate r, (%)
0.5	0.9707	6.036
1.0	0.9443	5.809
1.5	0.9175	5.824
2.0	0.8913	5.839
2.5	0.8644	5.914
3.0	0.8378	5.989

The 3-year swap rate is given by:

$$\text{swap rate } = 2 \times \frac{1 - D_6}{D_1 + D_2 + \ldots + D_6} \times 100 = 5.980\%$$

2. Next, we study a so-called step-up swap with the following data:

Maturity: 2 years
Notional principal: 100
Swap rate: 4% the first year, C% the second
Floating rate leg: 100
Semi-annual payments
Spot rates as above

The value of the fixed rate leg:

$$(D_1 + D_2) \times 2 + (D_3 + D_4) \times \frac{C}{2} + D_4 \times 100$$

This gives

$$100 = (D_1 + D_2) \times 2 + (D_3 + D_4) \times \frac{C}{2} + D_4 \times 100$$

$$\Rightarrow$$

$$\frac{C}{2} = \frac{100 \times (1 - D_4) - (D_1 + D_2) \times 2}{D_3 + D_4} = 3.894$$

That is,

$$C = 7.778\%$$

3. Next, we look at an amortizing swap with the following data:

Maturity: 2 years
Notional principal: 100 the 1st year, 50 in 2nd year
Swap rate: C%
Semi-annual payments
Spot rates as above

In our valuation, we use a principal of 50 and add another 50 after a year. The floating rate leg is then a sum of two floating rate notes, each with notional principal of 50.

To calculate the floating leg, consider the relation between the discount factor and the forward rate

$$D(t_i) = \frac{D(t_{i-1})}{1 + \tau \cdot F_i} \Rightarrow F_i = \frac{D(t_{i-1})/D(t_i) - 1}{\tau} = \frac{D(t_{i-1}) - D(t_i)}{\tau \cdot D(t_i)}, \quad \tau = t_i - t_{i-1}$$

giving

$$\tau \cdot F_i \cdot D(t_i) = D(t_{i-1}) - D(t_i) \qquad \text{especially} \qquad \tau \cdot F_1 \cdot D(t_1) = 1 - D(t_1)$$

Since the notional will change we write the value of the floating leg as

$$\begin{aligned}
&100 \cdot \frac{F_1}{2} \cdot D_1 + 100 \cdot \frac{F_2}{2} \cdot D_2 + 50 \cdot \frac{F_3}{2} \cdot D_3 + 50 \cdot \frac{F_4}{2} \cdot D_4 \\
&\quad = 100 \cdot (D_0 - D_1) + 100 \cdot (D_1 - D_2) + 50 \cdot (D_2 - D_3) + 50 \cdot (D_3 - D_4) \\
&\quad = 100 - 100 \cdot D_2 + 50 \cdot D_2 - 50 \cdot D_4 \\
&\quad = 100 - 50 \cdot (D_2 + D_4)
\end{aligned}$$

were $D_i = D(t_i)$. The value of the fixed rate leg (where we amortize 50 at t_2) is

$$100 \cdot (D_1 + D_2) \cdot \frac{C}{2} + 50 \cdot (D_3 + D_4) \cdot \frac{C}{2}$$

Since both legs have to be equal we get the swap rate C from

$$\frac{C}{2} = \frac{100 - 50 \cdot (D_2 + D_4)}{100 \cdot (D_1 + D_2) + 50 \cdot (D_3 + D_4)} = 0.02915$$

so

$$C = 5.831\%$$

4. Finally, we study a forward-starting swap with the following data:

Maturity: 2 years, starting in 1 year
Notional principal: 100
Semi-annual coupons (in 18,24,30 and 36 months)
Swap rate is C%
Spot rates as in vanilla swap above.

In our valuation we have a floating rate leg in one year. This will be a floating rate note ⇒ Value = $D_2 \times 100 = 94.435$. The fixed rate leg is given by

$$(D_3 + D_4 + D_5 + D_6) \times \frac{C}{2} + D_6 \times 100$$

Giving the market rate as

$$\begin{aligned}
&D_2 \times 100 = (D_3 + D_4 + D_5 + D_6) \times \tfrac{C}{2} + D_6 \times 100 \\
&\Rightarrow \\
&\frac{C}{2} = \frac{(D_2 - D_6) \times 100}{D_3 + D_4 + D_5 + D_6} = 3.036
\end{aligned}$$

That is,

$$C = 6.072\%$$

Such a swap is used as underlying for a swaption (an option on a swap).

Example 4.1.8.2

Comparative advantage

Swaps were first created to exploit comparative advantage. This is when two companies who want to borrow money face different quoted fixed and floating rates so that by exchanging payments between themselves they benefit, at the same time benefiting the intermediary who puts the deal together. Here is an example.

Two companies A and B want to borrow \$50 Million, to be paid back in 2 years. They are quoted the interest rates for borrowing at fixed and floating rates as

Borrowing rates for companies A and B		
	Fixed	Floating
A	7%	Six month LIBOR + 30 bp
B	8.20%	Six-month LIBOR + 100 bps

Note that both must pay a premium over LIBOR to cover the risk of default, which is perceived to be greater for company B. Ideally, company A wants to borrow at floating and company B at fixed. If they each borrow directly then they pay the following

A	Six month LIBOR + 30 bp
B	8.2% (fixed)

The total interest they are paying is

$$\text{six-month LIBOR} + 30\text{ bps} + 8.2\% = \text{six-month LIBOR} + 8.5\%.$$

If only they could get together, they would only be paying:

$$\text{six-month LIBOR} + 100\text{ bps} + 7\% = \text{six-month LIBOR} + 8\%$$

That is a saving of 0.5%.

Let us suppose that A borrows fixed and B floating, even though that is not what they want. Their total interest payments are six-month LIBOR plus 8%. Now let us see what happens if we throw a swap into the pot.

A is currently paying 7% and B six-month LIBOR plus 1%. They enter into a swap in which A pays LIBOR to B and B pays 6.95% to A. They have swapped interest payments.

Looked at from A's perspective they are paying 7% and LIBOR while receiving 6.95%, a net floating payment of LIBOR plus 5 bps. Not only is this floating, as they originally wanted, but it is 25 bps better than if they had borrowed directly at the floating rate. There's still another 25 bps missing and, of course, B gets this. B pays LIBOR plus 100 bps and also 6.95% to A while receiving LIBOR from A. This nets out at 7.95%, which is fixed, as required and 25 bps less than the original deal.

To see that this is a general principle, let us do the same calculations with x instead of 6.95.

A is currently paying 7% and B six-month LIBOR plus 1%. They enter into a swap in which A pays LIBOR to B and B pays x% to A. They have swapped interest payments. Looked at from A's perspective they are paying 7% and LIBOR while receiving x%, a net floating payment of LIBOR plus $7 - x$%. Now we want A to benefit by 25 bps over the original deal, this is half of the 50 bps advantage. If they decide to divide the advantage equally in this way, with 25 bps each, we can solve for x from

$$LIBOR + 7 - x + 0.25 = LIBOR + 0.3,$$

That is,

$$x = 6.95\%.$$

Not only does A now get floating, as originally wanted, but it is 25 bps better than if they had borrowed directly at the floating rate. There's still another 25 bps missing and, of course, B gets this. B pays LIBOR plus 100 bps and also 6.95% to A while receiving LIBOR from A. This nets out at 7.95%, which is fixed, as required and 25 bps less than the original deal.

In practice, the two counterparties would deal through an intermediary who would take a piece of the action.

Although comparative advantage was the original reason for the growth of the swaps market, it is no longer the reason for the popularity of swaps. Swaps are now very vanilla products existing in many maturities and more liquid than simple bonds.

Parity Relation

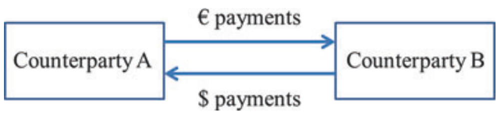
When valuing caps, floors and swaps we can use the following parity relation

$$\text{swap} = \text{cap} - \text{floor}.$$

This relationship can also be used for hedging.

4.1.8.2 Currency Swaps and FX Swaps

Cross currency swaps are powerful instruments which can transfer assets or liabilities from one currency into another. It is an agreement to swap a series of specified payments denominated in one currency for a series of specified payments in a different currency. The market charges for this a liquidity premium, the cross currency basis spread, which should be taken into account by the valuation methodology. The valuation methods for cross currency swaps are based upon using two different discounting curves.



Basically, such a swap has the following components: There are two currencies, say USD ($) and Euro (€). The swap is initiated at time t_0 and involves:

1. An exchange of a principal amount $N_{\$}$ against the principal $M_{€}$.
2. A series of floating interest payments associated with the principals $N_{\$}$ and $M_{€}$, respectively. Payments are settled at settlement dates, $\{t_1, t_2, \ldots, t_n\}$. One party will pay the floating payments $L^{\$}_{t_i} N^{\$} d_i$ and receive floating payments of size $L^{€}_{t_i} N^{€} d_i$ where d_i is the day-count adjustment

$$d_i = \frac{t_i - t_{i-1}}{D}$$

and D denotes the number of calendar days in the year according to the current day-count convention. The two Libor rates (L) will be determined at set dates

$$\{t_1, t_2, \ldots, t_{n-1}\}.$$

A **Circus swap** is a fixed-rate currency swap against floating US dollar LIBOR payments (Fig. 4.16).

An **FX-swap** is made of a money market deposit and a money market loan in different currencies written on the same "ticket".

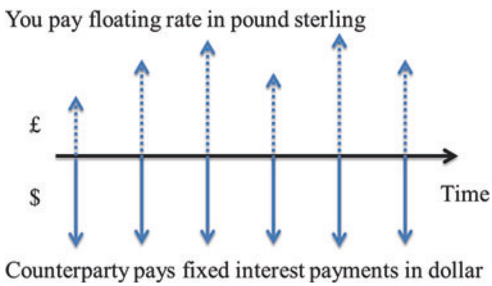


Fig. 4.16 A swap with fixed rate pound sterling against floating US dollar LIBOR.

An **Asset swap** is an interest rate swap used to alter the cash flow characteristics of an institution's assets in order to provide a better match with its liabilities.

An **Equity swap** is a swap in which the cash flows exchanged are based on the total return on some stock market index and an interest rate (either a fixed rate or floating rate).

When dealing with FX instruments, it is common to measure the changes in the price by pips. One pip is the smallest price change that a given exchange rate can make. Since most major currency pairs are priced to four decimal places, the smallest change is that of the last decimal point – for most pairs this is the equivalent of 1/100 of 1 per cent, or one bp. So a pip is actually an acronym for percentage in point, sometimes also called a price interest point.

> **Example 4.1.8.3**
> If you buy the EUR/USD pair at 1.40 and sell it at 1.41 you have gained 100 pips.

Pips are sometimes also used when bootstrapping yield curves in currencies when you do not have liquid market data. Then you might use the USD curve as a proxy and then add a spread calculated from FX prices given in pips.

Cross Currency Basis Swaps

Cross currency swaps differ from single currency swaps by the fact that the interests rate payments on the two legs are in different currencies. So on one-leg interest rate payments are in currency 1 on a notional amount N_1 and on the other leg interest rate payments are in currency 2 calculated on a notional amount N_2 in that currency.

At inception of the trade the notional principal amounts in the two currencies are usually set to be fair given the spot foreign exchange rate X, that is, $N_1 = X \cdot N_2$, that is, the current spot foreign exchange rate is used for the relationship of the notional amounts for all future exchanges. Contrary to single currency swaps there is usually an exchange of principals at maturity. So a cross currency swap can be seen as an exchange of payments from two bonds, one in currency 1 with principal N_1, and the other in currency 2 with principal N_2 (Fig. 4.17).

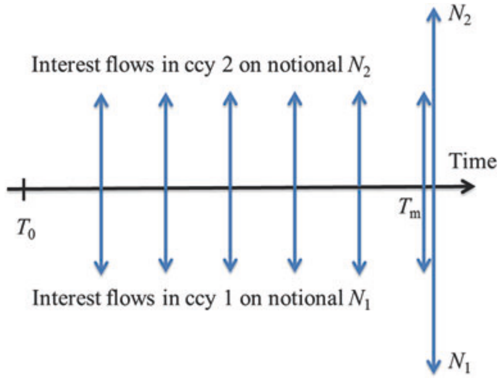


Fig. 4.17 A fix-fix cross currency swap. In most cases there is also an exchange of notionals when entering the swap.

Sometimes an opposite exchange of principals is made when entering the contract. If a leg is floating the variable reference rate refers to the payment currency of that leg, otherwise this would be a so-called **quanto swap**. From the possible types of cross currency swaps: *fixed versus fixed*, *fixed versus floating* and *floating versus floating*, the latter type is particularly important and it is called a **basis swap**. Combining a basis swap with a single currency swap the other types can be generated synthetically.

A **basis swap** is basically an exchange of **two floating rate bonds**. Following the arguments of the previous section the price of a floater is always par (at the beginning of each interest rate period disregarding credit risks). For a cross currency basis swap this means that the two legs should have a value of N_1 and N_2, respectively. Consequently, if the two principal amounts are linked by today's foreign exchange rate $X : N_1 = X \cdot N_2$, the basis swap is fair. This is theoretically true, but in practice the market quotes basis swaps to be fair if there is a certain spread, called *cross currency basis spread*, on top of the floating rate of one leg of the basis swap. Theoretically this would imply an arbitrage opportunity. However, cross currency swaps are powerful instruments to transfer assets or liabilities from one currency into another one and the market is charging a liquidity premium of one currency over the other. The market usually quotes cross currency basis spreads usually relative to some liquidity benchmark, for example USD or EUR Libor.

Basis Swaps - All currencies vs. 3m USD LIBOR - Also see <ICAB2>

	REC/PAY	REC/PAY	REC/PAY	REC/PAY	REC/PAY
	EUR	JPY	GBP	CHF	SEK
1 Yr	+3.125/+1.125	-02.00/-05.00	+02.50/-01.50	+1.50/-1.50	-3.75/-6.75
2 Yr	+3.000/+1.000	-02.00/-05.00	+02.50/-01.50	+1.00/-2.00	-2.50/-5.50
3 Yr	+3.000/+1.000	-01.75/-04.75	+02.25/-01.75	+0.25/-3.00	-1.00/-4.00
4 Yr	+3.000/+1.000	-01.75/-04.75	+02.00/-02.00	-0.75/-3.75	-0.25/-3.25
5 Yr	+2.750/+0.750	-02.00/-05.00	+01.75/-02.25	-1.25/-4.25	+0.25/-3.25
7 Yr	+2.750/+0.750	-02.50/-05.50	+00.75/-03.25	-1.50/-4.50	+0.25/-2.75
10Yr	+2.750/+0.750	-04.50/-07.50	-00.75/-04.75	-1.50/-4.50	+0.25/-2.75
15Yr	+4.125/+0.125	-10.50/-13.50	-02.25/-06.25	-0.75/-4.75	+2.50/-2.50
20Yr	+4.125/+0.125	-15.25/-18.25	-02.50/-06.50	-0.25/-4.25	+2.50/-2.50
30Yr	+4.125/+0.125	-23.25/-26.25	-02.50/-06.50	*FOR 3M V 6M EUR/EUR <ICAB4>*	
	DKK	NOK	CAD	CZK	PLN
1 Yr	-1.00/-5.00	-4.00/-8.00	+12.50/+08.50	+02.00/-07.00	+05.00/-12.00
2 Yr	-1.50/-4.50	-4.00/-8.00	+13.50/+09.50	+01.50/-06.50	+05.00/-12.00
3 Yr	-1.25/-4.25	-4.00/-8.00	+14.25/+10.25	+01.50/-06.50	+05.00/-12.00
4 Yr	-1.00/-4.00	-3.75/-7.75	+15.25/+11.25	+01.50/-06.50	+03.00/-10.00
5 Yr	-0.25/-3.25	-3.75/-7.75	+16.25/+12.25	+01.50/-06.50	+03.00/-10.00
7 Yr	+0.25/-3.00	-3.75/-7.75	+17.00/+13.00	+01.50/-06.50	+03.00/-10.00
10Yr	+0.25/-3.00	-3.75/-7.75	+17.00/+13.00	+01.50/-06.50	+03.00/-10.00
15Yr	+1.25/-3.75	-3.50/-8.50	+17.25/+13.25		
20Yr	+1.25/-3.75	-3.50/-8.50	+17.25/+13.25		

Fig. 4.18 Cross currency basis swap quotes against USD

Here is an example of cross currency basis swap quotes against the liquidity benchmark USD (Fig. 4.18).

For example, a 10-years cross currency basis swap of three-months USD Libor flat against JPY Libor is fair with a spread of -4.5 bps if USD Libor is received and with a spread of -7.5 bps if USD Libor is paid.

Evaluating cross currency swaps requires discounting the cash flows with the discount factors for the respective currency of the flow. But clearly, a valuation of those instruments would show a profit or loss, which is not existent. It is therefore necessary to incorporate the cross currency basis spread into the valuation methodology to be consistent with the market.

First of all, one has to agree on a liquidity reference currency (benchmark) which is usually chosen to be USD or EUR. swap cash flows in the liquidity reference currency are valued exactly as described above since there is no need for liquidity adjustments there. For all currencies different from the liquidity benchmark the idea is to use two different discount factor curves depending on whether to forecast or value variable cash flows or to discount cash flows.

Denote by s_m the market quoted fair cross currency basis spread (usually the mid) for maturity T_m on top of the floating rate for the given currency relative to the chosen liquidity reference. The fact that

s_m is the fair spread is equivalent to saying that a floating rate bond with maturity T_m in the given currency which pays Libor plus spread s_m values to par.

We will here show how to value FX Forwards and FX swaps (for tenors up until 2 years), using discount rates in non-USD. The value is implied from the FX Spot price (versus USD), the USD discount curve and FX points. Any legs in USD are discounted using the standard interbank curve for USD. If both legs in a product are non-USD we can use this methodology with synthetically replications by the two trades via USD.

First, forward USD rates, r_{USD} are calculated based on the USD zero-coupon yield curve. Then we calculate the cash-flow amount in the non-USD currency as:

$$a_f = \left(x_s + \frac{p}{s_f}\right)(1 + r_{USD})^{(d_m - d_v)/365}$$

where

d_m	is days to maturity date
d_v	is days to value date
r_{USD}	is the USD forward interest rate from today until maturity
p	is swap points
s_f	is swap point factor, usually 10000 or 100
x_s	is the spot exchange rate, expressed as number of units in foreign currency per 1 USD
x_f	is the forward exchange rate
a_f	is the cash-flow amount

this gives the implied deposit rate in the non-USD currency as

$$r_f = \frac{365}{(d_m - d_v)}\left[\left(\frac{a_f}{x_s}\right) - 1\right]$$

For maturities of 2 years and above, Currency Basis Spreads are more actively quoted in the market than FX points. Therefore an implied swap rates are used for curve building purposes.

Currency Basis Spreads are quoted in the market as a spread on the non-USD currency when swapping 3M USD Libor to or from 3M non-USD XIBOR.[3] That gives the implied Currency Basis swap rate in the non-USD currency for tenor T as

$$IRS_{CIRS}^{T} = IRS_{3M}^{T} + CBS^{T}$$

For most currencies, the standard swap curve is based on 6M fixings, because an adjustment for the spread between 3M and 6M fixings has to be made. Such spreads are quoted as a spread on the 3M leg when swapping 3M XIBOR to or from 6M XIBOR. In case only when a swap against 6M fixings and tenor basis spreads are available for the currency in question, the implied Currency Basis swap rate can be calculated as;

$$IRS_{CIRS}^{T} = IRS_{6M}^{T} - TBS_{3Mv6M}^{T} + CBS^{T}$$

Valuation of Cross Currency Basis swaps

A popular methodology among practitioners is to use two discount factor curves, one for projecting forward rates according and the other for finally discounting all cash flows. This is unfortunately inconsistent with the standard single currency swap valuation method. This will result in arbitrage opportunities between single currency and cross currency swaps.

Therefore we will make another approach, which is able to handle both types of swaps consistently in one and the same framework. The major drawback of this approach is that mark-to-market valuation of single currency swaps can be slightly different from the results of the current standard valuation method, in particular, for off market positions.

We will use two discount factors curves

1. the first one, $D(t)$, will be used to discount all fixed cash flows,
2. the second one, $DF^{*}(t)$, will be applied to completely value floating cash flows.

The two conditions on the two discount factor curves are

[3] XIBOR is used here to denote a generic Interbank Offering Rate, examples are LIBOR, EURIBOR or STIBOR.

- the value of a coupon bond with coupon equal to the swap rate C_n is identical to the value of a floating rate bond,
- a floating rate bond which pays Libor plus cross currency basis spread s_n values to par.

Combining these conditions a fixed coupon bond paying the coupon C_n plus the cross currency basis spread s_n should have a value of par

$$\sum_{i=1}^{n} C_n \cdot \overline{\Delta}_i \cdot D(\overline{T}_i) + \sum_{j=1}^{m} s_m \cdot \Delta_j \cdot D(T_j) + D(\overline{T}_n) = 1$$

From this equation the curve $D(t)$ can be extracted. If in particular, the floating and fixed legs admit the same frequency ($\overline{T}_i = T_i$), then we have the following simple bootstrapping equation

$$D(\overline{T}_n) = \frac{1 - \sum_{i=1}^{n-1} \left(C_n \cdot \overline{\Delta}_i + s_n \cdot \Delta_i\right) \cdot D(\overline{T}_i)}{1 + \overline{\Delta}_n \cdot C_n + s_n \cdot \Delta_i}, \qquad n = 1, \ldots$$

Now, in order to determine the second curve of discount factors, we define the value today of the floating interest rate (Libor) cash flow for period $[T_{i-1}, T_i]$ as given by

$$D^*(T_{i-1}) - D^*(T_i)$$

This also implies the desirable property that the value of a series of subsequent floating rate cash flows over the time interval $[T_0, T_m]$ is just $D^*(T_0) - D^*(T_m)$ and is thus independent of the payment frequency. The second condition above now implies the following requirement

$$D^*(T_0) - D^*(T_m) + \sum_{j=1}^{m} s_m \cdot \Delta_j \cdot D(T_j) + D(T_m) = 1$$

Setting, $D^*(T_0) = 1$, $T_0 = 0$, this gives

$$D^*(T_m) = s_m \cdot \sum_{j=1}^{m} \Delta_j \cdot D(T_j) + D(T_m)$$

This defines the second curve of discount factors $D^*(t)$.

In the current approach also cash flows in standard single currency swaps are discounted differently compared to the standard approach.

This gets even more pronounced when it comes to mark-to-market valuation of off market swaps. Consider a 10-year single currency swap with 200 bps off market with a fixed rate $C = 7.9\%$. In the standard approach its net present value is 1499.15 bps compared to 1515.32 bps in the current approach. This difference is equivalent to a fixed rate difference of 2.157 bps. Clearly, for a swap which is not too far from being fair the differences are much smaller. Obviously this has consequences, for example, on the fair values for unwinding an off market swap position with a counterparty.

4.1.9 Overnight Index Swaps (OIS)

Overnight Index Swaps (OIS) are interest rate swaps based on a specific currency that exchanges fixed rate interest payments for floating rate payments based on a **notional swap principal** at regular intervals over the life of the swap contract. The floating rate is based on a specified published index of the daily overnight rate for the OIS currency. For swaps based on the US dollar (USD), the referenced floating rate is the daily effective federal funds rate.

Introduced in 1995, overnight index swaps are used to either hedge or speculate on changes in the overnight interest rate. As a hedge, overnight index swaps are used to manage interest rate risk and liquidity. The terms of OISs range from 1 week to 10 years or more, with spreads typically ranging from 1.5 to 5 bps. At maturity, the parties determine the net payment by calculating the difference between the accrued interest of the fixed rate and the geometric averaging of the floating index rate on the notional swap principal. Because there is no exchange of principal and only the net difference in interest rates is paid at maturity, OISs have little credit risk exposure.

The **LIBOR-OIS spread** is the difference between the LIBOR and the overnight index swap rate, and it is the measure of the credit risk in the interbank lending market. Normally, when the central banks lower their rates of interest, both the LIBOR and the OIS rates decline with it. However, when banks are unsure of the creditworthiness of other banks, they charge higher interest rates to compensate them for the greater credit risk. The LIBOR-OIS spread is a better measure of credit risk in the interbank deposit market than the LIBOR itself because the LIBOR is influenced by

1. the rates set by central banks;
2. the credit risk in lending to other banks.

Because the overnight index swap rate is based on the rates set by central banks, subtracting it from the LIBOR shows the amount of the interest rate that is being charged for the credit risk.

In Fig. 4.19 and Fig. 4.20 we illustrate the 1-month US dollar LIBOR rate and the US dollar OIS rate and the spread between the rates during the financial crises. The increase in the difference between the two rates is evident starting at the beginning of the subprime credit crisis in August, 2007, with a wider spread in September, October and November of 2008 indicating worsening conditions. As you can see in the graph, prior to August, 2007, both the LIBOR and the OIS rates were high because the Federal Reserve, which is the central bank of the United States, raised their target rates. After the beginning of the credit crisis, the Federal Reserve started lowering targets, and the OIS rate has declined with it. The LIBOR, however,

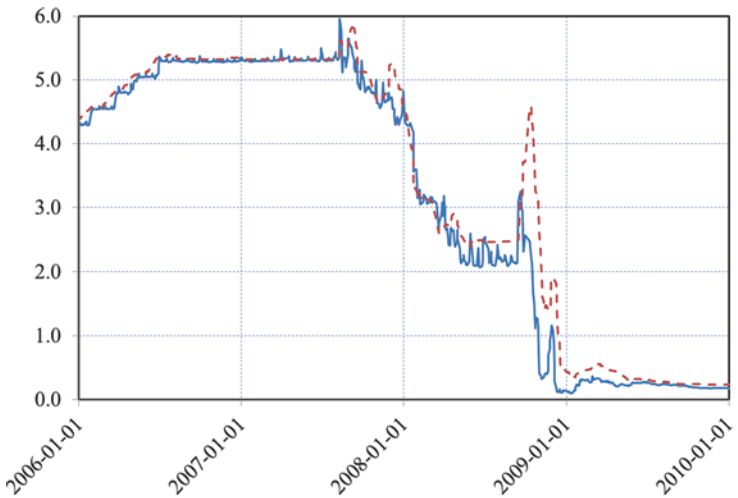


Fig. 4.19 USD LIBOR 1 month (dashed line) and USD OIS (solid line)[4].

[4] Source, FRED, https://fred.stlouisfed.org/ Federal Reserve Economic Data – St. Louis Fed

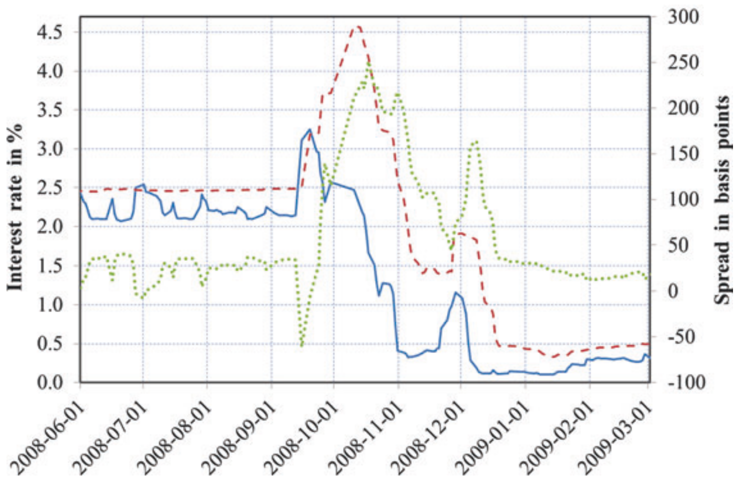


Fig. 4.20 USD LIBOR 1 month (dashed line) and USD OIS (solid line) and the spread in bps (dotted line). This is a zoomed in view from Fig. 4.19

has declined sporadically and not nearly as much as the OIS rate, because banks couldn't be sure which banks were creditworthy; hence, they charged higher interbank lending rates, which is what the LIBOR measures.

4.1.10 Asset Swap and Asset Swap Spread

A plain vanilla asset swap transaction entails purchasing a fixed rate asset and simultaneously entering into a swap to convert fixed interest payments to floating. The incentive for the investor is to earn a credit spread on a fixed rate security (for example a bond), while minimizing interest rate or market risk. Asset swaps are closely connected to credit derivatives.

Through asset swaps, investors can purchase and isolate credit risk in a wide variety of securities, including domestic and foreign corporate obligations, convertible bonds, and other financial assets. Asset swaps also have some disadvantages:

- Many investors cannot enter into derivative transactions due to regulations.
- Unlike a credit default swap, the asset swap is not coupled with the credit performance of the underlying bond in the event of default. This means that if the bond defaults, the investor remains subject to the swap and must fulfil the obligations to pay the rest of the payments to the counterpart. However, the asset swap can be terminated at the current market value.
- The counterpart has a credit exposure to the investor.

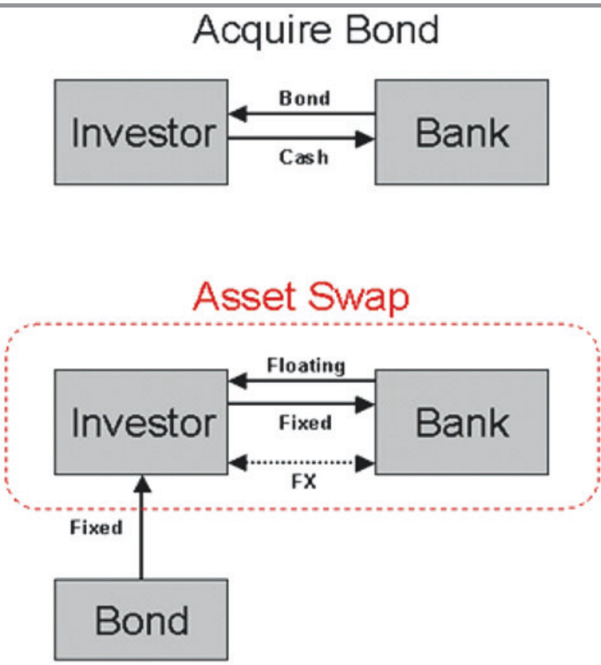
An asset swap enables the investor to minimise market risk while still maintaining exposure to credit risk. It is a method to convert a risky bond to a synthetic FRN. The market for synthetic securities is largely driven by the presence of arbitrage. The result of creating a synthetic structure is a higher yield (LIBOR + spread) than the existing market security.

Some investors refer to asset swaps as being a "single swap", while other asset swap dealers consider asset swaps to be a package consisting of the underlying bond and the swap. Some dealers trade asset swaps by using their currently held bonds as references while others buy the whole package. Some investors book the packages in different trading books and some use the same.

Example 4.1.10.1

Assume an investor owns (or acquires) a bond that pays an annual coupon of 8%. (The bond is often quoted in terms of asset swap spread bps.) The investor does not want to (or cannot) sell the security and wants the market risk to be as low as possible. The investor therefore enters into an asset swap transaction with a counterpart.

Example 4.



The FX effect (if any) occurs if the bond is denominated in a currency different from the swap. In that case we have a currency asset swap. The fixed coupons of the bond are passed through to the bank counterpart and the bank pay floating coupons, which, for example, could be LIBOR plus spread. The spread is referred to as the asset swap spread.

4.1.10.1 Asset Swap Spread

The idea behind asset swaps is, as we have seen, to convert bonds into synthetic FRN's where we want to a structure that only has credit risk and no interest rate risk. Normally, a payment, an up-front premium swap is made in order to bring the structure to par (the initial investment is 100). The asset swap spread, S, is the spread on the floating leg of the swap that gives this structure a PV of 0 at inception.

Given a bond price P, we can calculate the corresponding assets swap spread. If the bond has a market price, for example 95, we have to pay an initial premium of five (the leftmost arrow downwards in the Fig. 4.21) to make the FRN price equal to 100.

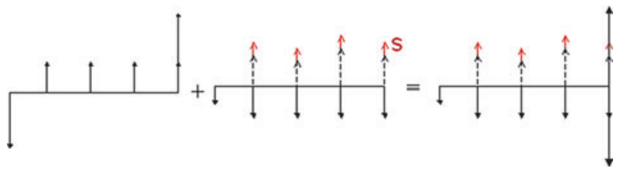


Fig. 4.21 Illustration of the asset swap spread.

If we identify the cash flows (see Fig. 4.21) on the receiving side we see that the floating cash flows plus the redemption amount is simply a risk free FRN pricing to par. The present value (*PV*) of the asset swap spread payments (the small upward arrows) is

$$PV = S \cdot \sum_i p_i d_i$$

where p_i is the length of period i and d_i the discount factors. In the pay side we have (all cash flows except the initial payment) the coupons from the bond plus the redemption amount. The initial swap premium payment is 100.0 – the market price of the bond.
We then have the following values:

- Floating cash flows + redemption: 100.0
- Asset swap spread: $S \, \Sigma p_i d_i$
- Theoretical price of the bond: P_{bond}
- Initial premium: $100 - P_{swap}$

Totally this should sum up to zero, that is,

$$100 + S \cdot \sum_i p_i d_i - P_{bond} - (100.0 - P_{swap}) = 0$$

Giving the spread as

$$S = \frac{P_{bond} - P_{swap}}{\sum_i p_i \cdot d_i}$$

We can also use the formula above to price a bond with a given asset swap spread.

4.1.11 Swaptions

As their name indicates, swaptions are options to enter into swaps. In a **payer swaption**, the investor has the option to enter into a swap with given date, paying the fixed rate (strike) and receiving floating. A **receiver swaption** gives the right to enter into a swap receiving fixed and paying floating. swaptions can be both European and American. In the latter case, the option can be exercised usually on cash-flow dates, rather than "continuously" like American stock options. American swaptions that can be exercised only at reset dates are often called "mid-Atlantic" or "Bermudan" because they are somewhere between American and European. The parameters that define a swaption are therefore

- Notional
- Maturity of the option
- Payer or receiver
- Type: American or European
- Maturity of the swap
- Cash-flow dates of the swap
- Floating rate.

For instance, a European **in-5-for-10** LIBOR payer swaption with strike 6.50% is an option to enter into a 10-year swap paying a fixed rate of 6.50% 5 years from now.

4.1.12 Credit Default Swaps

Credit derivatives include a range of instruments designed to transfer credit risk without requiring the sale or purchase of bonds and/or loans. They were originally designed by JP Morgan in 1994 so that banks could manage credit risk in their loan portfolios while preserving important customer relationships.

In the group of instrument known as *credit derivatives* the most commonly traded are:

- Credit Default swaps
- Credit Default Indices
- Total Return swaps

- Credit Default swaptions
- Credit Spread Options (Eurobond Options).

A credit default swap (CDS) is a financial swap where the seller of the CDS will compensate the buyer in the event of a default or other credit event. The buyer of the CDS makes a series of payments, the CDS "fee" or "spread", to the seller and, in exchange, receives a payoff on defaults.

The "spread" of a CDS is the annual amount the protection buyer must pay the protection seller over the length of the contract, expressed as a percentage of the notional amount. For example, if the CDS spread of Risky Corporation is 50 bps, or 0.5% (1 bp = 0.01%), then an investor buying $10 million worth of protection from AAA-Bank must pay the bank $50,000. Payments are usually made on a quarterly basis, in arrears. These payments continue until either the CDS contract expires or a credit event occurs.

A CDS on one single Risky Corporation is sometimes called **Single Name CDS**. Other CDS contracts are constructed on a market index, such as the North American CDX index or the European iTraxx index. Such CDS are usually called a Credit Default Index (CDI).

Many investors buy a CDS protect themselves against a default if they own an Corporate bond. But, anyone can purchase a CDS, even buyers who do not hold the bond or a loan in the company (these are called "naked" CDSs). If there are more CDS contracts outstanding than bonds in existence, a protocol exists to hold a credit event auction where the payment received will usually be significantly less than the face value of the loan.

The evolution of credit derivatives was prompted by the increased demand for asset-backed deals backed by credit instruments. The credit derivatives market had been growing rapidly since the early 1990s. By the end of 2007, the outstanding CDS amount was $62.2 trillion, but this increase stopped during the financial crisis and in early 2012 the amount had fallen to $25.5 trillion.

The evolution of credit derivatives allows domestic banks to provide the following benefits to their clients:

- Customized exposure to credit risk.
- Enable users to take short positions in credits previously not possible in the underlying securities.

- Provide institutional investors access to the interbank market, generally on a leveraged basis.
- Increase diversification in concentrated credit portfolios.
- Extract and hedge specific sections of credit risk.

In essence a default swap is a bilateral OTC agreement, which transfers a defined credit risk from one party to another. Contracts are typically standardized as documented by the **International swap and Derivatives Association** (ISDA).

4.1.12.1 Cash flows

Under a typical CDS the buyer of protection pays to the seller a regular premium (usually quarterly), which is specified at the beginning of the transaction. If no credit event, such as default, occurs during the life of the swap, these premium payments are the only cash flows.

Just like in many other swaps there is no exchange of the underlying principal. Following a credit event the protection seller makes a payment to the protection buyer. The protection buyer stops paying the regular premium following the credit event.

The cash-flow representation of a CDS are sown in Fig. 4.22.
The first leg is the credit default leg. The cash flow will only take place in the case of default of the credit reference. The value of this cash flow will be calculated using the yield curve for the credit reference, and the yield curve for the currency. In the case of default, the payment will be settled on a settlement date not known in advance. This

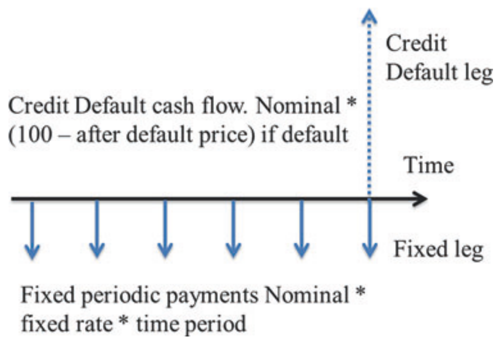


Fig. 4.22 Cash flows for a CDS.

settlement date does not have anything to do with the end date of the contract as is depicted in the figure.

The second leg is a normal Fixed Leg. The valuation of this leg is done using the yield curve for the credit reference. This incorporates the fact that these fixed payments will only be paid if no credit default has taken place.

Just because a credit event has occurred it does not necessarily mean that the claim on the Reference Entity will be worthless. Credit default contracts are structured to replicate the experience of a cash market holder of an obligation of the Reference Entity. The recovery values (or the market value of debt following default) are typically at a deep discount to par, for example 10 cents on the dollar.

4.1.12.2 Settlement of a CDS Contract

As described in an earlier section, if a credit event occurs then CDS contracts can either be **physically settled** or **cash settled**.

- *Physical settlement*: The protection seller pays the buyer par value, and in return takes delivery of a debt obligation of the reference entity.
- *Cash settlement*: The protection seller pays the buyer the difference between par value and the market price of the debt obligation of the reference entity.

The development and growth of the CDS market has meant that on many companies there now is a much larger outstanding sum of CDS contracts than the outstanding notional value of its debt obligations. This is due to speculations. For example, at the time of bankruptcy on September 14, 2008, Lehman Brothers had approximately $155 billion of outstanding debt but around $400 billion notional value of CDS contracts. Clearly not all of these contracts could be physically settled, since there was not enough outstanding Lehman Brothers debt to fulfil all of the contracts, demonstrating the necessity for cash settled CDS trades.

4.1.12.3 Auctions

When a credit event occurs for a major company on which a lot of CDS contracts are written, an auction (also known as a credit-fixing event) may be held to facilitate the settlement of a large number of contracts at once, at a fixed cash settlement price.

There are two consecutive parts to the auction process. The first stage involves requests for physical settlement and the dealer market process sets the inside market midpoint (IMM). Dealers place orders for the debt of the company that has undergone a credit event. The range of prices received is used to calculate the IMM. The IMM is published for viewing and then used in the second stage of the auction.

After the IMM is published, participants can decide if they would like to submit limit orders for the auction. Limit orders submitted are then matched to open interest orders.

The Lehman Brothers Auction

The Lehman Brothers failure in September 2008 provided a true test of the procedures and systems developed to settle credit derivatives. The auction, which occurred on October 10, 2008, set a price of 8.625 cents on the dollar for Lehman Brothers debt. It was estimated that between \$6 billion and \$8 billion changed hands during the cash settlement of the CDS auction. Recoveries for Fannie Mae and Freddie Mac were much higher 91.51 and 94.00, respectively.

The price of 8.625 means that the sellers of protection on Lehman CDS will have to pay 91.375 cents on the dollar to buyers of protection to settle and terminate the contracts via the Lehman Protocol auction process. In other words, if you had held Lehman Brothers bonds and had bought protection via a CDS contract, you would have received 91.375 cents on the dollar. This would offset your losses on the cash bonds you held. You would have expected to receive par, or 100, when they matured, but would have only received their recovery value after the bankruptcy process concluded. Instead, since you bought protection with a CDS contract, you receive 91.375.

4.1.12.4 Risk

When entering into a CDS, both the buyer and the seller of credit protection takes a counterparty risk

- The buyer takes the risk that the seller may default. If the seller of the CDS and the Risky Corporation default simultaneously (a "double default"), the buyer loses the protection against default by the reference entity. If the seller defaults but not the Risky Corporation, the buyer might need to replace the defaulted CDS at a higher cost.
- The seller takes the risk that the buyer may default so that the seller don't get the expected revenue stream. More important, a seller normally limits its risk by buying offsetting protection from another party – that is, it hedges its exposure. If the original buyer drops out, the seller squares its position by either unwinding the hedge transaction or by selling a new CDS to a third party. Depending on market conditions, that may be at a lower price than the original CDS and may therefore involve a loss to the seller.
- As with other kinds of over-the-counter derivative, CDS might involve liquidity risk. If one or both parties to a CDS contract must post collateral (which is common), there can be margin calls requiring the posting of additional collateral. Many CDS contracts even require payment of an upfront fee composed of "reset to par" and an "initial coupon.".
- Another kind of risk for the seller of CDS is the default risk. A CDS seller could be collecting monthly premiums with little expectation that the reference entity may default. A default creates a sudden obligation to pay millions, if not billions, of dollars to protection buyers. This risk is not present in other over-the-counter derivatives

4.1.12.5 Pricing and valuation

There are two competing theories for the pricing of CDS. The first, is a probability model, which takes the present value of a series of cashflows weighted by their probability of non-default. This method suggests that CDS should trade at a considerably lower spread than corporate bonds.

The second model, proposed by Darrell Duffie, but also by John Hull and Alan White, uses a no-arbitrage approach.

The CDS consists of two legs, a Premium leg and a protection leg. They are calculated as

$$PV(premium\_leg) = -\sum_{i=1}^{n} D_i\,(1-x)^i\,\frac{s}{10000}\cdot T_i$$

and

$$PV(protection\_leg) = N\cdot\sum_{i=1}^{n}\frac{D_{i-1}+D_i}{2}\,(1-x)^{i-1}\cdot x\cdot(1-R)$$

Here R is the recovery rate (= 0.40 as market praxis), x the default probability, N the notional amount and Di the discount factors. The spread s is given in basis points. If a trade is made at a certain price, given at the percentage of the notional, the default probability can be calculated by solving

$$PV(premium\_leg) + PV(protection\_leg) + (100 - price)\cdot N/100 = 0$$

With a known default probability the spread might be calculated as the par value

$$PV(premium\_leg) + PV(protection\_leg) = 0$$

The Probability Model

In the probability model, a CDS is priced using a model that takes four inputs:

- the issue premium,
- the recovery rate (percentage of notional repaid in event of default),
- a credit curve for the reference entity and
- the XIBOR interest rate curve.

If default events never occurred the price of a CDS would simply be the sum of the discounted premium payments. So the CDS pricing models have to take into account the possibility of a default at some time between the effective date and the maturity date of the contract.

We can imagine the case of a 1-year CDS with effective date t_0 with four quarterly premium payments occurring at times t_1, t_2, t_3, and t_4. If the nominal of the CDS is N and the issue premium is c, then the size of the quarterly premium payments is $Nc/4$. If we assume for simplicity that defaults can only occur on one of the payment dates then there are five ways the contract could end.

- either there is no default, so the four premium payments are made and the contract survives until maturity, or
- a default occurs on the first, second, third or fourth payment date.

To price the CDS we now need to assign probabilities to the five possible outcomes and calculate the present value of each outcome. The present value of the CDS is then simply the present value of the five payoffs multiplied by their probability of occurring.

This is illustrated in the tree diagram Fig. 4.23, where at each payment date, either the contract has a default event, with a payment of $N(1-R)$ where R is the recovery rate, or it survives without a default, in which case a premium payment of is made. At either side of the diagram are the cashflows up to that point in time.

The probability of surviving over the interval t_{i-1} to t_i without a default payment is p_i and the probability of a default being triggered is $(1-p_i)$. The calculation of present value, given discount factor of D_1 to D_4 is shown in Table 4.7.

The probabilities above can be calculated using the credit spread curve. The probability of no default occurring over a time period from t to $t+\Delta t$ decays exponentially with a time-constant determined by the

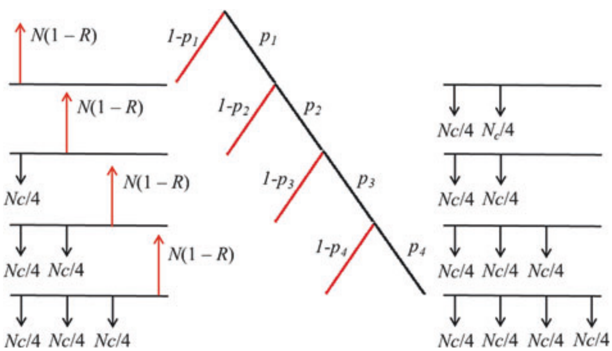


Fig. 4.23 Cash flows and default probabilities for a CDS.

Table 4.7 Payments and default probabilities for a CDS

Defaul Time	Premium Paymen PV	Default Payment PV	Probability
t1	0	$D_1(1-R)N$	$1-p_1$
t2	$-D_1\frac{N_c}{4}$	$D_2(1-R)N$	$P_1(1-p_2)$
t3	$-(D_{1+}D_2)\ \frac{N_c}{4}$	$D_3(1-R)N$	$p_1p_2(1-p_3)$
t4	$-(D_{1+}D_{2+}D_3)\frac{N_c}{4}$	$D_4(1-R)N$	$p_1p_2p_3(1-p_4)$
-	$-(D_{1+}D_2+D_3+D_4)\frac{N_c}{4}$	0	$p_1p_2p_3p_4$

credit spread, or mathematically

$$p(t) = \exp\left\{-\frac{s(t)\cdot\Delta t}{1-R}\right\}$$

where $s(t)$ is the credit spread zero curve at time t. The riskier the reference entity the greater the spread and the more rapidly the survival probability decays with time.

To get the total present value of the CDS we multiply the probability of each outcome by its present value to give

$$\begin{aligned}
PV = {} & D_1(1-p_1)N(1-R) \\
& + p_1(1-p_2)\left[D_2N(1-R) - D_1\frac{Nc}{4}\right] \\
& + p_1p_2(1-p_3)\left[D_3N(1-R) - (D_1+D_2)\frac{Nc}{4}\right] \\
& + p_1p_2p_3(1-p_4)\left[D_4N(1-R) - (D_1+D_2+D_3)\frac{Nc}{4}\right] \\
& - p_1p_2p_3p_4(D_1+D_2+D_3+D_4)\frac{Nc}{4}
\end{aligned}$$

The No-Arbitrage Model

In the **no-arbitrage** model proposed by both Duffie and Hull-White, it is assumed that there is no risk free arbitrage. Duffie uses the LIBOR as the risk free rate, whereas Hull and White use US Treasuries as the risk free rate. Both analyses make simplifying assumptions (such as the assumption that there is zero cost of unwinding the fixed leg of the swap on default), which may invalidate the no-arbitrage assumption. However the Duffie approach is frequently used by the market to determine theoretical prices.

Under the Duffie construct, the price of a CDS can also be derived by calculating the asset swap spread of a bond. If a bond has a spread of 100, and the swap spread is 70 bps, then a CDS contract should trade at 30. However, there are sometimes technical reasons why this will not be the case, and this may or may not present an arbitrage opportunity for the canny investor. The difference between the theoretical model and the actual price of a CDS is known as the basis.

In terms of cash-flow profile, a CDS is most readily comparable with a par floating rate note funded at LIBOR or an asset swapped fixed-rate bond financed in the repo market.

Though default protection should logically trade at a spread relative to a risk-free asset, in practice it trades at a level that is benchmarked to the asset swap market. Most banks look at their funding costs relative to LIBOR and calculate the net spread they can earn on an asset relative to their funding costs. LIBOR represents the rate at which AA-rated banks fund each other in the interbank market for a period of three to six months. Although this is a useful pricing benchmark it is not a risk free rate.

Intuitively, the price of a CDS will reflect several factors. The key inputs would include the following:

- probability of default of the reference entity and protection seller
- correlation between the reference entity and protection seller
- joint probability of default of the reference entity and protection seller
- maturity of the swap and
- expected recovery value of the reference asset.

Though several sophisticated pricing models exist in the market, default swaps are primarily valued relative to asset swap levels. This assumes that an investor would be satisfied with the same spread on a CDS as the spread earned by investing the cash in the asset.

The Asset Swap Approach

Default swap pricing can be based on arbitrage relationships between the derivative and cash instruments. Rather than using complicated pricing models to estimate default probabilities, we can use a simpler pricing mechanism which assumes that the expected value of credit risk is already Captured by the cash market credit spreads.

A CDS is equivalent to a fully funded purchase of a bond with an interest rate hedge.

CDS is an unfunded transaction requiring no initial cash outlay. As a result, the relative value of a CDS is compared to an asset swap rather than a bond's underlying spread over treasuries. An unfunded position in the bond would have to be financed in the repo market.

In a simplified model, the default swap should trade at the same level as an asset swap on the same bond. The asset swap provides a context for relative value because reference assets have transparent prices.

CDS exposure on the LIBOR market can be replicated in the following ways:

- Purchase a cash bond with a spread of $T + S_C$ for par.
- Pay fixed on an interest rate swap ($T + S_S$) with the maturity of the cash bond and receive Libor (L).
- Finance the bond purchase in the repo market. The repo rate is quoted at a spread to Libor ($L - x$).
- Pledge bond as collateral and be charged a haircut by the repo counterparty.

The interest rate swap component eliminates the duration and convexity exposure of the cash bond.

Without this hedge, the trade would be equivalent to a leveraged long position in the fixed rate corporate asset ($T + S_C - (L - x)$). Since a CDS is an unfunded transaction, the bond purchase needs to be fully financed. This financing is achieved with the bond repo. In a repo, collateral is traded for cash. The collateral "seller" borrows cash and lends collateral (a repo)

The collateral "buyer" borrows the collateral and lends cash (a reverse repo). The repo bid/offer refers to the rate at which the collateral can be bought. The bid is higher than the offer since it is the cost of buying cash and selling collateral.

Two important components of a repo trade are:

- Haircut: This is defined as the difference between the securities purchased and the money borrowed. The lender of cash charges a haircut for the loan in order to compensate for market risk of collateral as well as counterparty risk.
- Repo rate: This is the financing cost for the collateral. It varies according to the demand to borrow (or lend) the security. This rate has been denoted as L – x, since several liquid credits have repo rates that are usually, but not always, less than Libor.

The haircut represents the Capital in the trade. As a result, institutions with the cheapest cost of Capital will be able to assume this credit exposure for the lowest net cost. If we assume a haircut of 0 for simplicity, then Table 4.8 shows that the net cash flow is

$$(S_C - S_S) + x$$

If the repo rate for the bond was LIBOR flat ($x = 0$) the exposure would simply be the asset's swap spread ($S_C - S_S$).

This cash flow is similar to that received by a protection seller on a CDS, that is, a simple annuity stream expressed in basis points for the life of the trade.

If the bond defaulted, the repo would terminate and the investor would lose the difference between the purchase price and recovery price of the bond. In efficient markets, arbitrage relationships should drive default swap levels towards the asset swap level. Any mispricing between the markets would be arbitraged away by market makers. For example, if the default premium is greater that the asset swap level, protection sellers would enter the market and drive the default swap premium down towards the asset swap level.

Table 4.8 Cash flows in the asset swap approach

Investor Trade	Receive	Pay
Buy Cash bond	$T + S_c$	100
Swap Hedge	L	$T + S^S$
Repo	100	L^{-X}
Total Cash Flows	$T + S_C + L + 100$	$100 + T + S_S^{-X}$

4.1.13 Hazard rate models

Hazard rate model has become increasingly important when it comes to the pricing of CDS. The primary reason is the enhanced possibility to model the recovery rate associated with the reference entity and the probability of defaults in an explicit way. The financial mathematics behind the hazard rate models will be described here.

The following assumptions are necessary for this model to be applied:

- No counterparty default risk
- No dependence between interest rates, default probabilities and recovery rates.

With a hazard rate model, it is possible to strip out the recovery rate and trade it separately. It is also possible to price more exotic CDS, such as digital CDS, where a hazard rate model is superior to a replication-based model.

In the event of default, it is important to understand the value of the settlement amount. That is, the amount the reference entity owner will receive at default. This amount is calculated by multiplying recovery rate times the claim value. When the recovery rates are non-zero, one must make an assumption about the claim value bondholders will claim in the event of default.

The recovery rate is the percentage rate of the outstanding credit that will be recovered in the event of default. The claim is defined as the outstanding credit.

Normally in the market, the claim is always the same as the face value of the reference obligation. Using this definition, the claim will remain constant over time. It is important that the recovery rate and the claim value uses the same underlying assumption. That is, the recovery rate should be perceived as the percentage rate of the claim value and therefore, as the percentage rate of the face value.

A **Basket Credit Default Swap** is a derived from a credit default swap. The difference between these instrument types is the underlying instrument. Instead of a bond as the underlying instrument as in the case with CDS, the basket credit default swap has a basket of bonds as the underlying.

4.1.13.1 Hazard Rate and Credit Spreads

Let τ denote the uncertain time of default of a firm having issued debt: this can be modelled as a random variable, for which we are interested in finding its probability distribution. The main variable in reduced-form models is the risk-neutral rate of default per unit time, also called *the hazard rate*. This is defined as the probability per unit time of a default occurring, given the knowledge that the firm has not yet defaulted. If $\tau > t$, then the probability that a default occurs during the time interval $[t, t + \Delta t]$ is given by

$$\lambda(t)\,\Delta t + o(\Delta t)$$

where $o(\Delta t)$ denotes a term negligible with respect to Δt. Typically, if time is measured in annual units, λ is a small number: default rates $\lambda(t)$ are between 0.1% and, say, 3%.
The probability of survival, that is, the absence of default between t and $t + \Delta t$, is therefore

$$1 - [\lambda(t)\,\Delta t + o(\Delta t)]$$

Assume now that, conditionally on no default before t, the event of default during $[t, t + \Delta t]$ is independent of the past. Dividing $[0, t]$ into n periods of length Δt, we obtain the probability of no default between 0 and t as follows

$$\prod_{k=1}^{n} (1 - \lambda(t_k)\Delta t), \quad \text{with} \quad t_k = k\Delta t$$

Taking logarithms and using the fact that $\log(1 + x) = x + \mathrm{o}(x)$ for x close to zero, we obtain

$$\log \prod_{k=1}^{n} (1 - \lambda(t_k)\Delta t) = \sum_{k=1}^{n} \log(1 - \lambda(t_k)\Delta t) = -\sum_{k=1}^{n} \lambda(t_k)\Delta t + o(\Delta t) \rightarrow -\int_0^t \lambda(s)ds \quad \text{as} \quad \Delta t \rightarrow 0$$

So, as $\Delta t \to 0$, the expression for the probability of survival up to t becomes

$$\prod_{k=1}^{n} (1 - \lambda(t_k)\Delta t) \to P(\tau > t) = \exp\left[-\int_0^t \lambda(s)ds\right]$$
$$\Rightarrow$$
$$P(\tau > T|\tau > t) = \exp\left[-\int_t^T \lambda(s)ds\right]$$

Let us now introduce the benchmark discount rate $r(t)$ relative to which credit spreads will be computed: typically this will be Libor or the risk-free Treasury rate. The value of a default-free zero-coupon bond is given by

$$B(t, T) = \exp\left[-\int_t^T r(s)ds\right]$$

Denote by $D(t, T)$ the value at time t of a corporate bond with nominal \$1 and maturity T. Denoting by R the recovery rate at default, the corporate bond will pay \$1 at maturity (T) in the absence of default between 0 and T, and 0 otherwise (adjusted for R). The value of this bond is given by the discounted risk-neutral expectation of its terminal payoff

$$\begin{aligned}
D(t, T) &= \exp\left[-\int_t^T r(s)ds\right] E\left[1 \cdot I_{\{\tau>t\}} + R \cdot I_{\{\tau\leqslant T\}}|\tau > t\right] \\
&= \exp\left[-\int_t^T r(s)ds\right] E\left[I_{\{\tau>t\}} + R\left(1 - I_{\{\tau>T\}}\right)|\tau > t\right] \\
&= \exp\left[-\int_t^T r(s)ds\right] \left[(1-R)\,E\left[I_{\{\tau>t\}}|\tau > t\right] + R\right] \\
&= (1-R) \cdot \exp\left[-\int_t^T (r(s) + \lambda(s))\,ds\right] + R \cdot \exp\left[-\int_t^T r(s)ds\right]
\end{aligned}$$

Comparing this expression with the value of a default-free bond, we observe that the principal effect of default risk is to modify the discount factor by adding a spread to the short rate: this spread is none other than the hazard rate $\lambda(t)$. This is one of the principal results of reduced-form models. The hazard rate can be identified from the term structure of credit spreads, a quantity that is observable in the market. Please note, however, that the above relation requires knowledge of the recovery rate, which is typically quite uncertain. In the case of zero recovery

$$D(t,T) = \exp\left[-\int_t^T (r(s) + \lambda(s))\,ds\right] = \exp\left[-\int_t^T r(s)ds + (T-t)s(t,T)\right]$$

so the term structure $s(t,T)$ of credit spreads is simply given by

$$s(t,T) = \frac{1}{T-t}\exp\left[-\int_t^T \lambda(s)ds\right]$$

The hazard rate $\lambda(t)$ can thus be chosen to reproduce an arbitrary term structure of credit spreads. This is an important advantage of reduced-form models when compared to structural models, where the form of the credit spread term structure is imposed. In particular, it is possible to obtain arbitrary credit spreads for short maturities, which was not possible in the Merton or Black–Cox models.[5]

As noted above, retrieving hazard rates from credit spreads requires knowledge of the recovery rates. Since these are unknown in practice, the market standard is to use data collected by rating agencies for estimates of the average recovery rate by seniority and instrument type.

4.1.13.2 Pricing and Hedging of Credit Derivatives in Reduced-Form Models

To illustrate how the above concepts can be used to price credit derivatives, let us apply them to the pricing of CDSs. As discussed above, the

[5] http://www.larrylisblog.net/WebContents/Financial%20Models/BlackCoxModel.pdf

breakeven spread in a CDS should be the spread at which the present values of premium payments and the probability-weighted payoff of the protection legs are equal. Assume for simplicity that the hazard rate λ and the discount rate r are constant. The premium leg pays continuously a spread S and its present value is given by

$$S \cdot \int_0^T \exp\{-(r+\lambda)t\}\, dt = S \cdot \frac{1-\exp\{-(r+\lambda)T\}}{r+\lambda}$$

The protection leg pays $(1-R)$ in case of default: this occurs with probability $\lambda(t)dt$ during $[t, t+dt]$. So the discounted value of the protection leg is

$$(1-R) \cdot \int_0^T \lambda \cdot \exp\{-(r+\lambda)t\}\, dt = (1-R) \cdot \lambda \cdot \frac{1-\exp\{-(r+\lambda)T\}}{r+\lambda}$$

Equalizing the two expressions yields the following expression for the fair value of the CDS spread

$$S = \lambda\,(1-R)$$

The spread risk can then be calculated as

$$\begin{aligned} CV01 &= \frac{\partial}{\partial S}\left(S \cdot \frac{1-\exp\{-(r+\lambda)T\}}{r+\lambda}\right) = \frac{1-\exp\{-(r+\lambda)T\}}{r+\lambda} \\ &= \frac{1-\exp\left\{-\left(r(T)+\frac{S}{1-R}\right)T\right\}}{r(T)+\frac{S}{1-R}} \end{aligned}$$

The relation $S = \lambda\,(1-R)$ states that the credit spread compensates the investor for the risk of default per unit time. Note that in this simple case where the term structure of interest rates is flat, the interest rate does not appear in the relation between the CDS spread and the default parameters. This corresponds to the idea that a position in CDS is a pure "credit exposure" and, at least at first order, does not incorporate interest rate risk. When the term structure is not flat, the above computations can be carried out similarly but the interest rate dependency does not simplify the CDS spread which is then exposed to movements in the yield curve.

The main idea of the above-mentioned relation between hazard rates and CDS spreads is that once the CDS-spreads are known, it is possible to deduce the hazard rate using the relation above and to price any credit derivative. Note that this hazard rate is a risk-neutral rate of default, implied by market-quoted CDS spreads. In principle, and also in practice, it can be quite different from historically estimated default rates or from those implied by transition rates of credit ratings. In the language of arbitrage pricing theory, the hazard rate represents the risk-neutral probabilities of default while historical studies estimate "objective" default probabilities. Once the (risk neutral) hazard rate has been calibrated to spreads on credit-sensitive bonds or CDSs, it can be used to value a credit-sensitive payoff H, by taking the discounted risk-neutral expectation of H

$$V_t(H) = \exp\left[-\int_t^T r(s)ds\right] E\left[H|\mathcal{F}_t\right]$$

Denoting by H_0 the payoff in case of default and H_1 the payoff in case of survival, we obtain

$$\begin{aligned} V_t(H) = {} & \exp\left[-\int_t^T r(s)ds\right]\left(1 - \exp\left[-\int_t^T \lambda(s)ds\right]\right) E\left[H_0|\mathcal{F}_t\right] \\ & + \exp\left[-\int_t^T (r(s) + \lambda(s))\, ds\right] E\left[H_1|\mathcal{F}_t\right] \end{aligned}$$

This pricing approach guarantees the absence of arbitrage opportunities between, on the one hand, the instruments priced in this manner and, on the other hand, liquid instruments to which the models are calibrated (CDSs, corporate bonds, etc.). However, it cannot give us an indication of the "fair value" of a credit spread in absolute terms. Also, while it is still possible to compute "sensitivities" of the model price of a credit derivative with respect to, for instance, CDS spreads on the underlying credits, there is no element in the model that allows us to interpret such sensitivities as hedge. In fact, as opposed to structural models, which typically lead to complete markets, hedging in reduced-form models cannot be done by simple delta hedging. Delta hedging will only neutralize a derivative with respect to small movements in an underlying spread but not to losses incurred in case of jump-to-default (which is actually the main concern!) or unknown recovery rates. These need to be measured independently, typically through stress-test scenarios. Arrangers are then led to calculate reserves on their initial profit and loss, which are released over time, in order to account for these risks.

4.1.14 Total Return Swaps

Total return swaps are bilateral financial transactions (OTC) where the total return (which equals the coupon plus price change) on a fixed income security is exchanged for a funding cash flow, usually LIBOR plus a basis spread. Total return swaps are the historic precursors to CDSs. Investors in risky assets started swapping their risky returns for safe returns to others which in case of a default also offered to compensate the investor for any loss of principal. Note that this meant swapping the total returns which explains the name TRS. The problem with this was that besides the credit risk both parties also were subject to interest rate risk. This led to the creation of a cleaner instrument which only hade credit risk. Hence the CDS was born. In a CDS the investors only pay a premium to the party that is accepting the risk. This is like an insurance policy. And just like in insurance the counterparty of the CDS only pays out if a "credit event" occurs. Exactly what is meant by a credit event is specified in the CDS contract. Unlike CDS, in a TRS payments to balance the underlying credit's price depreciation or appreciation are always exchanged without requiring the occurrence of a specific credit event. Total Return swaps are beneficial to investors as it involves a leveraged participation in a fixed income instrument without the origination cost. Total Return swaps are particularly attractive to investment firms that want to diversify their portfolio credit exposure.

4.1.15 Caps, Floors and Collars

Interest rate caps and floors are basic products for hedging floating rate risk. A cap is a call option on the future realisation of a given underlying LIBOR rate. More specifically, it is a collection (or strip) of caplets, each of which is a call option on the LIBOR level at a specified date in the future. Similarly, a floor is a strip of floorlets, each of which is a put option on the LIBOR level at a given future date. caps and floors are widely traded OTC instruments. As explained below, they provide protection against widely fluctuating interest rates – a cap, for instance, is insurance against rising interest rates. caps and floors also reflect market views on the future volatility of LIBOR rates. caps and floors can be compared to Call and Put options in the equity markets.

The parameters of caps and floors are:

- Notional
- Cash-flow dates
- Floating rate
- Strike rate.

4.1.15.1 Caps and Caplets

A plain vanilla **(interest rate) Cap** is, as mentioned above, a series of European interest rate call options (called Caplets), with a particular interest rate strike, each of which expire on the date when a floating loan/swap rate will be reset.

For concreteness, suppose the underlying interest rate is the τ – maturity LIBOR. Let $L(t, T, \tau)$ denote[6] the forward LIBOR at time t for the accrual period $[T, T + \tau]$. The spot LIBOR at time T is then, by definition, $L(T, T, \tau)$. This rate fixes at time T, and one cash unit invested at this rate which pays $1 + \tau \cdot L(T, T, \tau)$ at time $T + \tau$. The maturity τ is expressed in terms of fractions of a year – for example, $\tau = 0.25$ for a 3-month LIBOR.

At each interest payment date the holder of a cap will exercise the current caplet if the strike rate is below the LIBOR swap rate. The seller have the obligation to compensate the buyer for the differences between the strike and the swap rate. Therefore, caps are often used by borrowers in order to hedge a floating interest rate.

Banks and financial institutions will use caps to limit their risk exposure to upward movements in the floating interest rate. caps are equally attractive to speculators which can get a profit due to the volatility in market interest rates (Fig. 4.24).

[6] In $L(t, T, \tau)$, the first argument is current time, the second argument is the start date for the accrual period, and the third argument is the length of the accrual period.

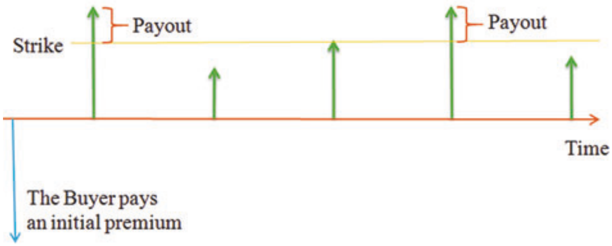


Fig. 4.24 The payout from a cap when the floating rate exceeds the cap-rate (strike level).

The price of a cap is the sum of its constituent caplet prices, and so we focus on these first. A caplet is a call option on L. Specifically, a caplet with maturity date T and strike rate K has the following payoff: at time $T+\tau$, the holder of the caplet receives at $T+\tau$

$$\Pi_T = (L(T,\ T, \tau) - K)+$$

Note that the caplet expires at time T, but the payoff is received at the end of the accrual period, that is, at time $T+\tau$. The payoff is day-count adjusted. The liabilities of the holder of this caplet are always bounded above by the strike rate K, and clearly if interest rates increase, the value of the caplet increases, so that the holder benefits from rising interest rates.

By the usual arguments, the price of this caplet is given by the discounted risk-adjusted expected payoff. If $\{p(t,\ T) : T \geqslant t\}$ represents the observed term structure of zero-coupon bond prices at time t, then the price of the caplet is given by

$$\Pi_t = \tau \cdot p(t, T+\tau)E^Q\left[(L(T, T, \tau) - K)^+\right]$$

In this equation, the only random term is the future spot LIBOR, $L(T, T, \tau)$. The price of the caplet therefore depends on the distributional assumptions made on $L(T, T, \tau)$. One of the standard models for this is the Black model, described in a later section. According to this model, for each maturity T, the risk-adjusted relative changes in the forward LIBOR $L(t, T, \tau)$ are normally distributed with specified constant volatility σ_T, that is,

$$\frac{dL(t, T, \tau)}{L(t, T, \tau)} = \sigma_T dW(t)$$

This implies a lognormal distribution for $L(t, T, \tau)$, and under this modelling assumption the price of the T – maturity caplet is given by

$$C(t) = \tau \cdot p(t, T + \tau) \left[L(t, T, \tau) \cdot N(d_1^T) - K \cdot N\left(d_2^T\right)\right]$$

where

$$d_{1,2}^T = \frac{\ln\left\{\frac{L(t,T,\tau)}{K}\right\} \pm \frac{1}{2}\sigma_T^2(T-t)}{\sigma_T\sqrt{T-t}}$$

For low interest rates, the Black model above can't be used. With really low or negative rates, the logarithm above is not defined. Even for small positive rates, regulators want banks to stress a decrease of the rate, so that the stressed rate might be negative. At the writing moment (February 2015) the interest rates are negative for short maturities in many currencies. The solution is to use a normally distributed rate with the following process under the risk neutral measure

$$dL(t, T, \tau) = \sigma_T^N dW(t)$$

Under this modelling assumption the price of the T – maturity caplet is given by

$$C(t) = \tau \cdot p(t, T + \tau) \left[(L(t, T, \tau) - K) \cdot N(d) + \frac{\sigma_T^N \sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2}\right]$$

where

$$d = \frac{L(t, T, \tau) - K}{\sigma_T^N \sqrt{T-t}}$$

As we can see, this model, called Normal Black or the Bachelier's model are valid also for negative rates (or strikes).

4.1.15.2 Normal Volatility to Black Volatility

When using different models, Black or Normal Black, it is possible to convert from one volatility type to another. In particular this is easy when you have an at-the-money (ATM) caplet. A formula also valid for strikes not given ATM is presented in a section below. This is important when rates are low (near zero or negative) and when the market praxis moves to normally distributed models. The reason is, as seen above, that the Black model cannot be used. If the strike or the forward rate is zero or below, the variable d_1 (and d_2) are not defined due to the logarithm ($\ln(L/K)$, where L here is the (forward) LIBOR rate and K the strike rate). The market will then quote the prices in normal volatility.

We will now show how we can translate between the normal and log-normal volatilities for ATM prices, that is, where the strike and forward rate are equal.

The normal model is, as we have seen above, derived from the risk-neutral stochastic process for the forward rate F

$$dF = \sigma_N dW_t$$

giving a price function for a call option (i.e. a Caplet) as

$$C_N = e^{-rT}\left[(F - K) \cdot N(d) + \frac{\sigma_N \sqrt{T}}{\sqrt{2\pi}} e^{-d^2/2}\right]$$

where

$$d = \frac{F - K}{\sigma_N \sqrt{T}}$$

If $K = F$, $d = 0$ giving

$$C_N = e^{-rT} \frac{\sigma_N \sqrt{T}}{\sqrt{2\pi}}$$

Similarly, the Black model is derived from the process:

$$dF = \sigma_B F dV_t$$

giving a price function for a call option as

$$C_B = e^{-rT} \cdot \{F \cdot N(d_1) - K \cdot N(d_2)\}$$

where

$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma_B^2 \cdot T}{\sigma_B \sqrt{T}}, \quad d_2 = d_1 - \sigma_B \sqrt{T}$$

If $K = F$, we get

$$C_B = e^{-rT} \cdot \left\{ F \cdot N\left(\frac{1}{2}\sigma_B\sqrt{T}\right) - F \cdot N\left(-\frac{1}{2}\sigma_B\sqrt{T}\right) \right\}$$
$$= e^{-rT} F \cdot \left\{ 2 \cdot N\left(\frac{1}{2}\sigma_B\sqrt{T}\right) - 1 \right\}$$

Since the prices must be the same (independent of the model), that is, $C_N = C_B$ we must have

$$\sigma_N\sqrt{\frac{T}{2\pi}} = F\left\{2 \cdot N\left(\frac{1}{2}\sigma_B\sqrt{T}\right) - 1\right\}$$

or

$$\sigma_N = F \cdot \sqrt{\frac{2\pi}{T}} \cdot \left\{2 \cdot N\left(\frac{1}{2}\sigma_B\sqrt{T}\right) - 1\right\}$$

This can be inverted to find the Black (log-normal) volatility from the corresponding normal volatility as

$$\sigma_B = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{\sigma_N}{2F}\sqrt{\frac{T}{2\pi}} + \frac{1}{2}\right)$$

where we have to invert the normal distribution. The argument in this function must be a probability, so the normal volatility must satisfy the following condition

$$\sigma_N \leqslant F\sqrt{\frac{2\pi}{T}}$$

4.1.15.3 Caps as a Strip of Caplets

Let's go back to caplets and look at some details.

Suppose we have a loan with the face value of N and payment dates $t_1 < t_2 < \cdots < t_n$, where $t_{i+1} - t_i = \tau$ for all i. In practice, there will not be exactly the same number of days between successive reset dates, and the calculations below must be slightly adjusted by using the relevant *day-count convention*. The interest rate to be paid at time t_i is determined by the τ-period money market interest rate prevailing at time $t_i - \tau$, that is, the payment at time t_i is equal to $N\tau L(t_i, t_{i\delta})$. Note that the interest rate is set at the beginning of the period, but paid at the end. Define $t_0 = t_1 - \tau$. The date's $t_0,\ t_1,\ \ldots,\ t_{n-1}$ where the rate for the coming period is determined are called the **reset dates** of the loan or swap.

A cap with a face value of N, payment dates $t_i (i = 1,\ \ldots,\ n)$ as above and a so-called cap rate K yields a time t_i payoff of $N\tau\ \max\{L(t_i,\ t_i - \tau) - K,\ 0\}$, for $i = 1,\ 2,\ \ldots,\ n$. If a borrower buys such a cap, the total payment at time t_i cannot exceed $N\tau K$. The period length τ is often referred to as the **frequency** or the **tenor** of the cap. In practice, the frequency is typically either 3, 6 or 12 months. Note that the time distance between payment dates coincides with the "maturity" of the floating interest rate. Also, note that while a cap is tailored for interest rate hedging, it can also be used for interest rate speculation.

A **caplet** can be characterized as a put option on a zero-coupon bond. The payoff at time t_i is equivalent to

$$\frac{N\tau}{1 + \tau L(t_i, t_i - \tau)} \max(L(t_i, t_i - \tau) - K, 0)$$

or

$$\max\left(N - \frac{N(1 + \tau K)}{1 + \tau L(t_i, t_i - \tau)}, 0\right)$$

The expression $N(1 + \tau K) \big/ (1 + \tau L(t_i, t_i - \tau))$ is the value of a zero-coupon bond that pays $N(1 + \tau K)$ at time t_i. The expression above is therefore the payoff from a put option, with maturity t_i, on a zero-coupon bond with maturity t_i when the face value of the bond is $N(1 + \tau K)$ and the strike price is K. It follows that an interest rate cap can be regarded as a portfolio of European put options on zero-coupon bonds.

In the following, we will find the value of the i'th caplet before time t_i. Since the payoff becomes known at time $t_i - \tau$, we can obtain its value in the interval between $t_i - \tau$ and t_i by a simple discounting of the payoff, that is,

$$C_t^i = p(t, t_i) N\tau \max\{L(t_i, t_i - \tau) - K, 0\}, \qquad t_i - \tau \leqslant t \leqslant t_i$$

In particular,

$$C^i_{t_i-\tau} = p(t_i, t_i - \tau)N\tau \max\{L(t_i, t_i - \tau) - K, 0\}.$$

The relation between the price of a zero-coupon bond at time t with maturity at T and the forward rate $L(t, T)$ is

$$p(t, T) = \frac{1}{1 + L(t, T) \cdot (T - t)}$$

or

$$L(t, T) = \frac{1}{T - t}\left(\frac{1}{p(t, T)} - 1\right)$$

We then have

$$\begin{aligned} C^i_{t_i-\tau} &= p(t_i, t_i - \tau)N \max\{1 + \tau L(t_i, t_i - \tau) - (1 + \tau K), 0\} \\ &= p(t_i, t_i - \tau)N \max\left\{\frac{1}{p(t_i, t_i - \tau)} - (1 + \tau K), 0\right\} \\ &= N(1 + \tau K) \max\left\{\frac{1}{1 + \tau K} - p(t_i, t_i - \tau), 0\right\} \end{aligned}$$

We can now see that the value at time $t_i - \tau$ is identical to the payoff of a European put option expiring at time $t_i - \tau$ that has an exercise price of $1/(1 + \tau K)$ and is written on a zero-coupon bond maturing at time t_i. Accordingly, the value of the i'th caplet at an earlier point in time $t \leqslant t_i - \tau$ must equal the value of that put option. If we denote the price of a call option on a zero-coupon bond at time t, with the strike price K, expiry T and where the bond expires at time S with $\pi(t, K, S, T)$, that is,

$$\pi(t, K, S, T) = \max\{p(T, S) - K, 0\}$$

We can write

$$C^i_t = N(1 + \tau K)\pi\left(t, \frac{1}{1 + \tau K}, t_i - \tau, t_i\right).$$

To find the value of the entire cap contract we simply have to add up the values of all the caplets corresponding to the remaining payment dates of the cap. Before the first reset date, t_0, none of the cap payments are known, so the value of the cap is given by

$$C_t = \sum_{i=1}^{n} C_t^i = N(1+\tau K) \sum_{i=1}^{n} \pi \left(t, \frac{1}{1+\tau K}, t_i - \tau, t_i \right), \qquad t < t_0.$$

At all dates after the first reset date, the next payment of the cap will already be known. If we again use the notation $t_{i(t)}$ for the nearest following payment date after time t, the value of the cap at any time t in $[t_0, t_n]$ (exclusive of any payment received exactly at time t) can be written as

$$\begin{aligned} C_t &= Np(t, t_{i(t)})\tau \max\left\{L(t_{i(t)}, t_{i(t)} - \tau) - K, 0\right\} \\ &\quad + N(1+\tau K) \sum_{i=i(t)+1}^{n} \pi \left(t, \frac{1}{1+\tau K}, t_i - \tau, t_i \right), \quad t_0 \leqslant t \leqslant t_n \end{aligned}$$

If $t_{n-1} < t < t_n$, we have $i_{(t)} = n$, and there will be no terms in the sum, which is then considered to be equal to zero. In later sections, we will discuss models for pricing bond options. From the results above, cap prices will follow from prices of European puts on zero-coupon bonds.

Note that the interest rates and the discount factors appearing in the expressions above are taken from the money market, not from the government bond market. Also note that since caps and most other contracts related to money market rates trade OTC, one should take the default risk of the two parties into account when valuing the cap. Here, default simply means that the party cannot pay the amounts promised in the contract. Official money market rates and the associated discount function apply to loan and deposit arrangements between large financial institutions, and thus they reflect the default risk of these corporations. If the parties in an OTC transaction have a default risk significantly different from that, the discount rates in the formulas should be adjusted accordingly. However, it is quite complicated to do that in a theoretically correct manner, so we will not discuss this issue any further.

4.1.15.4 Floors and Floorlets

A plain vanilla **(interest rate) floor** (as opposite to a cap) is represented as a series of European interest rate options (called floorlets), with a particular interest rate strike, each of which expire on the date the floating loan/swap rate will be reset.

At each interest payment date the seller of a floor agrees to compensate the buyer for a rate falling below the specified rate during the contract period. The difference occurs in that on each date the writer pays the holder if the reference rate drops below the floor. Lenders often use this method to hedge against falling interest rates. The step up cap counteracts this by raising the strike of the later caplets to reflect the higher forward rates (see Fig. 4.25).

The Buyer of a floor receives pay-outs when the floating rate falls below the floor-rate.

Therefore, a floor is designed to protect an investor who has lent funds on a floating rate basis against receiving very low interest rates. It is similar to a cap except that it is structured to hedge against decreasing interest rates (or downside risk). Interest rate floors can be purchased OTC from a bank. As a contract, when a chosen reference rate falling below the floor's interest rate level (the floor rate), the interest floor seller agrees to reimburse the buyer for the difference, calculated on a notional principal amount and for a certain period. Therefore, the chosen reference rate must drop below the floor rate before any cash payment takes place between the two parties.

An interest rate floor closely resembles a portfolio of put option contracts. The key elements of a floor are maturity, floor rate, reference floating rate, reset period and the notional principal amount.

A typical interest rate floor can be considered as a portfolio of interest rate floorlets, which only yield payment on one period of time. We can consider floorlets as a European put on the chosen reference rate with delayed payment of the payoff.

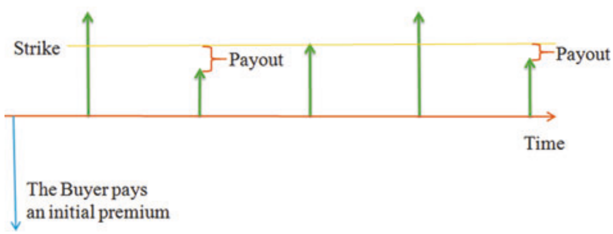


Fig. 4.25 The payout from a floor when the floating rate falls below the floor-rate (strike level).

The contract is constructed just as a cap except that the payoff at time $t_i (i = 1, \ldots, n)$ is given by

$$F^i_{t_i-\tau} = N\tau \max \{K - L(t_i, t_i - \tau), 0\} .$$

where K is called the floor rate. Buying an appropriate floor, an investor who has provided another investor with a floating rate loan will in total at least receive the floor rate. Of course, an investor can also speculate in low future interest rates by buying a floor.

The (hypothetical) contracts that only yield one of the payments above are called **floorlets**. Analogously to the analysis for caps, we consider each floorlet as a European call on a zero-coupon bond, and hence a floor is equivalent to a portfolio of European calls on zero-coupon bonds. More precisely, the value of the i'th floorlet at time $t_i - \tau$ is

$$F^i_{t_i-\tau} = N(1 + \tau K) \max \left\{ p(t_i, t_i - \tau) - \frac{1}{1 + \tau K}, 0 \right\} .$$

The total value of the floor contract at any time $t < t_0$ is therefore given by

$$F_t = N(1 + \tau K) \sum_{i=1}^{n} \pi \left(t, \frac{1}{1 + \tau K}, t_i - \tau, t_i \right), \qquad t < t_0.$$

and later the value is

$$\begin{aligned} F_t &= Np(t, t_{i(t)})\tau \max \left\{ K - L(t_{i(t)}, t_{i(t)} - \tau), 0 \right\} \\ &\quad + N(1 + \tau K) \sum_{i=i(t)+1}^{n} \pi \left(t, \frac{1}{1 + \tau K}, t_i - \tau, t_i \right), \quad t_0 \leqslant t \leqslant t_n \end{aligned}$$

4.1.15.5 Pros and Cons

The major advantages of caps are that the buyer limits his potential loss to the premium paid, while retaining the right to benefit from favourable rate movements. The borrower buying a cap limits his exposure to rising interest rates, while retaining the potential to benefit from falling rates. An upper limit is therefore placed on borrowing costs.

Interest rate options like caps and floors are highly geared instruments and, for a relatively small outlay of Capital, purchasers can make considerable profits. At the same time, a seller with a decay strategy in mind (i.e. where he would like the option's value to decay over time so that it can be bought back cheaper at a later stage or even expire worthless) can make a profit amounting to the option premium, without having to make a Capital outlay.

The disadvantages of caps are that the premium is a non-refundable cost, which is paid upfront by the buyer, and the negative impact of an immediate cash outflow. Caps can theoretically lose all their value (i.e. the premium paid) if they expire as out-the-money or start to approach their expiry dates. In addition, there are high potential losses for writers (sellers) of option-type interest rate derivatives if market movements are contrary to market expectations. Also, one needs to keep in mind that the bid/offer spreads on most option-type interest rate derivative products are quite wide.

4.1.15.6 Strategies

The cap is a guarantee of a future rate. The implied forward rate will change over time as the market changes its view of future rates. The price of the cap will therefore depend on the likelihood that the market will change its view. This likelihood of change is measured by volatility. An instrument expected to be volatile between entry and maturity will have a higher price than a low volatility instrument. The volatility used in calculating the price should be the expected future volatility. This is based on the historic volatility.

As time goes by, the volatility will have less and less impact on the price, as there is less time for the market to change its view. Therefore, in a stable market, the passing of time will lead to the cap falling in value. This phenomenon is known as Time Decay. This increases in severity, as we get closer to maturity. The higher the strike compared to prevailing interest rates the lower the price of the cap. High strike ("*out-of-the-money*") caps will be cheaper than "*at-the-money*" or low strike ("*in-the-money*") because of the reduced probability of the caplets being in the money during the life of the option. The price of the cap will increase with the length of the tenor, as it will include more caplets to maturity. The further the strike is set *out-of-the-money*, the cheaper the cap, as the probability of payout is less, therefore the cap is considered to be more leveraged. As rates rise, the cap will increase in value as it becomes closer to the money.

It is therefore an interesting strategy to buy out-of-the-money caps for a small premium, which will increase in value dramatically (due to the leverage) if rates rise. This is a trading strategy rather than a buy-and-hold strategy. Sophisticated Investors or Borrowers may like to sell caps to benefit from time decay. This is also known as writing caps. In this case, the seller is providing the guarantee and therefore has an unlimited loss potential. The profit from this strategy is limited to the premium earned and will occur only when there are no claims against the cap.

In the market, traders will use volatility to quantify the probability of changes around interest rate trends. Higher volatility will increase the probability of a Caplet being in the money and therefore the price of the cap.

Corridor

This is a strategy where the cost of purchasing a cap is offset by the simultaneous sale of another cap with a higher strike. It is possible to offset the entire cost of the cap purchase by increasing the notional amount on the cap sold to match the purchase price. The inherent risk in this strategy is that if short-term rates rise through the higher strike the purchaser is no longer protected above this level and will incur considerable risk if the amount of the cap sold is proportionately larger.

Step up Cap

In steep yield curve environments the implied forward rates will be much higher than spot rates and the strike for caplets later in the tenor may be deep in the money. The price of a cap, being the sum of the caplets, may prove prohibitively expensive. The step up cap counteracts this by raising the strike of the later caplets to reflect the higher forward rates. This may provide a more attractive combination of risk hedge at a lower price. The payoff diagram is shown as in the Fig. 4.26.

After purchasing the cap, the buyer can make "claims" under the guarantee should Libor be above the level agreed on the cap on the

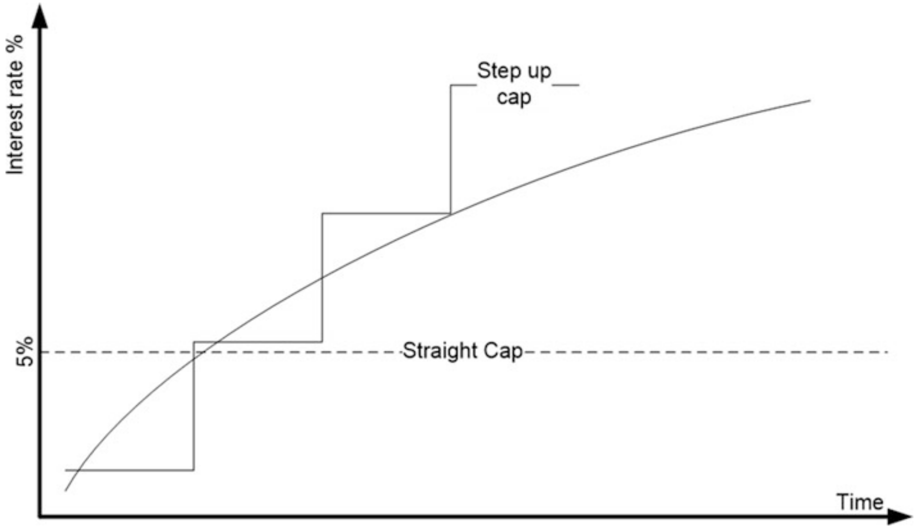


Fig. 4.26 A step-up cap strategy.

settlement dates. A cap is NOT a continuous guarantee; claims can only be made on specified settlement dates. The purchaser selects these dates.

4.1.15.7 Numerical Example using a Binomial Tree

We will consider a binomial tree for a 2-year semi-annual floor on \$100 notional amount with strike rate $K = 4.5\ \%$, indexed to the 6-month rate. At time 0, the 6-month rate is 5.54 % so the floor is out-of-the-money, and pays \$0 at time 0.5. The later payments of the floor depend on the path of interest rates. Suppose rates follow the path in the tree below. The value of the floor is the sum of the values of the 4 puts on the six-months rates at times 0, 0.5, 1, and 1.5. We begin the evaluation at time 1.5 (Fig. 4.27).

As in binomial valuation of put options, we calculate backwards from time 1.5. At time 1.5, the only possible floating interest rate below the floor rate of 4.5 per cent is 3.823 per cent. And, only when the floating interest rate falls to this level, will the buyer of the floor be compensated with the cash of \$33.85 at time 2. For the calculation,

$$\$0.3385 = \$100 \times (4.5\% - 3.823\%)/2$$

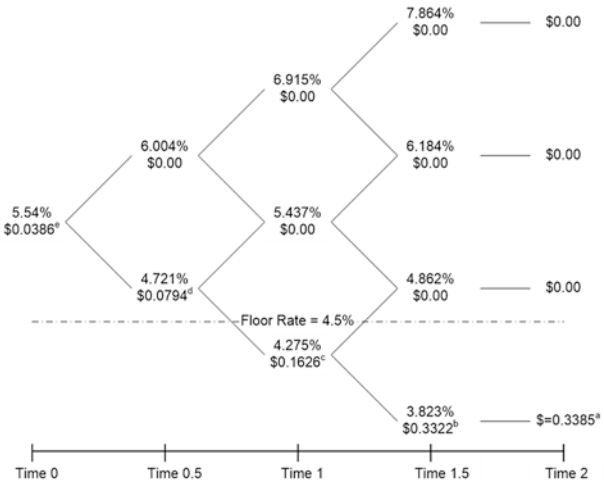


Fig. 4.27 The binominal tree for a floor at time 1.5 year.

For the value of $33.85, at time 1.5, its present values is,

$$\$0.3322 = \$0.3385/(1 + 3.823\%/2)$$

For time 1, 0.5 and 0, assume that in this example, the probabilities to raise and fall are both 50 per cent. The calculations for the values at these time spots will be

$$\$0.1626 = 0.5 \times (0 + \$0.3322)/(1 + 4.275\%/2)$$
$$\$0.0794 = 0.5 \times (0 + \$0.1626)/(1 + 4.721\%/2)$$
$$\$0.0386 = 0.5 \times (0 + \$0.0794)/(1 + 5.54\%/2)$$

Then, we calculate the floorlet due at Time 1 as below (Fig. 4.28).

Base on the same calculation method, we simply provide here the calculation for each node as

$$\$0.1125 = \$100 \times (4.5\% - 4.275\%)/2$$
$$\$0.1101 = \$0.1125/(1 + 4.275\%/2)$$
$$\$0.0538 = 0.5 \times (0 + \$0.1101)/(1 + 4.721\%/2)$$
$$\$0.0262 = 0.5 \times (0 + \$0.0538)/(1 + 5.54\%/2)$$

At time 0.5 and 0, the floorlets never get in the money, so the value of the floor will be \$ 0.0648 = \$0.0386 + \$0.0262.

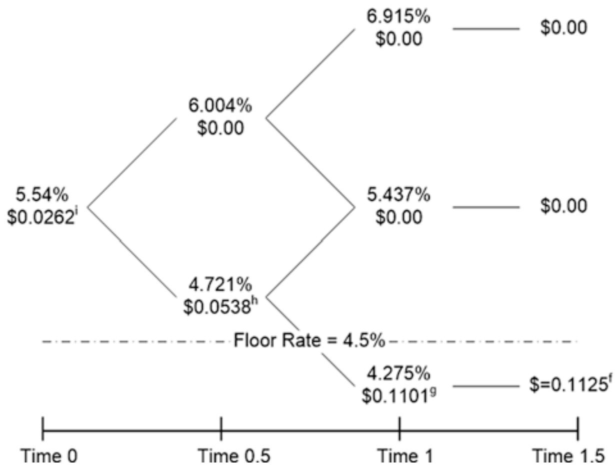


Fig. 4.28 The binominal tree for a floor at time 1 year.

4.1.15.8 Collars

Combining a cap and a floor into one product creates a "collar". A **collar** is a contract designed to ensure that the interest rate payments on a floating rate borrowing arrangement stays between two pre-specified levels. Collars can benefit both borrowers and investors. In the case of a borrower, the collar protects against rising rates but limits the benefits of falling rates. In the case of an investor, the collar protects against falling rates but limits the benefits from rising rates. Similar to caps and floors the customer selects the index, the length of time, and strike rates for both the cap and the floor. However, unlike a single cap and a floor, an up-front premium may or may not be required, depending upon where the strikes are set. In either scenario, the customer is a buyer of one product, and a seller of the other.

The buyer and the seller agree upon the term (tenor), the cap and floor strike rates, the notional amount, the amortization, the start date and the settlement frequency. If at any time during the tenor of the collar, the index moves above the cap strike rate or below the floor strike rate, one party will owe the other a payment. The payment is calculated as the difference between the strike rate and the index times the notional amount outstanding times the day's basis for the settlement period.

A typical collar can be seen as a portfolio of a long position in a cap with a cap rate K_c and a short position in a floor with a floor rate of $K_f < K_c$ (and the same payment dates and underlying floating rate). The payoff of a collar at time $t_i, i = 1, 2, \ldots, n$, is thus

$$\begin{aligned} P^i_{t_i} &= N\tau \left[\max\left\{L(t_{i(t)}, t_{i(t)} - \tau) - K_c, 0\right\} - \max\left\{K_f - L(t_{i(t)}, t_{i(t)} - \tau), 0\right\}\right] \\ &= \begin{cases} N\tau \left[K_f - L(t_{i(t)}, t_{i(t)} - \tau)\right], & \text{if} \quad L(t_{i(t)}, t_{i(t)} - \tau) \leqslant K_f \\ 0, & \text{if} \quad K_f \leqslant L(t_{i(t)}, t_{i(t)} - \tau) \leqslant K_c \\ N\tau \left[L(t_{i(t)}, t_{i(t)} - \tau) - K_c\right] & \text{if} \quad K_c \leqslant L(t_{i(t)}, t_{i(t)} - \tau) \end{cases}. \end{aligned}$$

The value of a collar with cap rate K_c and floor rate K_f is of course given by

$$L_t\left(K_c, K_f\right) = C_t\left(K_c\right) - F_t\left(K_f\right),$$

where the expressions for the values of caps and floors derived earlier can be substituted in. An investor who has borrowed funds on a floating rate basis will by buying a collar ensure that the paid interest rate always lies in the interval between K_f and K_c. Clearly, a collar gives cheaper protection against high interest rates than a cap (with the same cap rate K_c), but on the other hand the full benefits of very low interest rates are sacrificed. In practice, K_f and K_c are often set such that the value of the collar is zero at the inception of the contract.

Fig. 4.29 illustrates the payoff from buying a one-period zero-cost interest rate collar. If the index interest rate r is less than the floor rate r_f on the interest rate reset date, the floor is in-the-money and the collar buyer (who has sold a floor) must pay the collar counterparty an amount equal to. When r is greater than r_f but less than the cap rate r_c, both the floor and the cap are out-of-the-money and no payments are exchanged. Finally, when the index is above the cap rate the cap is in-the-money and the buyer receives $N \times (r - r_c) \times d_t \div 360$.

Fig. 4.30 illustrates a special case of a zero-cost collar that results from the simultaneous purchasing of a one-period cap and sale of a one-period floor when the cap and floor rates are equal. In this case, the combined transaction replicates the payoff of an FRA (Forward

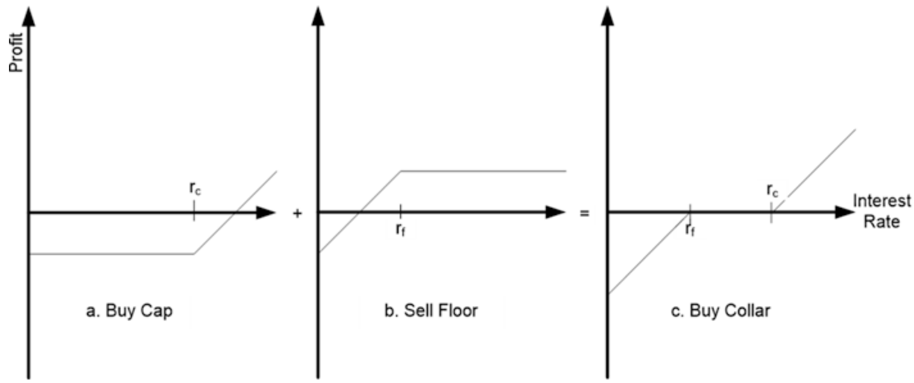


Fig. 4.29 The payoff from buying a one-period zero-cost interest rate collar.

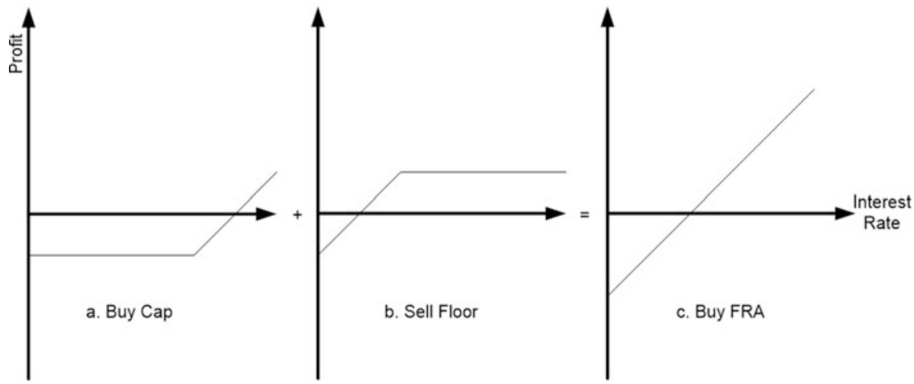


Fig. 4.30 The put-call parity between a long cap, a short floor and a forward rate agreement (FRA).

Rate Agreement) with a forward interest rate equal to the Cap/floor rate. This result is a consequence of a property of option prices known as put-call parity.

More generally, the purchase of a cap and sale of a floor with the same notional principle, index rate, strike price and reset dates produces the same payout stream as an interest rate swap with an All-In-Cost equal to the cap or floor rate.

Example 4.1.15.1

If a manufacturing firm wanted to "swap out" its floating rate debt, its all-in-cost of fixed rate debt would be quoted at 7.50% assuming the following information.

Term of floating debt	5 years
5-year Treasury yield	5.70 %
Loan spread	150 bps
3-month Libor	3.50 %
Swap spread	30 bps

All in cost then approximately $= 5.70 + 0.30 + 1.50 = 7.50\%$

The manufacturing firm will now pay 7.50% over the next 5 years and receive a floating three-months Libor over the same term.

Since caps and floors can be viewed as a sequence of European call and put options on FRAs, buying a cap and selling a floor with the same strike price and interest rate reset and payment dates effectively creates a sequence of FRAs, all with the same forward rate. But note that an interest rate swap can be viewed as a sequence of FRAs, each with a forward rate equal to the All-In-Cost of the swap. Therefore, put-call parity implies that buying a cap and selling a floor with the same contract specifications results in the same payment stream that would be obtained from buying an interest rate swap. That is,

$$\text{Cap} - \text{floor} = \text{Payers swap}$$

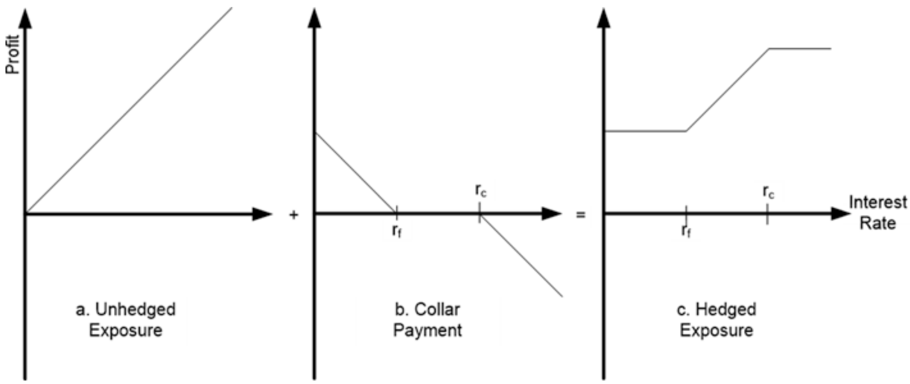


Fig. 4.31 The effect of buying an interest rate collar on interest expense.

Finally, in this section, we will show how interest rate collars can be used for hedging. Fig. 4.31 illustrates the effect that buying a one-period, zero-cost collar has on the exposure to changes in market interest rates faced by a firm with outstanding variable-rate debt. Panel (a) depicts the firm's inherent or unhedged interest exposure, while the panel (b) illustrates the effect that buying a collar has on interest expense. Finally, panel (c) combines the borrower's inherent exposure with the payoff to buying a collar to display the effect of a change in market interest rates on a hedged borrower's interest expense. Note that changes in market interest rates can only affect the hedged borrower's interest expense when the index rate varies between the floor and cap rates. Outside this range, the borrower's interest expense is completely hedged.

Pros and Cons

Pros

1. Collars provide you with protection against unfavourable interest rate movements above the Cap Rate while allowing you to participate in some interest rate decreases.
2. Collars can be structured so that there is no up-front premium payable. While you can also set your own cap rate and floor rate, a premium may be payable in these circumstances.
3. The term of a collar is flexible and does not have to match the term of the underlying bill facility. A collar may be used as a form of short-term interest rate protection in times of uncertainty.
4. Collars can be cancelled (however there may be a cost in doing so – see the Early termination section for further details.

Cons

1. While a collar provides you with some ability to participate in interest rate decreases, your interest rate cannot fall to less than the floor Rate.
2. To provide a zero cost structure or a reasonable reduction in premium payable under the cap, the floor rate may need to be set at a high level. This negates the potential to take advantage of favourable market rate movements.

4.1.15.9 Hedging Caps

A cap is a basket of options on a strip of forward LIBORs, and so is sensitive to changes in these. By the nature of the payoff of each caplet, this sensitivity is similar to that of call options on the underlying stock. A long cap position benefits from rising interest rates, and so a hedging instrument must lose value if interest rates rise. Appropriate hedging instruments include FRAs (receive fixed pay floating), futures strips (long) and swaps (receive fixed). The amount of hedge depends on the delta or hedge ratio. The broad strategy is to allocate more money to the hedge if interest rates rise and unwind the hedge if interest rates fall.

To illustrate the basic concept of delta-hedging, consider hedging with a futures strip. We shall focus on hedging an individual caplet. The payoff from the caplet is determined by

$$V(T, L_T) = (L(T, T, \tau) - K)^+ = (L_T - K)^+$$

By no-arbitrage, the present value of the caplet is $V(t, L_t)$, where we have adopted the shorthand notation $L_t = L(T, T, \tau)$. By the Black model assumption,

$$dL_t = \sigma L_t dW$$

The terminal value of a futures contract is $F_T = 1 - L_T$. One of the consequences of the Black modelling assumptions is that we approximate the present value of the futures contract with

$$F_t = 1 - L_t$$

Consider a portfolio consisting of one caplet and long Δ units of the T–maturity futures contract. The value of this portfolio is

$$\Pi_t = V(t, L_t) + \Delta(1 - L_t)$$

By Ito's lemma,

$$d\Pi_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 L_t^2 \frac{\partial^2 V}{\partial L^2}\right) dt + \sigma L_t \left(\frac{\partial V}{\partial L} - \Delta\right) dW.$$

By choosing

$$\Delta = \frac{\partial V}{\partial L}.$$

we can net out the randomness in the value of the portfolio. This is the delta-hedge. Additionally, since we are hedging two sets of cashflows which are initialized to be equal, we must have $d\Pi_t = 0$, and solving the resultant PDE (which is just a Kolmogorov backward equation) gives another equivalent way of calculating the price of a caplet.

The owner of a cap is always short the market, that is, as bond prices rally (increase), the price of the cap decreases, as does its delta, and caplets become more out-of-the-money (OTM).

A cap is, however, long Vega – its value increases with volatility. The value of a cap also increases with increasing maturity, as the holder now owns a basket containing more caplets, and hence has more options.

4.1.15.10 Exotic Caps and Floors

Above we have considered standard, *plain vanilla* caps, floors and collars. In addition to these instruments, several contracts trade on the international OTC markets with cash flows that are similar to plain vanilla contracts, but differ in one or more aspects. These deviations complicate the pricing methods considerably. Let us briefly look at a few of these exotic instruments.

- A **bounded cap** is like an ordinary cap except that the cap owner will only receive the scheduled payoff if the sum of the payments received so far due to the contract does not exceed a certain pre-specified level. Consequently, the ordinary cap payments $C^i_{t_i}$ are to be multiplied with an indicator function. The payoff at the end of a given period will depend not only on the interest rate in the beginning of the period but also on previous interest rates. As many other exotic instruments, a bounded cap is therefore a path-dependent asset.
- A **dual strike cap** is similar to a cap with a cap rate of K_1 in periods when the underlying floating rate $l(t+\delta, t)$ stays below a pre-specified level l, and similar to a cap with a cap rate of K_2, where $K_2 > K_1$, in periods when the floating rate is above l.
- A **cumulative cap** ensures that the accumulated interest rate payments do not exceed a given level.

- A **digital cap** is a strip of digital caplets, each of which is a digital call on the underlying LIBOR rate. Consider a digital caplet maturing at time T. The payoff from this caplet, received at the end of the accrual period $T + \tau$, is

$$DC(T) = \tau \cdot \theta(L(T,\ T,\ \tau) - K)$$

where θ is the Heaviside function, that is, $\theta(x) = 0$ if $x < 0$ and 1 else. The price of the digital cap is therefore given by

$$DC(t) = \sum_{i=0}^{n-1} p(t, T_{i+1}) \cdot N(d_2^{T_i})$$

- A **knock-out cap** will at any time t_i give the standard payoff $C_{t_i}^i$ unless the floating rate $l(t + \delta, t)$ during the period $[t_i - \delta,\ t_i]$ has exceeded a certain level. In that case, the payoff is zero. Similarly, there are knock-in caps. They are named as: *down and out*, *down and in*, *up and out*, and *up and in*.

Other exotic caps and floors are:

- **Ratchet Cap**
A ratchet cap is like a plain vanilla cap except that the strike is given by:

$$K_i = \begin{cases} \min\left[K, m\right] & i = 1 \\ \min\left[K_{i-1} + X, m\right] & i > 1 \end{cases}$$

where K and K_i are the strikes and m a given limit. In a ratchet cap there are rules for determining the cap rate for each caplet. The cap rate equals the LIBOR rate at the previous reset date plus a spread see Fig. 4.32. A limit, m is set on the strike level, above which a strike cannot be set.

- **Sticky Cap**
A sticky cap is like a plain vanilla cap except that the strike is given by

$$K_i = \begin{cases} \min\left[K, m\right] & i = 1 \\ \min\left[\min\left\{K_{i-1}, L_{i-1}\right\} + X, m\right] & i > 1 \end{cases}$$

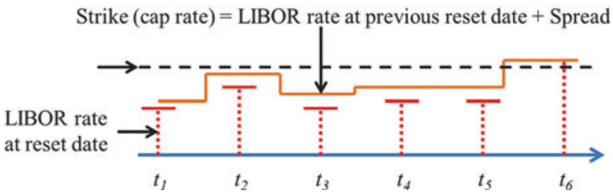


Fig. 4.32 A ratchet cap.

The sticky cap rate equals the previous capped rate plus a spread. A limit is set on the strike level, above which a strike cannot be set.

- **Flexi Cap or Auto Cap**
  An auto cap or a flexi cap has the same structure as a vanilla cap, that is, it consists of n appropriately constructed caplets, except that the holder only receives payoff from $m < n$ of these. The cap disappears once the specified number of exercises is reached. Exercises are mandatory. These caps are cheaper than regular caps with n caplets and more expensive than regular caps with m caplets. Markov functional models, which we do not discuss here, are used to price these heavily path-dependent instruments.
- **Chooser Cap**
  Chooser caps have the same structure as auto caps, except that the holder has the right to choose which caplets to exercise. The added optionality makes them more expensive than auto caps. Once the reset of a Caplet has taken place, it can no longer be chosen.
- **Momentum Cap**
  A Momentum Cap is like a plain vanilla Cap except that the strike is given by

$$K_i = \begin{cases} \min[K, m] & i = 1 \\ \min[K_{i-1} + X, m] & i > 1, L_i - b > L_{i-1} \\ \min[K_{i-1}, m] & i > 1, L_i - b > L_{i-1} \end{cases}$$

The sticky cap rate equals the previous capped rate plus a spread. A limit is set on the strike level, above which a strike cannot be set.

4.1.15.11 Options on Caps and Floors

Options on caps and floors are also traded. Since caps and floors themselves are (portfolios of) options, the options on caps and floors are so-called *compound options*. An option on a cap is called a **Caption** and provides the holder with the right at a future point in time, t_0, to enter into a cap starting at time t_0 (with payment dates $t_1, \ldots, t_n$) against paying a given exercise price.

4.1.16 Interest Rate Guarantees – IRG

An **interest rate guarantee (IRG)** is an option on a forward rate agreement (FRA), sometimes called a Fraption, that is traded over-the-counter (OTC). The holder of an IRG has the right to enter into an FRA at a specific strike rate during a predetermined amount of time. A buyer of an IRG is therefore protected against a falls in the interest rates for which he pays a premium. An IRG can also be seen as a single-period interest rate cap, that is, a caplet. Historically IRGs are the precursors to both caps and floors.

There are two types of IRGs, a call-on-IRG also called a borrower's IRG and a put-on-IRG called a lender's IRG. When exercising a call-on IRG, the holder has the right (but not the obligation) to take a loan with a predetermined amount at a predetermined interest rate (the strike rate) during the predetermined time period. When exercising a put-on IRG, the holder has the right (but not the obligation) to make a loan with a predetermined amount at a predetermined interest rate (the strike rate) during the predetermined time period. Of course, the seller has always the obligation to fulfil his obligations if the holder exercises his option.

Consider the pricing model of a call-on-IRG whose strike rate is L_{IRG}. The holder of the IRG receive at time t an amount equal to $N \cdot \max\{L_{i-1}(T_{i-1}) - L_{IRG}, 0\}$. The present value of this payment at T_{i-1} is

$$\begin{aligned}
&\frac{N}{1 + (T_i - T_{i-1})\, L_{i-1}(T_{i-1})} \max\{L_{i-1}(T_{i-1}) - L_{IRG}, 0\} \\
&= N \cdot \max\left\{1 - \frac{1 + \alpha \cdot L_{IRG}}{1 + \alpha \cdot L_{i-1}(T_{i-1})}, 0\right\}
\end{aligned}$$

Here $\alpha = T_i - T_{i-1}$ is the time period for the IRG (Caplet) and L_{i-1} the forward rate settled at the beginning of the period. Remember the value of a pure discount bond

$$p(T_{i-1}, T_i) = \frac{1}{1 + \alpha \cdot L_{i-1}(T_{i-1})}$$

Therefore, we get the same simple formula for the IRG as for a Caplet

$$C(t) = \frac{N \cdot \alpha}{1 + L_{i-1} \cdot \alpha} e^{-r(T-t)} \left[L_{i-1} \cdot N(d_1) - L_{IRG} \cdot N(d_2) \right]$$

where

$$d_1 = \frac{\ln\left\{\frac{L_{i-1}(t)}{L_{IRG}}\right\} - \frac{1}{2}(\sigma)^2 \alpha}{\sigma\sqrt{\alpha}}$$

and

$$d_2 = d_1 - \sigma\sqrt{\alpha}$$

Here α is the tenor and N the face value of the fictive loan.

4.1.17 Repos and Reverses

A **Repo/Reverse** (repurchase agreement) involves the sale of assets and a simultaneous agreement to repurchase the same or similar equivalent assets at a future date (Fig. 4.33).

- A **Repo** involves lending securities with a simultaneous agreement to repurchase at some time in the future.
- A **Reverse** involves borrowing securities with a simultaneous agreement to sell them back at some time in the future.

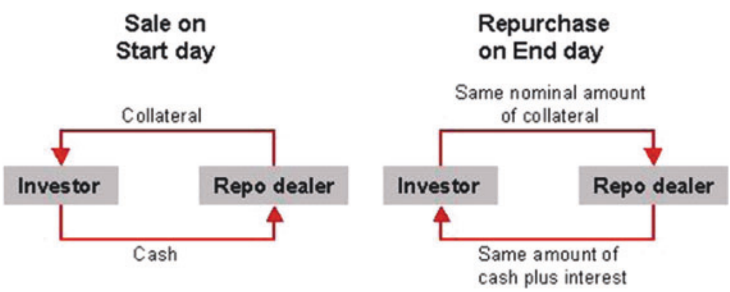


Fig. 4.33 Illustration of a repo transaction.

Other key features of a Repo/Reverse deal:

- Provides a legal sale of collateral, meaning that the legal title is transferred to the buyer. This provides protection against default for the buyer. Other implications of this characteristic are that the buyer has the right to sell the collateral short, and that the voting rights (on equity collateral) are transferred by the Repo/Reverse deal.
  - Provides an economic loan of cash and collateral, meaning that the risk and return is retained by the seller because sale and repurchase are for the same value.
  - Coupons that are paid out during the life of the loan should be transferred to the original owner of the underlying instrument. The present value of a repo trade is the present value of all future cash flows included in the repo transaction. The cash flows are the initial and final payments.

  The general cash-flow structure of Repo/Reverse instruments depend on the following factors (Fig. 4.34):
  - Is the deal a Repo or Reverse transaction?
  - Is the required rate of return fixed, or based on a floating reference rate?

When you are entering into a typical fixed rate "reverse", you face the following cash-flow structure:

You pay a premium on the start day and receive (borrow) securities as collateral. After the term, you return the securities and receive a cash amount equal to your initial premium plus interest earned during the repo term.

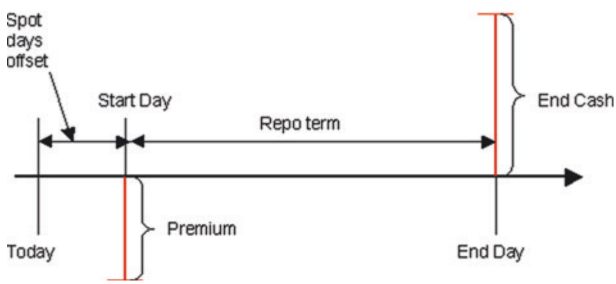


Fig. 4.34 The cash-flow structure of a repo transaction.

4.1.17.1 Special's and GC's

There are two types of assurance in repo transactions, **Specials** and **GCs**. In a Special, a specific security is used as assurance in the deal. The seller needs a cheap financing, a low rate with the use of the demand for a specific instrument (bond, bill or note). The buyer is short in the same instrument and is not that sensitive in the change in repo rate. If many actors on the market are short in the same instrument, the repo rate will decrease.

A GC (general collateral) is a basket of securities (e.g. bonds) used in the repo contract. The seller chooses the securities to deliver with the purpose to get a low cost for the loan.

In a pure repo, the seller keeps the coupon pay-outs during the lifetime of the contract. If instead the buyer gets the coupons, this has to be considered in the repurchase of the securities.

4.1.17.2 Other repos

An **open-date repo** is a contract that can be closed at any time by any of the parties. The repo rate is negotiated on daily basis. The advantage is that the collateral security does not have to be sent back and forth.

A **cross-currency repo** involves two different currencies, one for the collateral security and another one for the loan. The advantage is that you can avoid exchange rates.

Two variants that reduce the transaction costs are:

- *Holdin-costody*: Here, the issuer (the bank) keeps the collateral security in a separate account for the buyer. Therefore, the costs for deliveries are eliminated.
- *Tri-party-repo*: Here, a third party (security bank) handles the transactions and the margin requirements.

Four variants are used to increase the income yield:

- *High-yield repo*: Here, the buyer is given collateral security with low credit ranking.
- *Multi-collateral repo*: A repo with many bonds with small principals.
- *Floating-rate repo*: A repo rate based on a certain formula or some index.

- *Options involving repos*: This can be caps or floors on the repo rate or a freedom to choose a fixed repo rate.

Also, central banks use repos. These are used in the money markets to control liquidity and short rates. If a central bank is borrowing securities, it is said to do a reverse repo.

There are a number of named repos. Some of them are:

- O/N (Over-Night): The overnight repo is a loan from today until the close of the next banking day. On a Friday, the next banking day will be on next Monday if this is not a holiday.
- T/N (Tomorrow-Next or just TomNext): The amount of money is paid out tomorrow §(or if this is not a banking day the day after tomorrow etc.) and paid back one banking day§thereafter.
- C/W (Corporate-Week): This is a loan for a week starting two bank days from today.
- S/N (Spot-Next): The amount of money is paid two bank days from today and paid back one bank day thereafter.

4.1.18 Loans

A **security loan** is a contract where a security is temporarily lent to a party. A lending fee is paid to the party that originally owned the security (Fig. 4.35).

The lending fee payment is based on the lending rate, the day-count method and the time period. Coupons paid out during the life of the loan should be transferred to the original owner of the security.

A **Promissory Loan** has the property that the coupon is split between the historical owners of the security.

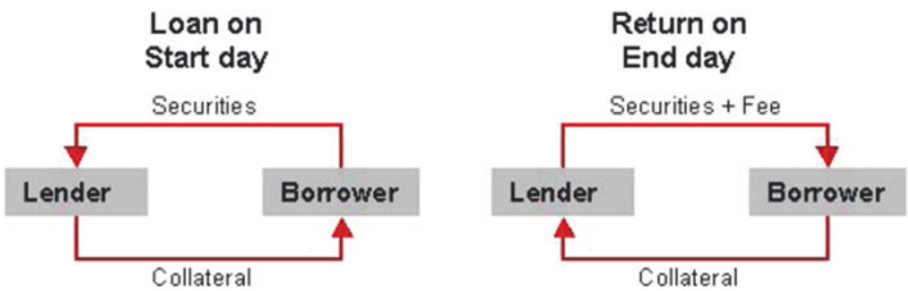


Fig. 4.35 Illustration of a security loan.

For example, assume yearly coupons and that Party A owns the security for the first nine months of the current cash-flow period and then Party B buys the security and keeps it at least three months. 75% of the coupon will be paid out to Party A and 25% of the coupon will be paid out to Party B. Obviously, the size of the next coupon depends on the acquire day of the trade. The coupon cash flows for trades are presented exactly as they will be, taking the acquire day into account.

4.1.19 CPPI – Constant-Proportions-Portfolio-Insurance

4.1.19.1 Introduction – Funds

Mutual funds, hedge funds and funds of funds have a number of features that are markedly different from funds holding regular equities.

1. They can only be traded at discrete times. For example, mutual funds are often only tradable once a day. Some hedge funds or funds of funds can only be traded on a weekly or monthly basis.
2. Frequently notice has to be given before one intends to trade. For example, 5 days' notice has to be given before purchasing, while 30 days' notice has to be given before selling the fund.
3. One might agree on a value to trade, rather than a number of fund units to trade.
4. When using the cash in the fund for purchasing the cash might have to be delivered several days before trading, while on redeeming a part of the investment in the fund, it might take several days or weeks for delivery of the proceeds.
5. One might be limited in the volume that can be traded on a given day.

For pricing and risk management, the first feature is typically the most significant. A frequently used structure to reduce the risks of a fund is a CPPI. Only being able to trade at specific times makes the fund more vulnerable to sharp market moves. Consequently, we have to extend the models for the evolution of stock prices and include additional random jumps.

4.1.19.2 Hedging of funds

In a continuously tradable Black-Scholes world, the fair price and the cost of a delta hedge are equal. In the case of discrete hedging, this is no longer the case. It is simple to show that the discrete delta hedge might cost *more* than the equivalent continuous hedge. Consider the hedging of a call option over a month where the price of the underlying rises steadily:

1. In the case of continuous hedging, one steadily buys more delta.
2. In the case of discrete monthly hedging, one can only increase delta at the end of the month, which means that one must buy the additional delta at the highest price over the month, rather than on the way up.

In the case of the price dropping continuously over a month the discrete delta also costs more as one is forced to sell at the lowest point.

So when does the discrete case cost *less* than the continuous hedge? Consider the above case, with the modification that the underlying goes up and then returns to its original level by the end of the month. If one compares monthly hedging with bimonthly hedging, we know that bimonthly hedging will cost something on the way up, and then something on the way down. In contrast, monthly hedging will be approximately free.

Considering all the possible paths, one ends up with a distribution for the costs of the delta hedge. The average of this distribution is the continuous Black-Scholes fair price. The more frequent the re-hedging times, the narrower the distribution of delta hedging costs.

It is important to realize that this risk cannot be hedged. One can be lucky and make a profit, or unlucky and make a loss with respect to the Black-Scholes "fair" price. Of course, by taking sensible provisions with respect to the Black-Scholes "fair" price, the risk of loss for the fund can be significantly reduced.

4.1.19.3 CPPI Structures

A CPPI (Constant Proportions Portfolio Insurance) is a so-called structured product and we will now discuss CPPI structures as an alternative investment strategy. Such an investment strategy will protect against large losses. The CPPI can, as we will see, use any kind of underlying structure. The CPPI contract that we will study has a number of parameters, such as:

- a kernel
- a satellite
- a floor
- a multiplier and
- a pillow.

The kernel and satellite will consist of two different assets with different risks. The kernel consist of an asset with relatively low risk, while the satellite holds the risky assets that hopefully will give better profit than the kernel. The kernel is considered to be a passive component, typically a bond (or a zero-coupon bond) and the satellite is the active part of the portfolio. The reason for the kernel-satellite parts is to control the risk of the CPPI portfolio.

The strategy is to rebalance between the kernel and the satellite over time, due to changes in the value of the satellite. The floor is used as a protection (or a trigger level) of the CPPI. If the value of our portfolio will hit the floor, all investments are placed in the kernel, for the rest of the CPPI lifetime. The pillow is defined as the total portfolio value minus the floor; $P = V - F$.

The multiplier m is used to give the portfolio manager the possibility to invest more in the satellite if this will give a better profit than the kernel. The multiplier times the pillow gives the amount of the portfolio value that will be invested in the satellite. The invested amount in the satellite is $S = m\ x\ P$. The remaining amount is invested in the kernel;

$$K = V - m \times P.$$

The value of the floor F and the multiplier m is decided by the investor due to the current market condition and the investors view of the risk. A low value of the floor gives high risk and a positive view of the satellite. In some contracts, the floor and multiplier can vary over time. The difference between the floor and the initial invested amount gives the maximum loss.

To get a perfect protection against losses, one needs continuously rebalance between the kernel and the satellite. This would give high transaction costs and therefore the CPPI is rebalanced only at discrete times. If the performance of the satellite is better than the kernel, the multiplier and the pillow will act to increase the investments in the satellite. We will show this in a simple example:

A CPPI with multiplier. Suppose we have the following initial data

Initial market index, I	100
Initial portfolio value, V	100
Initial satellite, S	40
Initial kernel, K	60
Floor, F	90
Initial pillow, P	10

Now, suppose we have the following relative changes of the market index, as function of time: [+10%, -9,09%, +12%, +8%, -7%, +10%, -5%, 7%, -3%]. Assuming no transaction costs, we then get the following values:

Time	0	1	2	4	4	5	6	7	8	9	10
Index	100	110	100	112	120.96	112.49	106.87	117.56	111.68	119.5	115.91
Value	100	104	98.91	103.19	107.41	102.53	100.03	104.04	101.23	104.37	102.65
Satellite	40	56	35.64	52.74	69.62	50.13	40.1	56.14	44.92	57.49	50.59
Kernel	60	48	63.27	50.44	37.78	52.4	59.92	47.89	56.31	46.88	52.06
Pillow	10	14	8.91	13.19	17.41	12.53	10.03	14.04	11.23	14.37	12.65

Two other scenarios are shown below, one with a decreasing index and one with an increasing market index.

Time	0	1	2	4	4	5	6	7	8	9	10
Index	100	110	104.5	91.96	84.6	82.91	82.08	73.87	70.92	65.95	63.98
Value	100	104	101.2	95.82	93.96	93.64	93.5	92.1	91.76	91.27	91.12
Satellite	40	56	44.8	23.3	15.84	14.57	13.99	8.39	7.05	5.08	4.47
Kernel	60	48	56.4	72.53	78.12	79.07	79.51	83.7	84.71	86.19	86.65
Pillow	10	14	11.2	5.82	3.96	3.64	3.5	2.1	1.76	1.27	1.12

Time	0	1	2	4	4	5	6	7	8	9	10
Index	100	110	115.5	129.36	139.71	142.5	143.93	158.32	164.65	176.18	181.46
Value	100	104	106.8	114.86	122.82	125.45	126.86	141.61	149.87	166.63	175.83
Satellite	40	56	67.2	99.46	131.28	141.78	147.46	206.44	239.47	306.52	343.3
Kernel	60	48	39.6	15.41	-8.46	-16.34	-20.59	-64.83	-89.6	-139.89	-167.48
Pillow	10	14	16.8	24.86	32.82	35.45	36.86	51.61	59.87	76.63	85.83

As we can see, the result is a high profit in a bullish and a god protection on a bearish market. We also see that in a bullish market we can go short in the low-risk security, that is, the kernel and use the money to buy more of the risky asset, that is, invest in the satellite.

The values are calculated as

$$\begin{cases} I(t_i) = I(t_{i-1}) \cdot (1 + \Delta I(t_i - t_{i-1})) \\ V(t_i) = \max\{V(t_{i-1}) + S(t_{i-1}) \cdot \Delta I(t_i - t_{i-1});\ F\} \\ P(t_i) = V(t_i) - F \\ S(t_i) = m \cdot P(t_i) \\ K(t_i) = V(t_i) - S(t_i) \end{cases}$$

From the example above, we realize that Monte-Carlo simulations must be used to value a CPPI.

Example 4.1.19.1

A 6-year Capital Protected CPPI Note linked to a Basket of assets. (The Lehman Brothers International issued the specific CPPI structure discussed below in April 2006.)

A CPPI consist of a nominal amount in a European Quality Fund and a risk-free asset in cash. The final payment to the investor (buyer) of the CPPI is given by the following formula

$$P = SD \cdot \left[100\% \cdot \max\left\{ 100\%, \frac{CPPI_{Final}}{CPPI_{Initial}} \right\} \right]$$

where

SD = Value per Bond (SEK 500 000)

$CPPI_{Initial}$ = The value of the CPPI in April 2006

$CPPI_{Final}$ = The value of the CPPI in April 2012

A yearly fee of 0.75% is subtracted in the calculation of the final CPPI value. This is the same to say that the financial cost is 75 bp. The initial allocation of the CPPI is 100% in the Fund and 0% in the risk-free asset (cash). The target exposure in the Fund is

$$TE_t = \min\left[\max\left\{ \frac{CPPI_t - BF_t}{CPPI_t} \times m, E_{\min} \right\}, E_{\max} \right]$$

where

$CPPI_t$ is the value of the CPPI at time t.

BF_t is the interest rate floor at the potential rebalancing date t. This value starts at 80% of the nominal amount and increases linearly with 2.66667% per annum to 100% of the nominal amount on the final potential rebalancing date.

$m = 5$ is the multiplier

E_{min} = minimum exposure of the fund = 0%

E_{max} = maximum exposure of the fund = 150%

Remark! The maximum exposure of the fund > 100%. The reason for this is that an investment bank can add from their own Capital to the CPPI in order to increase the leverage.

Here are two numerical scenarios:

1.) Assets down by 10%

Initial CPPI Value:	100 SEK (Notional invested)
Distance = CPPI value – Bond floor (80%):	20%
Target Exposure = Distance × m (= 5):	100% in Premium and 0% in Riskless Asset
Fall in Premium Asset: $-10\% \times 100\,\%$	-10%
New Notional Value:	$100\% - 10\% = 90\% \rightarrow 90$ SEK
New Distance:	$90\% - 80\% = 10\%$
New Target Exposure:	$10\% \times 5 = 50\% \rightarrow 50$ SEK
New Risk-less Asset Allocation:	90 SEK - 50 SEK = 40 SEK

2.) Assets up by 10%

Initial CPPI Value:	100 SEK (Notional invested)
Distance = CPPI value – Bond floor (80%):	20%
Target Exposure = Distance × m (= 5):	100% in Premium and 0% in Riskless Asset
Increase in Premium Asset10% × 100 %	10 %
New Notional Value:	$100\% + 10\% = 110\% \rightarrow 110$ SEK
New Distance:	$110\% - 80\% = 30\%$
New Target Exposure:	$30\% \times 5 = 150\% \rightarrow 150$ SEK
Risk-less Asset Allocation:	0%
Leverage:	150 SEK - 110 SEK = 40 SEK

We find that, thanks to the multiplier we earned 50 % on an increase of 10%

An important CPPI Parameter is the Target Exposure. It denotes the maximum sustainable proportion of the CPPI Balanced Account, which can be invested in the Premium Asset without jeopardizing the Capital protection at maturity. It is calculated on daily basis, based on the multiplier/Crash Size, the bond floor and the Balanced Account Value. Crash Size denotes the maximum loss of the underlying Premium Asset between two potential rebalancing dates. The Distance denotes the Capital, which can be put at risk without jeopardizing the Capital protection at maturity.

5

Yield Curves

5.1 Introduction to Yield Curves

Ordering the current spot yields to maturities for any group of bonds. By maturity we get a so-called yield curve. This curve is often represented as a graph with time to maturity on the horizontal axis and yields on the vertical axis. The group is usually defined as bonds by the same issuer and/or the same credit rating. Thus, we speak of yield curves for government bonds, for mortgage bonds or for corporate bonds of the same credit rating. The word bond here is used in the academic sense which means bills, notes and bonds. Interest rates in international or domestic time deposit markets too can be ordered by maturity and credit class. Thus, we get London inter-bank offered rate (LIBOR) or XIBOR yield curves or yield curves for domestic deposits in any currency. There are many different yield curves. In general, yield curves may slope upwards or downwards, their shapes can be concave, convex or have humps.

In Table 5.1 and Fig. 5.1, we show the yield curve for UK government bonds. This data was taken from *The Financial Times* September 6, 2016.

Today, many countries have negative interest rates. A few years ago, many actors (if not all) thought that negative interest rates could not exist. But things have changed and nowadays it is a fact. In Table 5.2 we have market prices for the Swedish government securities[1] (bills and bonds) at 2016-09-09.

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[1] https://www.riksgalden.se/sv/For-investerare/Statspapper/Utestaende-statspapper/

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_5

Table 5.1 Government bond yields in UK 2016-09-06

Maturity	Yield	Today'schange	1 week ago	1 month ago
1 Month	0.18%	0	0.18%	0.20%
3 Month	0.23%	>-0.01	0.22%	0.27%
6 Month	0.31%	0.05	0.31%	0.28%
1 Year	0.15%	0.06	0.17%	0.16%
2 Year	0.10%	<0.01	0.16%	0.14%
3 Year	0.09%	0	0.14%	0.13%
4 Year	0.16%	>-0.01	0.19%	0.17%
5 Year	0.19%	>-0.01	0.22%	0.21%
7 Year	0.36%	0	0.36%	0.43%
8 Year	0.46%	>-0.01	0.44%	0.57%
9 Year	0.56%	-0.06	0.54%	0.59%
10 Year	0.66%	>-0.01	0.64%	0.67%
15 Year	1.03%	>-0.01	0.96%	1.14%
20 Year	1.18%	>-0.01	1.11%	1.32%
30 Year	1.30%	>-0.01	1.24%	1.49%

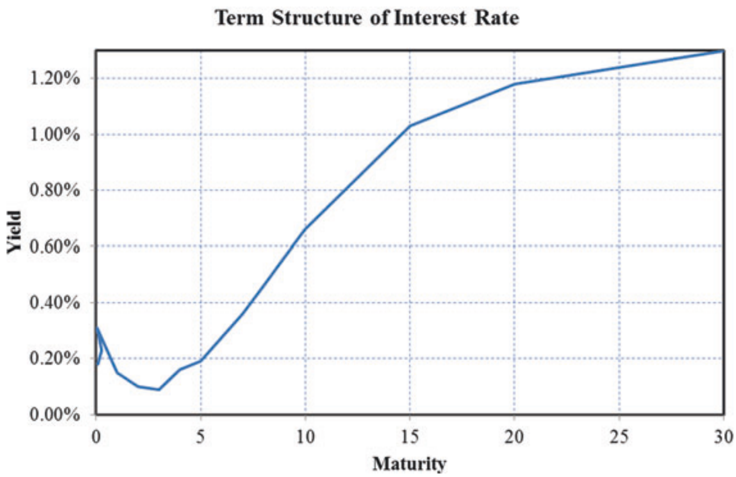

Fig. 5.1 Government bond yields in UK 2016-09-06

The bootstrapped zero-coupon rates from Table 5.2 are shown in Fig. 5.2. As we can see, the zero rate is negative up to nine years.

Term structure models are based on the assumption that the whole term structure of interest rates can be derived from the stochastic behaviour of one or many variables. The reason for modelling the entire term structure is to make all model prices internally consistent.

Table 5.2 Quotes of Swedish Government securities[2]

Securities	Issue date	Maturity	Coupon	Price
STB 21 Sep 16	2016-03-11	2016-09-21	0	100.02
STB 19 Oct 16	2016-07-01	2016-10-19	0	100.09
STB 16 Nov 16	2016-08-05	2016-11-16	0	100.15
STB 21 Dec 16	2016-06-03	2016-12-21	0	100.23
STB 15 Mar 17	2016-09-02	2017-03-15	0	100.42
SGB 1051 3.75% 12 Aug 17	2006-09-15	2017-08-12	3.75	104.45
SGB 1052 4.25% 12 Mar 19	2007-11-21	2019-03-12	4.25	114.47
SGB 1047 5% 1 Dec 20	2004-01-28	2020-12-01	5.00	127.47
SGB 1054 3.5% 1 Jun 22	2011-02-09	2022-06-01	3.50	123.18
SGB 1057 1.5% 13 Nov 23	2012-10-22	2023-11-13	1.50	113.68
SGB 1058 2.5% 12 May 25	2014-02-03	2025-05-12	2.50	123.27
SGB 1059 1.0% 12 Nov 26	2015-05-22	2026-11-12	1.00	109.57
SGB 1056 2.25% 1 Jun 32	2012-03-20	2032-06-01	2.25	124.84
SGB 1053 3.5% 30 Mar 39	2009-03-30	2039-03-30	3.50	153.64

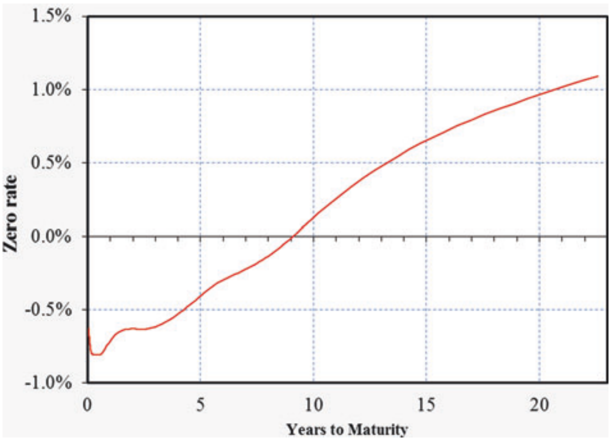

Fig. 5.2 The Swedish treasury zero-coupon rates per 2016-09-09

In categorising these models, two properties are significant:

- **Number of state variables**
  - Most models lack analytical solutions and have to be solved using numerical methods. The computing time increases dramatically for each new state variable.
- **External consistency**.
  - By external consistency, we mean coherence between the model term structure and the observed term structure. When the model

[2] Source, https://www.riksgalden.se/sv/For-investerare/Statspapper/Utestaende-statspapper/

is used to price derivative instruments, it is essential that the underlying instrument is priced in accordance with observed market prices.

Although this yield curve obviously includes T-bills, T-notes and T-bonds, we refer to these collectively as “bonds” in the usual academic sense. In parallel to the US government yield curve, there are yield curves for domestic time deposits between banks, for the international deposit market (LIBOR), for interest rate swaps, and for municipal and corporate bonds. Closest to the US government yield curve are curves for instruments with the highest credit rankings.

As an example consider the yield curve on the same day for USD interest rate swaps. The yields are given in buckets as seen next:

2010-05-10	1	0.5069
2010-05-11	2	0.5323
2010-05-12	3	0.5235
2010-05-16	7	0.4882
2010-06-08	30	0.3082
2010-08-07	90	0.2225
2010-11-05	180	0.2944
2011-02-03	270	0.4067
2011-05-09	365	0.5564
2012-05-08	730	1.1271
2013-05-08	1095	1.6642
2014-05-08	1460	2.0844
2015-05-08	1825	2.4084
2017-05-07	2555	2.8126
2019-05-07	3285	3.1287
2020-05-06	3650	3.2642
2025-05-05	5475	3.4964
2030-05-04	7300	3.4964
2040-05-01	10950	3.4964

This data uses a day count conversion 30/360.

To find the yield to maturity (*ytm*) for intermediate maturities, interpolation is used. In C/C++ the following function can be used to interpolate the y-values and return y for a given term/maturity x (pX and pY are arrays of length N),

```
double IPOL(double x, double, *pX, Double *pY, int N)
{
  if (x <= pX[0]) return pY[0];
  if (x >= pX[N-1]) return pY[N-1];
```

```
  for (int i = 1; i < N; i++)  {
    if (x == pX[i-1]) return pY[i-1];
    if (x == pX[i]) return pY[i];

    If (x > pX[i-1] && x < pX[i])
      return pY[i-1] + (x - pX[i-1])*(pY[i] - pY[i-1])/
                       (pX[i] - pX[i-1]);
  }
}
```

When you have the interpolated values, forward rates between two arbitrary future dates can be calculated.

```
double forwardRate(int Days1, double r_t1, int Days2, double r_t2)
{
  return pow((pow(1.0 + r_t2, Days2/365.0))/
             (pow(1.0 + r_t1, Days1/365.0)), 365.0/
             (Days2 - Days1)) - 1.0;
}
```

Both these examples show yield curves that are upwards sloping. This is the typical case. Why is this so? Why do *ytms* for instruments of the same credit rating differ because of maturity? The traditional explanations are:

- Expectations theory
- Liquidity preference theory
- Market segmentation theory

Briefly, the expectations theory argues that the slope of the yield curve (or equivalently the term premium) reflects the market's average expectations about future interest rates/yields. Lending short term while borrowing long term you can lock in yields on any forward starting loan today. This approach was used to evaluate interest rate swaps in the preceding chapter. In particular, it was argued that implied forward rates calculated from the current yield curve were unbiased forecasts of future spot rates, while the liquidity preference theory argues that forward rates are always biased high because investors prefer liquidity. Market segmentation based on credit ratings is clearly an empirical fact but it has also been used to explain why yield curves typically should slope upwards. The reason is that there was a chronic shortage of long-term investors in relation to the supply. Typically insurance companies prefer to invest their cash long term while banks rely more heavily on short-term funding.

5.1.1 Credit Ratings

Next, we see the notation used for credit rating *Moody's* and *Standard and Poor's*:

Standard and Poor's		Moody's
AAA		Aaa
AA		Aa
A	Investment grade	A
BBB		Baa
BB		Ba
B	Speculative grade	B
CCC	"Junk"	Caa
CC		Ca
C		C
D	Default	D

A few years ago, government bonds issued by the government of Argentina was classed as D.

Some models use a transition matrix to describe the probabilities for transitions between the different credit ratings. For the financial crises in 2008, see Table 5.3. The transition matrix is an implied matrix sampled from bond prices in emerging markets. It describes the transition probabilities between different credit ratings. By the same method, it is also possible to calculate a cumulative default probability matrix, which is illustrated in Table 5.4.

Table 5.3 The Transition Matrix in the beginning of 2008

1-Year Transition Matrix								
From	To Rating							
Rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	88,658%	10,294%	1,017%	0,000%	0,031%	0,000%	0,000%	0,000%
AA	1,079%	88,705%	9,553%	0,342%	0,145%	0,145%	0,000%	0,031%
A	0,063%	2,876%	90,205%	5,919%	0,740%	0,177%	0,010%	0,010%
BBB	0,053%	0,339%	7,069%	85,238%	6,053%	1,005%	0,085%	0,159%
BB	0,033%	0,077%	0,557%	5,680%	83,572%	8,083%	0,535%	1,464%
B	0,011%	0,044%	0,174%	0,652%	6,595%	82,703%	2,760%	7,062%
CCC	0,000%	0,000%	0,660%	1,050%	3,050%	6,110%	62,970%	26,160%
D	0,000%	0,000%	0,000%	0,000%	0,000%	0,000%	0,000%	100,000%

Table 5.4 The cumulative default probability matrix

Cumulative Default Probability Matrix							
	AAA	AA	A	BBB	BB	B	CCC
1 Year	0,00%	0,31%	0,1%	0,159%	1,464 %	7,062%	26160%
2 Years	0,004%	0,073%	0,056%	0,477%	347%	13722%	43111%
3 Years	0,12%	0,127%	0,145%	0,950%	5678%	19828%	54255%
4 Years	0,027%	0,198%	0,284%	1,568%	8,157%	25,339%	6172%
5 Years	0,050%	0,289%	0,477%	2,317%	10,750%	30270%	66,840%

5.2 Zero-coupon Yield Curves

An important subclass among the yield curves are so-called zero-coupon yield curves. These can be derived almost directly from money market instruments or calculated from interest rate swaps or groups of coupon paying bonds using special bootstrapping techniques. A zero-coupon yield curve defines a set of discount factors that can be used for the discounting of future cash flows. An older name for this construct was the term structure of interest rates.

While the previous examples showed *ytm*s for each bond individually, the zero-coupon yield curve shows a curve that when used for the discounting of the future cash flows of all the bonds in the curve replicate their market prices. Note that the yield curves in the examples were obtained by applying the present value (*PV*) formula to each individual bond in order to translate from price to *ytm*. Thus, different *ytm*s were being used for "discounting" cash flows at future dates depending on the bond. This is inconsistent. The proper way to discount future cash payments is to apply the same *ytm* to all the bonds which pay cash on the same future date. All the underlying cash flows from the whole set of bonds should be discounted with a unique yield that only depends on the future date of the cash flow. There should only be one yield per future date. Otherwise portfolios of bonds could be constructed that exploits any mispriced cash flow. This is a no arbitrage requirement.

So when the quoted *ytms* are given for a subset of bonds, we need to use a method called bootstrapping to calculate a matching zero-coupon yield curve. With this technique, we strip the bonds to create virtual zero-coupon bonds from the coupons and the principal. This is not a trivial exercise and the results will be different depending on the method used. One method is used by the US Treasury Department and the results are published as Separate Trading of Registered Interest and Principal of Securities (STRIPS).

The coupons and principal of normal bonds are split up, creating artificial zero-coupon bonds of longer maturity than would otherwise be available.

Example 5.2.0.1

Let us study the price of a bond maturing in exactly one year with semi-annual coupons of 10 % and a quoted *ytm* of 5.951%. Using the *PV* formula to derive the cash price, we get

$$\frac{5}{1+\frac{0.05951}{2}}+\frac{105}{\left(1+\frac{0.05951}{2}\right)^2}=103.874$$

This is not necessarily the one-year, zero-coupon yield. Suppose the six-months zero-coupon yield is 4%. Then the matching zero-coupon yield for one year, say s, must be given by

$$\frac{5}{1+\frac{0.04}{2}}+\frac{105}{\left(1+\frac{s}{2}\right)^2}=103.874$$

Solving the equation we get $s = 6.0\%$ which is close but not exactly equal to the given 12m yield on the coupon bond.

We know that the quoted *ytm* on a bond y can be used in the *PV* formula to calculate its market price. Vice versa, given the price P, we can find the *ytm* by solving the following equation

$$P=\sum_i \frac{c_i}{(1+ytm)^{t_i}}+\frac{100}{(1+ytm)^{t_n}}$$

where c_i is the coupons of the bond, t_i the time for the payouts and P the market price of the bond. For continuously compounded *ytms*, the formula is

$$P=\sum_i c_i\cdot e^{-t_i\cdot ytm}+100\cdot e^{-t_n\cdot ytm}$$

Remember that the *ytm* on a bond is only an adequate measure of the rate of return on this investment if all coupons can be reinvested at the same yield.

5.2.1 ISMA and Moosmüller

There exist a number of different methods to calculate the *ytm* and we will give two of them, International Securities Market Association (ISMA) and Moosmüller. ISMA is given by

$$\begin{aligned} P &= v^f \cdot \left[\frac{C}{H} \cdot \sum_{i=1}^{n} v^{i+1} + \left(Nom + g \cdot \frac{C}{H} \right) \cdot v^{n-1+g} \right] \\ &= v^f \cdot \left[\frac{C}{H} \cdot \frac{1-v^n}{1-v} + \left(Nom + g \cdot \frac{C}{H} \right) \cdot v^{n-1+g} \right] \end{aligned}$$

where

$$v = \frac{1}{1 + \frac{ytm_h}{H}}$$

ytm_h is the given H coupons per year. A common formula in Germany is the Moosmüller method that can also handle parts of coupons. The Moosmüller formula is given by

$$P = \frac{1}{1 + \frac{f \cdot ytm_h}{H}} \cdot \left[\frac{C}{H} \cdot \frac{1-v^n}{1-v} + \frac{Nom + g \cdot \frac{C}{H}}{1 + \frac{g \cdot ytm_h}{H}} \cdot v^{n-1+g} \right]$$

In the next section we'll show how zero-coupon yield curves can be derived from any given set of quoted market yields.

6

Bootstrapping Yield Curves

6.1 Constructing Zero-Coupon Yield Curves

We will now explain how to obtain zero-coupon yield curves from market data for coupon bonds or interest rate swaps. To do so, we begin with some simple examples and show how to use linear bootstrapping to find the spot rates and forward rates from a number of benchmark instruments. Also we will show how to use the derived zero-coupon yields to discount future cash flows. Finally, we will use some real market data, such as bonds, deposits, forward rate agreements (FRAs) and swaps in the bootstrap procedure.

As a first example suppose we have the following benchmark bond quotes

T	Yield	Coupon	Price
0.5	2.15 %	–	98.94
1.0	2.50 %	5.0 %	102.45
1.5	2.90 %	6.5 %	105.24
2.0	3.20 %	9.0 %	111.14

where T is time to maturity. This data can be given by a Government bill ($T = 0.5$) and three Government bonds. In some countries (like Sweden), both bills and bonds are quoted as yields-to-maturity (*ytms*). Therefore, we can calculate the prices as

$$P = \frac{100}{1 + ytm \cdot \frac{d}{360}}$$

J.R.M. Röman, *Analytical Finance: Volume II*,
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for instruments with no coupons and by

$$P = \frac{N}{(1+ytm)^T} + \sum_{i=1}^{n} \frac{C}{(1+ytm)^{t_i}}$$

for bonds paying coupons, where C is the coupon (i.e. 2.5, 3.25 and 4.5 in the previous example). We use the day-count convention *act/*360 and we suppose the bonds are paying the coupons semi-annually (i.e. with frequency $f = 2$).

6.1.1 The Matching Zero-Coupon Yield Curve

We start by stripping the instruments to find the corresponding zero-coupon rates. The zero-coupon yield curve will then be used to discount the future cash flows for all the given instruments which here have maturities from zero to 2 years. Since the 0.5-year bond does not pay any coupon (it is actually a T-bill), it can directly be considered to be a zero-coupon bond. From the aforementioned data we immediately find the first zero-coupon rate for borrowing today with a payback in six months. It is

$$s_1 = 2.15\%$$

For times to maturity less than six months the rates have to be calculated by extrapolation.

The next bond will pay a coupon of 2.50 after six months (5 % on a nominal amount of \$100). Using the zero-coupon rate for discounting the present value of this payout must be

$$PV = \frac{2.5}{1 + \frac{0.0215}{2}} = 2.47$$

After a year, the same bond will pay \$102.5 and we can solve for the second 1-year zero-coupon rate s_2. The following must hold

$$\frac{2.5}{1 + \frac{0.0215}{2}} + \frac{102.5}{\left(1 + \frac{s_2}{2}\right)^2} = 102.45$$

where \$102.45 is the given (quoted) market bond price from the market data. Solving for s_2 we get

$$s_2 = 2.504\%.$$

Plugging in the calculated values for the zero-coupon yields for maturities 0.5 and 1 year we can solve for the 1.5-year zero-coupon yield s_3 from the future cash flow for the 1.5-year bond in the market data. Its coupon rate is 6.5% per year which means \$3.25 every six months. So

$$\frac{3.25}{1+\frac{0.0215}{2}}+\frac{3.25}{\left(1+\frac{0.02504}{2}\right)^2}+\frac{103.25}{\left(1+\frac{s_3}{2}\right)^3}=105.24$$

Solving this equation we get: $s_3 = 2.923\,\%$.
In the same way, we can calculate s_4:$s_4 = 3.244\ \%$.

Another way to look at the previous formula is

$$\frac{103.25}{\left(1+\frac{s_3}{2}\right)^3}=105.24-3.22-3.17=98.85$$

As we can see, we subtract the discounted values of the coupons from the current market price of the bond. What we have left is a zero-coupon bond with the face value (nominal amount) of \$103.25 with a value of \$98.85. This gives a discount factor of 98.85/103.25 = 0.9574 or, equivalently, a zero-coupon yield of 2.923%.

We then have derived the following zero-coupon yield curve

T	Yield	Coupon	Price	Spot rate
0.5	2.15 %	–	98.94	2.150 %
1.0	2.50 %	5.0 %	102.45	2.504 %
1.5	2.90 %	6.5 %	105.24	2.923 %
2.0	3.20 %	9.0 %	111.14	3.244 %

As we can see this yield curve is slightly above the given quoted *ytm* curve which included three coupon bonds.

Once we have calculated the implied zero-coupon yield curve we also have the prices of the corresponding zero-coupon bonds (the discount function), and we can find the prices of all other bonds. If they are more risky than the treasury bonds given here, we can apply a spread-over-yield as was done in Chapter 3.

In order to get a smooth and nice curve from these four calculated points, we have to use some kind of interpolation method in the bootstrap model. The reason is that we need derivatives (the slope) of the yield curve for some calculations. This leads to a system of equations which will be described next.

6.1.2 Implied Forward Rates

From the derived zero-coupon curve, which consists of zero-coupon *ytms*, we can calculate implied forward rates as

$$r_{t_2-t_1}^{forward} = \left(\frac{(1 + r_{t_2}^{spot})^{t_2}}{(1 + r_{t_1}^{spot})^{t_1}} \right)^{\frac{1}{t_2-t_1}} - 1$$

or

$$r_{t_2-t_1}^{forward} = \frac{1}{t_2 - t_1} \cdot \left(\frac{(1 + r_{t_2}^{spot})^{t_2}}{(1 + r_{t_1}^{spot})^{t_1}} - 1 \right)$$

depending on the discounting method for the forward rates. We get the following values using the aforementioned "first" formula

T	**Yield**	**Coupon**	**Price**	**Spot rate**	**Forward rate**
0.5	2.15 %	–	98.94	2.150 %	2.150 %
1.0	2.50 %	5.0 %	102.45	2.504 %	2.860 %
1.5	2.90 %	6.5 %	105.24	2.923 %	3.766 %
2.0	3.20 %	9.0 %	111.14	3.244 %	4.214%

In Fig. 6.1, the zero-coupon spot-and-implied forward rates are plotted.

The previous example of bootstrapping was really simple because all cash flow payouts happened at the same date. We will therefore consider four bonds where the cash flows do not coincide. Suppose then, we have the following benchmark data

T	**Yield**	**Coupon**	**Price**
0.5	2.15 %	–	98.94
1.0	2.50 %	5.0 %	102.45
2.0	2.90 %	6.5 %	106.94
4.0	3.20 %	9.0 %	121.60

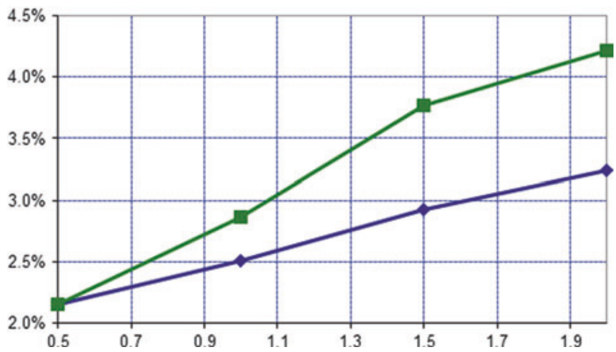

Fig. 6.1 The zero rate and the forward rate from bootstrapping

We start by stripping the instruments to find the corresponding zero-coupon spot rates. This yield curve can then be used as the risk-free yield for instruments with maturities from zero to 4 years. As before, we start with the six-month zero rate: $s_1 = 2.15\,\%$. On time less than six months the rate has to be calculated by extrapolation. The next bond is the same as in the previous example, so: $s_2 = 2.504\,\%$

To calculate s_3 we have to extrapolate. The reason is that we have one coupon at $t = 1.5$ and the nominal plus a coupon at $t = 2$. Therefore we cannot subtract all the coupons to find the zero bond price at $t = 2$. When we extrapolate we use the known points at $t = 0.5$ and $t = 1.0$. That is,

$$s_3' = \frac{s_2 - s_1}{t_2 - t_1} \cdot t_3 + s_2 - \frac{s_2 - s_1}{t_2 - t_1} \cdot t_2 = s_2 + \frac{s_2 - s_1}{t_2 - t_1} \cdot (t_3 - t_2)$$

$$= 2.504 + \frac{2.504 - 2.150}{1.0 - 0.5} \cdot (1.5 - 1.0) = 2.859$$

With s_3 known, we can calculate s_4 from:

$$\frac{3.25}{1 + \frac{0.0215}{2}} + \frac{3.25}{\left(1 + \frac{0.02504}{2}\right)^2} + \frac{3.25}{\left(1 + \frac{0.02859}{2}\right)^3} + \frac{103.25}{\left(1 + \frac{s_3}{2}\right)^4} = 106.94$$

giving $s_4 = 2.918$. Next, we have to perform a new extrapolation and finally get the result shown in Table 6.1.

In the leftmost columns (T, Coupon, Yield and Price) we have the given bond data. Then, in the rows for each bond we see the projected cash flow and its present value. That is, in the square with 103.25 and 97.44 we have the projected cash flow of 103.25 and PV(103.25) = 97.44. In the row Time we have the time for each cash flow and in

Table 6.1 The result of bootstrapping with linear interpolation

T	Coupon	Yield	Price								
0.5	0.00%	2.15%	98.94	100							
				98.94							
1	5.00%	2.50%	102.45	2.5	102.5						
				2.47	99.98						
2	6.50%	2.90%	106.94	3.25	3.25	3.25	103.25				
				3.22	3.17	3.11	97.44				
4	9.00%	3.20%	121.6	4.5	4.5	4.5	4.5	4.5	4.5	4.5	104.5
				4.45	4.39	4.31	4.25	4.16	4.08	3.98	91.98
			Time	0.5	1	1.5	2	2.5	3	3.5	4
			Spot	2.15%	2.50%	2.86%	2.92%	3.13%	3.33%	3.54%	3.22%
			dY/dT		0.708		0.41				
			m		1.80%		2.09%				
			Forw	2.15%	2.86%	3.57%	3.10%	3.96%	4.37%	4.79%	0.98%

Spot the calculated zero-coupon spot rate. The values at times 1.5, 2.5, 3.0 and 3.5 are calculated with extrapolation. The values *dY/dT* and *m* is the coefficients of the line used for extrapolation. That is,

$$\frac{dY}{dT} = \frac{2.504 - 2.150}{1.0 - 0.5} = 0.708$$

$$m = 2.504 - \frac{2.504 - 2.150}{1.0 - 0.5} \cdot 1.0 = 1.796$$

giving

$$r(1.5) = 1.796 + 0.708 \cdot 1.5 = 2.858$$

We have also calculated the implied forward rates from the arbitrage condition between the zero-coupon spot rates. These are listed in the last row in Table 6.1. We then have the zero-coupon spot-and-implied forward curves as shown in Fig. 6.2.

As we can see, these curves have "knees". We can get better results using a Newton-Raphson method. If we use the fact that the yield between times 1 and 2 should be connected with a line, then we can solve the following equation

$$\frac{103.25}{\left(1 + \frac{s_4}{2}\right)^{2 \cdot 2}} + \frac{3.25}{\left(1 + \frac{0.02504}{2} + \frac{1}{2} \cdot \left(\frac{s_4 - 0.02504}{2 - 1}\right) \cdot (1.5 - 1)\right)^{1.5 \cdot 2}}$$
$$+ 3.17 + 3.22 - 106.94 = 0$$

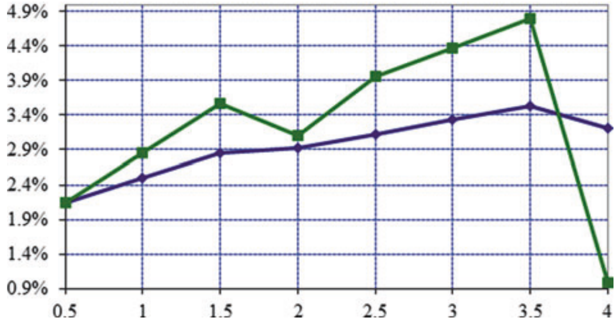

Fig. 6.2 The bootstrapped spot rate and forward rate

where

$$s_3 = 0.02504 + \left(\frac{S_4 - 0.02504}{2-1}\right) \cdot (1.5 - 1)$$

is given by interpolation. Solving s_4, we can use Newton-Raphson to solve the other rates as well. The final result is given in Table 6.2.

Finally, we can draw the graph in Fig. 6.3 which should be compared with Fig. 6.2.

Instead of straight lines, other functions can be used to connect the nodes, such as different kinds of polynomials, in order to get a smoother result. The Newton Raphson method can be derived as

$$\begin{aligned} df = \frac{\partial f}{\partial x} dx \quad &\Rightarrow \quad f(x_{n+1}) - f(x_n) = f'(x_n)(x_{n+1} - x_n) \\ f(x_{n+1}) = 0 \quad &\Rightarrow \\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} & \end{aligned}$$

6.1.3 Bootstrapping with Government Bonds

The data in **Error! Reference source not found.** is taken from NasdaqOMX at January 12, 2010 (Table 6.3)

We will use bootstrapping to find the zero-coupon curve. Swedish bonds are quoted in *ytm* with day-count conversion: 30/360. The coupon frequency is 1 (i.e. there is one coupon per year for bonds).

We start with the bills and immediately have the zero-coupon yields

$$\begin{aligned} r(1m) &= r(30d) = 1.25\ \% \\ r(2m) &= r(60d) = 1.22\ \% \\ r(3m) &= r(90d) = 1.30\ \% \\ r(4m) &= r(120d) = 1.31\ \% \\ r(6m) &= r(180d) = 1.49\ \% \end{aligned}$$

If we ignore business days (weekends and holidays), from RGKB 1045 we also have the rate at

$$\begin{aligned} t &= 360 \cdot (Y_t - Y_{t-1}) + 30 \cdot (M_t - M_{t-1}) + D_t - D_{t-1} \\ &= 360 \cdot (2011 - 2011) + 30 \cdot (3 - 1) + 15 - 12 \\ &= 63 \text{ days} \end{aligned}$$

Table 6.2 The result of bootstrapping with Newton Raphson

T	Coupon	Yield	Price								
0.5	0.00%	2.15%	98.94	100							
				98.94							
1	5.00%	2.50%	102.45	2.5	102.5						
				2.47	99.98						
2	6.50%	2.90%	106.94	3.25	3.25	3.25	103.25				
				3.22	3.17	3.12	97.43				
4	9.00%	3.20%	121.6	4.5	4.5	4.5	4.5	4.5	4.5	4.5	104.5
				4.45	4.39	4.32	4.25	4.18	4.11	4.03	91.88
			Time	0.5	1	1.5	2	2.5	3	3.5	4
			Spot	2.15%	2.50%	2.71%	2.92%	3.00%	3.08%	3.16%	3.24%
			d*Y*/d*T*		0.42		0.16				
			m		2.09%		2.60%				
			Forw	2.15%	2.86%	3.13%	3.55%	3.33%	3.49%	3.65%	3.81%

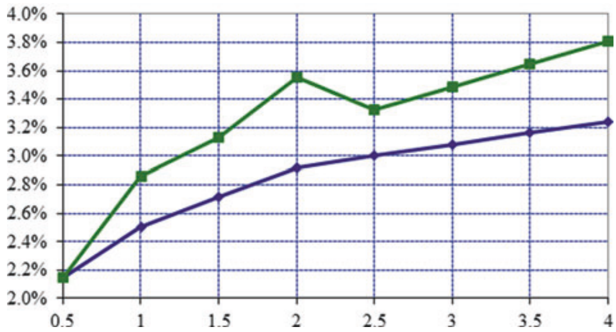

Fig. 6.3 The bootstrapped spot rate and forward rate using Newton Raphson

Table 6.3 Market data from Nasdaq OMX

Bonds	Coupon	YTM	Maturity
RGKB 1041	6.75	2.415	2014-05-05
RGKB1045	5.25	1.24	2011-03-15
RGKB1046	5.5	1.79	2012-10-18
RGKB1047	5	3.219	2020-12-01
Bills		YTM	Maturity
RGKT 1101		1.25	1 month
RGKT1102		1.22	2 month
RGKT1103		1.3	3month
RGKT1104		1.31	4month
RGKT1106		1.49	6 month

as

$$r\,(2011-03-15) = r\,(63d) = 1.25\ \%$$

We now start the bootstrap with the bond RGKB 1046. This bond has a coupon rate $c = 5.5\,\%$, $ytm = 1.79\,\%$ and maturity at $T = 2012$-10-18. That is, time to maturity $= 1y9m6d = 360 + 270 + 6 = 636$ days. Therefore, we also have a coupon payment of 5.5 at 276 days from now. We start by calculating the present value of this coupon. We do this by extrapolation using 180 and 120 days

$$r(276d) = 1.49 + \frac{1.49 - 1.31}{180 - 120}(276 - 180) = 1.778\ \%$$

The present value of the coupon is therefore:

$$PV = \frac{5.5}{(1 + 0.01778)^{276/360}} = 5.4262$$

The price of the bond is given by the quoted yield:

$$\begin{aligned} P &= \frac{5.5}{(1+0.0179)^{276/360}} + \frac{105.5}{(1+0.0179)^{1+276/360}} \\ &= 5.4257 + 102.2445 = 107.6702 \end{aligned}$$

This means that a zero-coupon bond with maturity $T = 636$ days from now, has the price $P = 107.6702 - 5.4262 = 102.2440$. This gives the zero-coupon rate at time $t = 636$ days:

$$\begin{aligned} 102.2440 &= \frac{105.5}{(1+r)^{1+276/360}} \\ r(636d) &= \left(\frac{105.5}{102.2440}\right)^{1/(1+276/360)} - 1 = 1.7903\ \% \end{aligned}$$

This is close to the *ytm* (*ytm* = 1.79) which shows that the result is correct.

We then continue with the next bond, RGKB 1041 with coupon rate $c = 6.75\,\%$, $ytm = 2.415\,\%$ and maturity at $T = 2014\text{-}05\text{-}05$. That is, time to maturity $= 3y3m23d = 1193$ days from today. We also have coupon payments of 6.75 at 833, 473 and 113 days from now. We start by calculate the present values of the coupons using inter- and extrapolation:

$$\begin{aligned} r(113d) &= 1.30 + \frac{1.31 - 1.30}{120 - 90}(113 - 90) = 1.3077\ \% \\ r(473d) &= 1.49 + \frac{1.7903 - 1.49}{636 - 180}(473 - 180) = 1.6830\ \% \\ r(833d) &= 1.49 + \frac{1.7903 - 1.49}{636 - 180}(833 - 180) = 1.9200\ \% \end{aligned}$$

Remark

$$r(833d) = 1.7903 + \frac{1.7903 - 1.49}{636 - 180}(833 - 636) = 1.9200\ \%$$

The present value of the coupons is therefore:

$$PV = \frac{6.75}{(1+0.013077)^{113/360}} + \frac{6.75}{(1+0.01683)^{1+113/360}} + \frac{6.75}{(1+0.0192)^{2+113/360}}$$
$$= 6.7225 + 6.6045 + 6.4594 = 19.7864$$

The price of the bond is

$$P = \frac{6.75}{(1+0.02415)^{113/360}} + \frac{6.75}{(1+0.02415)^{1+113/360}} + \frac{6.75}{(1+0.02415)^{2+113/360}} + \frac{106.75}{(1+0.02415)^{3+113/360}}$$
$$= \frac{1}{(1+0.02415)^{1193/360}} \left\{ 100 + 6.75\frac{(1+0.02415)^4 - 1}{0.02415} \right\} = 118.2621$$

Where we have used the formula

$$PV(ytm) = \frac{1}{(1+ytm)^T} \left\{ N + C\frac{(1+ytm)^M - 1}{(1+ytm)^{1/f} - 1} \right\}$$

We now have that a zero-coupon bond with maturity at $T = 1193$ days have the present value $P = 118.2621\text{-}19.7864 = 98.4757$. This gives the zero-coupon rate at $t = 1193$ days

$$98.4757 = \frac{106.75}{(1+r)^{1193/360}}$$

$$r(1193d) = \left(\frac{106.75}{98.4757}\right)^{360/1193} - 1 = 2.4645\ \%$$

This is close to the *ytm* (*ytm* = 2.451) which shows that the result is correct.

We now have the following zero-coupon yields:

$$
\begin{aligned}
r(30d) &= 1.25\% \\
r(60d) &= 1.22\% \\
r(90d) &= 1.30\% \\
r(120d) &= 1.31\% \\
r(180d) &= 1.49\% \\
r(276d) &= 1.778\% \\
r(636d) &= 1.7903\% \\
r(1193d) &= 2.4645\%
\end{aligned}
$$

The last bond has maturity 2020-12-01. This is in $9y10m19d$. That is, in 3559 days with coupons at 3199, 2839, 2479, 2119, 1759, 1399, 1039, 679 and 319 days from today.

We start by calculating the present value of the interpolated coupons (319, 679, 1039 days from today)

$$r(319d) = 1.778 + \frac{1.7903 - 1.778}{636 - 276}(319 - 276) = 1.7795\ \%$$

$$r(679d) = 1.7903 + \frac{2.4645 - 1.7903}{1193 - 636}(679 - 636) = 1.8423\ \%$$

$$r(1039d) = 1.7903 + \frac{2.4645 - 1.7903}{1193 - 636}(1039 - 636) = 2.2781\ \%$$

We continue by calculating the present value of the extrapolated coupons (3199, 2839, 2479, 2119, 1759, 1399 days from today)

$$r(1399d) = 1.7903 + 0.0012104 \cdot (1399 - 636) = 2.7138\ \%$$

$$r(1759d) = 1.7903 + 0.0012104 \cdot (1759 - 636) = 3.1496\ \%$$

$$r(2119d) = 1.7903 + 0.0012104 \cdot (2119 - 636) = 3.5853\ \%$$

$$r(2479d) = 1.7903 + 0.0012104 \cdot (2479 - 636) = 4.0211\ \%$$

$$r(2839d) = 1.7903 + 0.0012104 \cdot (2839 - 636) = 4.4568\ \%$$

$$r(3199d) = 1.7903 + 0.0012104 \cdot (3199 - 636) = 4.8926\ \%$$

The present value of the coupons is therefore

$$
\begin{aligned}
PV &= \frac{5}{(1+0.017795)^{319/360}} + \frac{5}{(1+0.018423)^{679/360}} \\
&\quad + \frac{5}{(1+0.022781)^{1039/360}} + \frac{5}{(1+0.027138)^{1399/360}} \\
&\quad + \frac{5}{(1+0.031496)^{1759/360}} + \frac{5}{(1+0.035853)^{2119/360}} \\
&\quad + \frac{5}{(1+0.040211)^{2479/360}} + \frac{5}{(1+0.44568)^{2839/360}} \\
&\quad + \frac{5}{(1+0.048926)^{3199/360}} \\
&= 4.9225 + 4.8308 + 4.6853 + 4.5059 + 4.2970 + 4.0637 + 3.8113 \\
&\quad + 3.5451 + 3.2706 \\
&= 37.9321
\end{aligned}
$$

The present value if the bond is

$$
P = \frac{1}{(1+0.03219)^{3559/360}} \left\{ 100 + 5.0 \frac{(1+0.03219)^{10} - 1}{0.03219} \right\} = 115.4397
$$

Therefore, a zero-coupon bond with maturity at $T = 3559$ days from today have the present value of $P = 115.4397 - 37.9321 = 77.5076$. This gives the zero-coupon rate at $t = 3559$ days as

$$
\begin{aligned}
77.5075 &= \frac{105.0}{(1+r)^{3559/360}} \\
r(3559d) &= \left(\frac{105.0}{77.5075} \right)^{360/3559} - 1 = 3.1185\ \%
\end{aligned}
$$

The resulting zero-coupon yield curve will have a knee, since we overestimated the interest rates with extrapolations with many cash flows.

To calculate the zero-coupon yield beyond the last maturity, we need to make an assumption. Here, the assumption is made that the implied forward rate remains constant. This is a reasonable assumption, because the forward rate is actually a kind of prediction about the future spot rate. We then use the following formula for the zero-coupon rates/yields for times > 14.6 years

$$
s_{n+1} = (1+s_n)^{t_n/t_{n+1}} (1+F)^{(t_{n+1}-t_n)/t_n} - 1
$$

The upper limit of the zero-coupon rates is given by $s_\infty = F$ where F is the constant forward rate.

If we do this for 10, 11, 12, 13, 14 and 15 years from today on the previous curve, we can calculate the forward rate between the last two points in the curve:

$$r(1193d) = 2.4645\%$$
$$r(3559d) = 3.1185\%$$

This is given by

$$r_{t_2-t_1}^{forward} = \left(\frac{(1+r_{t_2}^{spot})^{t_2}}{(1+r_{t_1}^{spot})^{t_1}}\right)^{\frac{1}{t_2-t_1}} - 1$$
$$= \left(\frac{(1+0.031185)^{3559/360}}{(1+0.024645)^{1193\,/\,360}}\right)^{\frac{360}{3559-1193}} - 1 = 3.4500\ \%$$

Then, using

$$(1+r_{t_2}^{spot})^{t_2} = (1+r_{t_1}^{spot})^{t_1} \cdot (1+r_{t_2-t_1}^{forward})^{t_2-t_1}$$

we get the 10 year rate as

$$(1+r_{10y})^{3600/360} = (1+0.031185)^{3559/360} \cdot (1.0345)^{(3600-3559)/360}$$
$$= 1.35996$$
$$r_{10y} = 1.35996^{1/10} - 1 = 3.1223\ \%$$

and

$$(1+r_{11y})^{11} = (1+0.031223)^{10} \cdot 1.0345 = 1.35996$$
$$r_{11y} = 3.1520\ \%$$
$$(1+r_{12y})^{12} = 1.35996 \cdot 1.0345 = 1.45541$$
$$r_{12y} = 3.1768\ \%$$
$$(1+r_{13y})^{13} = 1.45541 \cdot 1.0345 = 1.50562$$
$$r_{13y} = 3.1978\ \%$$
$$(1+r_{14y})^{14} = 1.50562 \cdot 1.0345 = 1.55756$$
$$r_{14y} = 3.2158\ \%$$
$$(1+r_{15y})^{15} = 1.55756 \cdot 1.0345 = 1.56113$$
$$r_{15y} = 3.2314\ \%$$

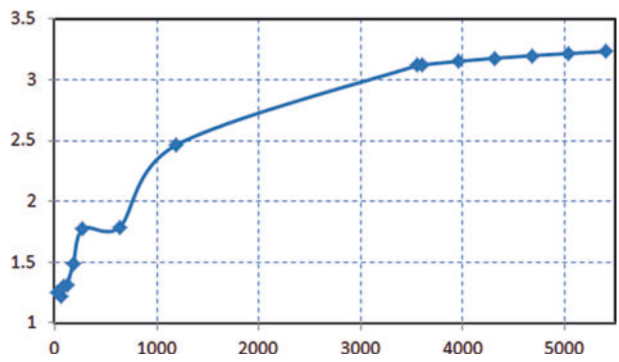

Fig. 6.4 The zero-coupon curve as function of days to maturity

Finally, we get the zero-coupon rate curve as in Fig. 6.4.

We can summarize the previous bootstrap method in the following schema:

1) Calculate the zero-coupon rates for all instruments, which have only one cash flow until maturity.
2) Take the bond with the least number of coupons.
3) Interpolate/extrapolate (or better, use a solver instead of an extrapolation) to calculate the yields between known zero-coupon rates at times when we have coupon payouts.
4) Calculate the present value, $PV(coupons)$ for all coupons with payout before the maturity of the bond in step 2.
5) Calculate the zero-coupon price: $ZCP = \left[P - PV(coupons)\right] \cdot \frac{100}{100+C}$.
6) Calculate the yield of a zero-coupon bond with the face value of 100 and price ZCP.
7) Add this yield to the known zero-coupon rates.
8) If there are more bonds, go to step 2, if not, go to 9.
9) Create a zero-coupon yield curve.

6.1.3.1 The Swap Curve

There are far more swaps of different maturities than there are bonds, so that in practice, swaps are used to build up the forward rates by bootstrapping. Fortunately, there is a simple decomposition of swaps prices into the prices of zero-coupon bonds so that bootstrapping is still relatively straightforward.

When the swap is first entered into, it is usual for the deal to have no value to either party. This is done by a careful choice of the fixed rate of interest. In other words, the present value, let us say, of the fixed side and the floating side both have the same value, netting out to zero.

Why should both parties agree that the deal is valueless?

There are two ways to look at this. One way is to observe that a swap can be decomposed into a portfolio of bonds and so its value is not open to question if we are given the yield curve. However, in practice the calculation goes the other way. The swap market is so liquid, at so many maturities, that it is the prices of swaps that drive the prices of bonds. The fixed leg of a *par swap* (the one having no value) is determined by the market. The rates of interest in the fixed leg of a swap are quoted at various maturities. These rates make up the swap curve.

6.1.3.2 The Relationship Between Swaps and Bonds

There are two sides to a swap, the fixed-rate side and the floating-rate side. The fixed interest payments, since they are all known in terms of actual dollar amount, can be seen as a sum of zero-coupon bonds. If the fixed rate of interest is r_s then the fixed payments add up to

$$r_s \sum_{i=1}^{N} p(t, T_i)$$

This is the discounted value today, at time t, of all the future fixed-rate payments. Here there are N payments, one at each T_i. Of course, this is multiplied by the notional principal, but assume that we have scaled this to one. To see the simple relationship between the floating leg and zero-coupon bonds we draw some schematic diagrams and compare the cash flows. A single floating leg payment is shown in Fig. 6.5.

At time T_i there is payment of r_τ of the notional principal, where r_τ is the period τ rate of LIBOR, set at time $T_i - \tau$. We add and subtract \$1 at time T_i to get the second diagram. The first and the second diagrams obviously have the same present value. Now recall the precise definition of LIBOR. It is the interest rate paid on a fixed-term time deposit in the interbank market. Thus, the \$1 + r_{τ}- at time T_i is the

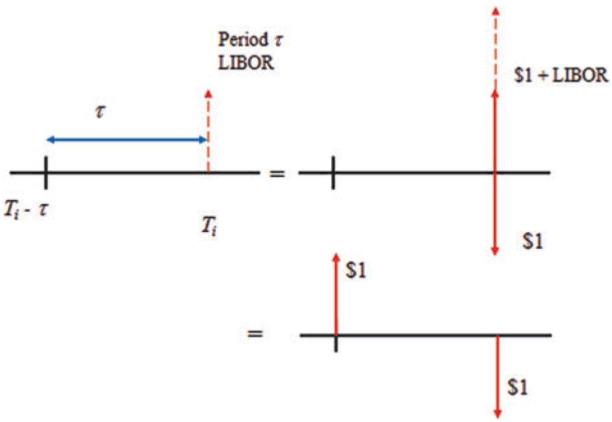

Fig. 6.5 A single floating swap cash flow in relation with bond cash flows

same as \$1 at time $T_i - \tau$. This gives the third diagram. It follows that the single floating rate payment is equivalent to two zero-coupon bond cash flows. A single floating leg of a swap at time T_i is *exactly* equal to a deposit of \$1 at time $T_i - \tau$ and a withdrawal of \$1 at timeτ.

Now adding up all the floating legs all \$1 cash flows cancel out except for the first and last one. Thus, the floating side of the swap has the current or discounted value

$$1 - p(t, T_N)$$

Bringing the fixed and floating sides together we find that the value of the swap, for the receiver of the fixed side, is

$$r_s \sum_{i=1}^{N} p(t, T_i) - 1 + p(t, T_N)$$

This result is *model-independent*, that is, relationship is independent of any mathematical model for bonds or swaps. At the start of the swap contract the rate r_s is usually chosen to give the contract par value (i.e. zero value initially). Thus the quoted swap rate is

$$r_s = \frac{1 - p(t, T_N)}{\sum_{i=1}^{N} p(t, T_i)}$$

We will discuss this in more detail later.

6.1.3.3 Bootstrapping with Swaps

Swaps are very liquid and there exists a wide range of maturities so that their prices determine the yield curve and not vice versa. In practice, the swap rates $r_s(T_i)$ are given in the market for many maturities T_i and 1 uses the aforementioned formula to calculate the prices of zero-coupon bonds and thus the yield curve. For the first point on the discount-factor curve we must solve

$$r_s(T_1) = \frac{1 - p\,(t, T_1)}{p\,(t, T_1)}$$

That is,

$$p\,(t, T_1) = \frac{1}{1 + r_s(T_1)}$$

After finding the first j discount factors the $j + 1$st is then found from

$$p\left(t, T_j\right) = 1 - r_s\,(T_i) \sum_{i=1}^{j} p\,(t, T_i)$$

=>

$$p\left(t, T_{j+1}\right) = \frac{1 - r_s\left(T_{j+1}\right) \sum_{i=1}^{j} p\,(t, T_i)}{1 + r_s\left(T_{j+1}\right)}$$

We will now show in practice how we can calculate a zero-rate yield curve using swaps. A typical data source for the underlying construction of the nominal interest rate term structure can be the European swap rates for 1–10-years maturities (yearly intervals) and 12-, 15-, 20-, 25-, 30-, 40- and 50-years maturities as they are listed on a daily basis by Bloomberg. In such interest rate swaps, 6-month EURIBOR is exchanged for a fixed interest rate.

Methodology
The rates quoted in the market are par yields and the interest day-count convention of the fixed-rate side of an ordinary swap is 30/360, meaning that a month is set at 30 days and a year at 360 days. We define the following (annually accrued) interest rates

r_t = the (par) swap rate at maturity t,

z_t = the zero-coupon swap rate at maturity t.

$f_{t1,t2}$ = the forward rate between t_1 and t_2.

The cash flows of the underlying fixed-rate bond included in a t-year swap are as follows

date (years)	1	2	...	$t-1$	t
cash flows	r_t	r_t	...	r_t	$1+r_t$

The value at the time the swap is made equals 1 (= 100%).

The zero-coupon rate is derived from the par swap rate by means of bootstrapping, starting with the 1-year swap rate. Since $(1+r_1)/(1+z_1) = 1$, it follows that $z_1 = r_1$. The 2-years zero-coupon interest is determined by calculating the present value, at the 1-and 2-years zero rate, of the cash flows from (the fixed-rate side of) the 2-years swap, and equating this present value to 1. The 1-year zero-rate is already known, so that this leaves an equation with a single unknown (the 2-years zero-coupon rate z_2):

$$\frac{r_2}{1+z_1} + \frac{1+r_2}{(1+z_2)^2} = 1$$

which may be rewritten as

$$z_2 = \sqrt{\frac{1+r_2}{1-\frac{r_2}{1+z_1}}} - 1$$

z_3 through z_{10} are derived analogously. By way of explanation, we also derive the 1-year forward over 1 year (i.e. the forward interest rate accruing between $t = 1$ and $t = 2$) via:

$$(1+z_2)^2 = (1+z_1)\left(1+f_{1,2}\right)$$

and hence

$$f_{1,2} = \frac{(1+z_2)^2}{(1+z_1)} - 1$$

From maturities of 10 years onwards, not all Bloomberg swap rates are used. Intervening rates are derived from the 12-, 15-, 20-, 25-, 30-, 40- and 50-years maturity points. To calculate, for instance, the 21-years zero-coupon rate, we need to make an assumption. Here, the assumption is made that the 1-year forward remains constant between 20 and 25 years. This is a reasonable assumption, because the forward rate can be seen as a kind of forecast for the 1-year rate that will be realized 20, 21 etc. years from now. And the market is not very likely to take substantially different views on 1-year interest rates 20 or 21 years forward. Now, based on the assumption that $f_{20,21} = f_{21,22} = f_{22,23} = f_{23,24} = f_{24,25} = f_{20,25}$, we may write the 21-, 22-, 23-, 24- and 25-year zero rates, respectively,

$$\begin{cases}(1+z_{21})^{21} = (1+z_{20})^{20}\left(1+f_{20,21}\right) = (1+z_{20})^{20}\left(1+f_{20,25}\right)\\(1+z_{22})^{22} = (1+z_{21})^{21}\left(1+f_{21,22}\right) = (1+z_{20})^{20}\left(1+f_{20,25}\right)^2\\(1+z_{23})^{23} = (1+z_{22})^{22}\left(1+f_{22,23}\right) = (1+z_{20})^{20}\left(1+f_{20,25}\right)^3\\(1+z_{24})^{24} = (1+z_{23})^{23}\left(1+f_{23,24}\right) = (1+z_{20})^{20}\left(1+f_{20,25}\right)^4\\(1+z_{25})^{25} = (1+z_{24})^{24}\left(1+f_{24,25}\right) = (1+z_{20})^{20}\left(1+f_{20,25}\right)^5\end{cases}$$

Consequently, we may formulate the present value of the 25-year swap as

$$\begin{aligned}&\frac{r_{25}}{1+z_1} + \frac{r_{25}}{(1+z_2)^2} + \ldots.\frac{r_{25}}{(1+z_{24})^{24}} + \frac{1+r_{25}}{(1+z_{25})^{25}}\\&= r_{25}\left[\sum_{t=1}^{20}\frac{1}{(1+z_t)^t} + \frac{1}{(1+z_{20})^{20}}\sum_{t=1}^{5}\frac{1}{\left(1+f_{20,25}\right)^t}\right]\\&+ \frac{1}{(1+z_{20})^{20}\left(1+f_{20,25}\right)^5} = 1\end{aligned}$$

A numerical procedure is needed to solve for $f_{20,25}$. Substitution of the result in the earlier equations will yield z_{21} through z_{25}.

For other maturities, the calculation is analogous. For points beyond 30 years, the 1-year forward is assumed to remain constant. The assumption of a constant forward rate may also be used in extrapolating beyond 50 years. Based on this latter forward rate, we may calculate spot rates for very long maturities. This method can also be used when bootstrapping bonds.

6.1.4 Bootstrapping a Swap Curve

When bootstrapping a zero-coupon curve it is very important to use liquid instruments. In the Swedish market these are typically: an overnight rate (O/N), a tomorrow-next rate (T/N), and interbank time deposit rates for one week, one month, two and three month maturities. For additional maturities we use some representative quotes for interbank Forward Rate Agreements (FRA) and finally, interest rate swap rates from 3 years up to 30 years. We do not use shorter swaps since the FRAs are more liquid. We also show the Swedish market due to the special handling of IMM FRA's and the negative interest rates. In the Euro and US market the FRA are note quoted at the IMM days and this simplify the bootstrap calculations.

6.1.4.1 Market Data

The bootstrap process will be demonstrated here with market data from 2017-07-17. Starting with the STIBOR fixings (also known as cash deposits) with maturities O/N, T/N, 1W, 1M,. . . 3M we have:

STIBOR Fixing

Maturity	STIBOR
T/N	–0.518
1W	–0.526
1M	–0.523
2M	–0.503
3M	–0.474

We also, for simplicity, suppose that the (O/N) is the same as the (T/N).

We continue with the short-term FRA's, FRA 3M, with maturities on IMM days.

FRA 3M Rate Maturity	Bid	Ask	Mid
2017/sep/20	–0.469	–0.449	–0.459
2017/dec/20	–0.539	–0.519	–0.529
2018/mar/21	–0.359	–0.339	–0.349
2018/jun/20	–0.264	–0.244	–0.254
2018/sep/19	–0.154	–0.124	–0.139
2018/dec/19	–0.049	–0.019	–0.034
2019/mar/20	0.081	0.111	0.096
2019/jun/19	0.196	0.226	0.211
2019/sep/18	0.310	0.350	0.330
2019/dec/18	0.430	0.470	0.450

Finally, we have the swap rates from SEK STIBOR A 3M.

Swap Maturity	Bid	Ask	Mid
3Y	–0.0375	0.0125	–0.0125
4Y	0.1975	0.2475	0.2225
5Y	0.4325	0.4825	0.4575
6Y	0.6450	0.6950	0.6700
7Y	0.8325	0.8825	0.8575
8Y	0.9975	1.0475	1.0225
9Y	1.1400	1.1900	1.1650
10Y	1.2625	1.3125	1.2875
12Y	1.4550	1.5150	1.4850
15Y	1.6775	1.7375	1.7075
20Y	1.8775	1.9575	1.9175
25Y	1.9575	2.0375	1.9975
30Y	1.9775	2.0775	2.0275

In the SEK market, rates and market quotes are the same since all instruments are quoted as yields to maturity. In most other markets, FRA contracts are quoted in clean price. In that case, we first have to calculate the yield to get the table above. We can also use the price to find the discount factors.

The data above results in the following table, where we will bootstrap only the mid curve.

Tenor	Start date	Maturity	Period days $d(t)$	Quote rate $q(t)$
O/N	2017-07-17	2017-07-18	1	–0.518
T/N	2017-07-18	2017-07-19	1	–0.518
1W	2017-07-19	2017-07-26	7	–0.526
1M	2017-07-19	2017-08-21	33	–0.523
2M	2017-07-19	2017-09-19	62	–0.503
3M	2017-07-19	2017-10-19	92	–0.474
sep_17	2017-09-20	2017-12-20	91	–0.459
dec_17	2017-12-20	2018-03-21	91	–0.529
mar_18	2018-03-21	2018-06-20	91	–0.349
jun_18	2018-06-20	2018-09-19	91	–0.254
sep_18	2018-09-19	2018-12-19	91	–0.139
dec_18	2018-12-19	2019-03-20	91	–0.034
mar_19	2019-03-20	2019-06-19	91	0.096
jun_19	2019-06-19	2019-09-18	91	0.211
sep_19	2019-09-18	2019-12-18	91	0.330
dec_19	2019-12-18	2020-03-18	91	0.450
3Y	2017-07-19	2020-07-20	1097	–0.0125
4Y	2017-07-19	2021-07-19	1461	0.2225
5Y	2017-07-19	2021-07-19	1826	0.4575
6Y	2017-07-19	2021-07-19	2191	0.6700
7Y	2017-07-19	2021-07-19	2557	0.8575
8Y	2017-07-19	2025-07-21	2924	1.0225
9Y	2017-07-19	2026-07-20	3288	1.1650
10Y	2017-07-19	2027-07-19	3652	1.2875
12Y	2017-07-19	2027-07-19	4383	1.4850
15Y	2017-07-19	2027-07-19	5479	1.7075
20Y	2017-07-19	2037-07-20	7306	1.9175
25Y	2017-07-19	2042-07-21	9133	1.9975
30Y	2017-07-19	2047-07-19	10957	2.0275

6.1.4.2 Cash Deposits

We start by calculating the discount factor and from here the zero rate. By using O/N, we get

$$D_{O/N} = \frac{1}{1 + q_{O/N} \cdot \frac{d_{O/N}}{360}} = \frac{1}{1 + \frac{-0.518}{100} \cdot \frac{1}{360}} = 1.000014389$$

From here, we get the zero rate as

$$Z_{O/N} = -100 . \frac{\ln(D_{O/N})}{\frac{d_{O/N}}{365}} = -100 . \frac{\ln(1.000014389)}{\frac{1}{365}} = -0.525198223$$

Zero rates are commonly given as continuously compounded rates, Act/365. If the day today is a Friday, $d_{O/N} = 3$ instead of 1 as above. Next, we use the T/N

$$D_{T/N} = \frac{D_{O/N}}{1 + q_{T/N} \cdot \frac{d_{T/N}}{360}} = \frac{1.000014389}{1 + \frac{-0.518}{100} \cdot \frac{1}{360}} = 1.000028778$$

From here, we get the zero rate as

$$Z_{T/N} = -100 \cdot \frac{\ln(D_{T/N})}{\frac{d_{T/N}}{365}} = -100 \cdot \frac{\ln(1.000028778)}{\frac{1}{365}} = -0.525198223$$

Discount rates greater than 1.0 give, as we can see, negative zero-coupon rates. Sometimes also the spot next (S/N) is used. Now we have the beginning of the curve. The above calculations must be done because of different “Start Dates”. We continue with the money-market instruments (1W, 1M, 2M and 3M), i.e. deposits. All of them have the same Start date: 2017-07-19 (two days from today because of the two settlement days)

$$D_i = \frac{D_{T/N}}{1 + q_i.\dfrac{d_i}{365}}; \qquad i = \{1W, 1M, 2M, 3M\}$$

From here, we get the Zero rate as

$$Z_i = -100.\frac{\ln(D_i)}{\dfrac{d_i}{365}}; \qquad i = \{1W, 1M, 2M, 3M\}$$

As seen above we use $D_{T/N}$ as the numerator. However, we have here three choices dependent on the **spot lag value**. The spot lag is defined as the number of business days between the trade date and the value date. For most currencies the spot lag is two days. The choices are:

1) Spot lag = 0 => The numerator = 1
2) Spot lag = 1 => The numerator = $DF_{O/N}$
3) Spot lag > 1 => The numerator = $DF_{T/N}$

6.1.4.3 Forward Rate Agreements – FRA

Next, we are ready to handle the FRA rates. Since these are forward contracts (with netted principal cash payments), the corresponding quoted rates are forward rates. Therefore, we need a so-called **stub rate**. The stub rate shall have its maturity on the same date as the start date of the first FRA contract. This stub has to be calculated only if we have IMM FRA contracts. We can calculate that rate using (linear) interpolation on the discount factors

$$D_{stub}(t) = D(t_0) + \frac{D(T) - D(t_0)}{T - t_0}(t - t_0)$$

In our example, we get:

$$\begin{aligned} D_{stub} &= D(2017-09-20) \\ &= D(2017-09-19) + (D(2017-10-19) - D(2017-09-19)) \\ &\quad \cdot \frac{(T_{2017-09-20}) - (T_{2017-09-19})}{(T_{2017-10-19}) - (T_{2017-09-19})} \\ &= 1.000907358 \end{aligned}$$

The stub rate is then given by

$$z_{stub} = -\frac{100 \cdot \ln(D_{Stub})}{65/365} = -0.509285585$$

where 65 is the number of days between today (2017-07-17) and the stub maturity. Now, we can handle the FRA rates as given below:

$$D^{i}_{FRA} = \frac{D^{i-1}_{FRA}}{1 + q^{i}_{FRA} \cdot \frac{d^{i}_{FRA}}{360}}; \qquad i = \{\text{sep\_17}, \text{dec\_17}, \ldots, \text{dec\_19}\}$$

where $D^{0}_{FRA} = D_{Stub}$ From here we get the zero rate as

$$Z^{i}_{FRA}(T) = -100 \cdot \frac{\ln D^{i}_{FRA}}{\frac{d^{i}_{FRA}}{365}}; \qquad i = \{\text{sep}/11, \text{dec}/11, \ldots, \text{dec}/14\}$$

If the FRA contracts are quoted in price, the first FRA discount factor is calculated as

$$D_{FRA}(t) = \frac{D_{Stub}(t)}{1 + \left(\frac{100 - P_{FRA}}{100}\right) \cdot \frac{91}{360}}$$

where P_{FRA} is the quoted price of the FRA contract and 91, the days between the two IMM dates. The implied par rates are calculated as

$$r^{implised}_{par}(t_{FRA}) = 100 \cdot \left(\frac{D_{T/N}}{D_{FRA}} - 1\right) \cdot \frac{1}{t_{FRA} - t_{TN}}$$

6.1.4.4 Swaps

Now, that the zero-rates up to 2014-09-17 have been calculated we can continue with the rest of the curve which is given by swap rates. Since they start at 3Y, we first calculate approximate swap rates for 1Y and 2Y by (linear) interpolation

Year	Start date	Maturity	DF	Zero rate
1	2017-07-19	2018-07-19	1.004503375	–0.446877879
2	2017-07-19	2019-07-19	1.004962055	–0.246813006

We now recall how to calculate the par swap rates r_T^{par}:

$$r_T^{par} = \frac{D_{T/N} - D_T}{\sum_{t=1}^{T} Y_t \cdot D_t} = \frac{D_{T/N} - D_T}{\sum_{t=1}^{T-1} Y_t \cdot D_t + Y_T.D_T}$$

We then have

$$D_T = \frac{D_{T/N} - r_T^{par} \sum_{t=1}^{T-1} Y_t \cdot D_t}{1 + Y_t.r_T^{par}}$$

where Y_t is the year fraction at time t, given by:

$$Y_t = \frac{360 \cdot (y_t -) y_{t-1} + 30 \cdot (m_t - mt - 1) + d_t - d_{t-1}}{360}$$

where y_{t-1} is the previous year where a rate exist, mt the month for the rate and d_t the days. For the years where swap rates are not quoted in the market (11Y, 13Y, 14Y, 16YĚ., 29Y) we use (linear) extrapolation to find the zero rate when calculating the discount factors. Suppose all tenor spreads are zero, then the swap rates can be considered as maturing once a year. This gives $Y_t = 1$ for all t. We then have

$$D_T = \frac{D_{T/N} - r_T^{par} \sum_{t=1}^{T-1} D_T}{1 + r_T^{par}}$$

Finally, we get the result

Tenor	Start date	Maturity	Quote rate	Discount	Zero rate
O/N	2017-07-17	2017-07-18	–0.518	1.00001439	–0.52519822
T/N	2017-07-18	2017-07-19	–0.518	1.00002878	–0.52519822
1W	2017-07-19	2017-07-26	–0.526	1.00013107	–0.53152514
1M	2017-07-19	2017-08-21	–0.523	1.00050844	–0.53009431
2M	2017-07-19	2017-09-19	–0.503	1.00089583	–0.51067561
3M	2017-07-19	2017-10-19	–0.474	1.00124162	–0.48181770
sep_17	2017-09-20	2017-12-20	–0.459	1.00207001	–0.48382869
dec_17	2017-12-20	2018-03-21	–0.529	1.00341177	–0.50330985
mar_18	2018-03-21	2018-06-20	–0.349	1.00429775	–0.46311196
jun_18	2018-06-20	2018-09-19	–0.254	1.00494298	–0.41952074
sep_18	2018-09-19	2018-12-19	–0.139	1.00529620	–0.37077179
dec_18	2018-12-19	2019-03-20	–0.034	1.00538261	–0.32068484
mar_19	2019-03-20	2019-06-19	0.096	1.00513870	–0.26649883
jun_19	2019-06-19	2019-09-18	0.211	1.00460288	–0.21137413
sep_19	2019-09-18	2019-12-18	0.33	1.00376557	–0.15518699
dec_19	2019-12-18	2020-03-18	0.45	1.00262509	–0.09814374
3Y	2017-07-19	2020-07-20	–0.0125	1.00040546	–0.01346347
4Y	2017-07-19	2021-07-19	0.2225	0.99111701	0.22260951
5Y	2017-07-19	2021-07-19	0.4575	0.97721010	0.46031544
6Y	2017-07-19	2021-07-19	0.67	0.96013564	0.67708428
7Y	2017-07-19	2021-07-19	0.8575	0.94084199	0.86978222
8Y	2017-07-19	2025-07-21	1.0225	0.91991457	1.04129128
9Y	2017-07-19	2026-07-20	1.165	0.89822023	1.19085400
10Y	2017-07-19	2027-07-19	1.2875	0.87616010	1.32061722
12Y	2017-07-19	2027-07-19	1.485	0.83187817	1.53216160
15Y	2017-07-19	2027-07-19	1.7075	0.76590232	1.77605784
20Y	2017-07-19	2037-07-20	1.9175	0.66854155	2.01107974
25Y	2017-07-19	2042-07-21	1.9975	0.59192602	2.09519836
30Y	2017-07-19	2047-07-19	2.0275	0.52919479	2.11958684

To get high accuracy in the calculated values where we have to use extrapolation, we also use a Newton-Raphson method. This is applied for the discount factors. The Newton-Raphson method calculates the discount factors so that we can reprice the swaps and find their quotes, i.e., the interest rates for the fixed legs.

If we have semi-annual data, the above formula is replaced by

$$D_T = \frac{D_{T/N} - r_T^{par} \sum_{t=0.5}^{T-0.5} Y_t \cdot D_t}{1 + Y_Y \cdot r_T^{par}}$$

From here, we get the zero rate as usual

$$Z_T = -100 \cdot \ln(D_T) \cdot \frac{365}{d_T}$$

If the zero-coupon rates are expressed as annual bond equivalent yields we have

$$D_T = \frac{1}{(1 + Z_T/100)^{days/365}}$$

Solving the previous equation, the zero coupon rate is

$$Z_T = \left\{ \left(\frac{1}{D_T} \right)^{365/days} - 1 \right\}$$

If you want continuously compounded zero rates, the discount factor will be calculated as

$$D_T = \exp\left\{ -\frac{Z_T}{100} \cdot \frac{days}{365} \right\}$$

From the latter equation, the zero- coupon rate becomes a function of the discount factor, as follows

$$Z_T = -\left(\ln(D_T). \frac{365}{days} \right) \cdot 100$$

If you prefer to represent zero coupon rates as simple annualized rates, the discount factor should be written as

$$D_T = \frac{1}{1 + Z_T/100}$$

6.1.5 A More General Bootstrap

In the risk system of a bank the bootstrap procedure is slightly more complicated than described earlier. Generally, in such a risk system there are many different currencies and therefore many different curves. Each curve will be bootstrapped from many different instruments with different maturities. Therefore it can be difficult to compare and aggregate the risk for a complex portfolio. Because of this added complexity, many risk systems use only specific nodes in time, where they specify the zero-coupon rates. These nodes can be something like

$$[1d, 2d, 3d, 1w, 1m, 2m, 3m, 6m, 9m, 1y, 2y, \\ 3y, 4y, 5y, 7y, 10y, 12y, 15y, 20y, 25y, 30y]$$

Here d denotes days, w weeks, m months and y years. Now, these dates will naturally not be the same as the quoted instruments that is used to bootstrap the curve. Especially, since the same nodes are used for all currencies. Also in the same currency, you might have multiple curves, like a Swap curve for the Interbank market, a Government curve from treasury bonds and a Mortgage curve for the valuation of real estate loans.

The risk system will also use the zero-coupon yield curves for risk calculations and shift the aforementioned nodes. This is usually made by triangular shifts to calculate the risk in each time bucket. Here the 1y bucket is the interval [9m, 1y]. By using specific nodes the risk can be aggregated to a total risk in each bucket. Also the sum of this risk will correspond to a parallel shift of the entire curve.

Banks also use to sets risk limits for each such buckets. This risk can be interest rate risk (Delta and Gamma), Vega risk (by shifting the volatility) etc. Different limits can be used for different size of the shifts. Usually interest rate shifts are calculated for 1 bp, +100 bp and -100 bp. Here 1 bp (one basis point) represent 1/100 of a %.

A problem by using “fixed” nodes (as shown earlier) is that you need an interpolation method between the nodes. Then, when you reprice the instruments used to bootstrap the curve, you will not replicate the input market quotes exactly. Therefore, when you bootstrap the curve, you must minimize the repricing error in order to find a “best fit”. Also, with different interpolation methods, you will get different curves. It is important to understand that there are no “true interest rate curves” since they always depend on what method is used to create the curve.

Also note that the Risk group in the bank shifts zero-coupon yield curves while the traders, on the other hand, typically shift market quotes. The reason is that the trades want to hedge their position. For the hedging purpose, they can only use instruments that exist and are quoted in the market.

Therefore the risk in the trading system and in the risk system (which can be the same, with different setups) may often differ. Subsequently we will describe a typical problem when the risk manager discusses the risk with a trader. This is taken from a real-life situation.

6.1.5.1 An Example of Risk Calculations of an FRA

Consider an FRA contract between 2016-03-16 and 2016-06-15. The today's date is 2015-05-05. The notional amount of the FRA is 1 500 000 000 SEK and day count method Act/360.

The trading desk calls the Group Risk and says that something is wrong! They see all the risk at 2016-03-16 when the FRA expires and the payments are made. They refer to their approximation of the risk.

$$Risk = Notional * Time * 1\ bp = 1\,500\,000\,000 * 0.25 * 0.0001 = 37\,500\ \text{SEK}$$

Here the time is a quarter of a year. They complain and say that the risk system gives a risk of – 70 000 in the 9m bucket and + 88 000 in the 1y bucket.

When Group Risk investigates their modelling system, they find the following.

The used nodes are:

$$\{1\text{d},\ 2\text{d},\ 3\text{d},\ 1\text{w},\ 1\text{m},\ 3\text{m},\ 6\text{m},\ 9\text{m},\ 1\text{y},\ 2\text{y},\ 3\text{y},\ \ldots\}.$$

In term of days, this corresponds to

$$\{1, 2, 3, 7, 30, 90, 180, 270, 365, 730, 1095, \ldots\}.$$

Since the FRA days will be (in days) 316d and 407d, the risk manager will find risk when nodes {270d, 365d and 730d} are shifted.

- With a triangular shift in the node 270d we will change the rates in the interval [180d, 365d].
- With a triangular shift in the node 365d we will change the rates in the interval [270d, 730d].

- With a triangular shift in the node 730d we will change the rates in the interval [365d, 1095d].

Suppose we make these shifts and use continuously compounded zero rates in the discount function

$$D(T) = exp\{-rT\}.$$

The Group Risk are satisfied if they can explain their calculations and find an approximation of the risk. So they just "guess" some zero rates at each node and calculate the values.

$r(T)$	**T**	$D(T)$
0.25000%	270	0.998152394
0.27000%	316	0.997665196
0.28000%	365	0.997203916
0.30000%	407	0.996660383
0.40000%	730	0.992031915

The forward rate (for the FRN) is then given by

$$-365 \cdot \ln(0.996660383/0.997665196)/(407-316) = 0.4042\%$$

and the value of the FRN by

$$0.997665196 \cdot 1\,500\,000\,000 \cdot 0.004043 \cdot 0.25 = 1\,512\,121$$

Now, we make a shift of 1bp on node 730d. The result is

$r(T)$	**T**	$D(T)$
0.25000%	270	0.998152394
0.27000%	316	0.997665196
0.28000%	365	0.997203916
0.30115%	407	0.996647595
0.41000%	730	0.991833528

The forward rate (for the FRN) is then given by: 0.4093% giving a new price = 1 531 375.

The risk on this node is therefore 1 531 375–1 512 121 = 19 254 SEK. A check in the Risk systems (with correct interest rates) shows 19 138 SEK.

Next, we make a shift of 1bp on node 635d. We then get

$r(T)$	**T**	$D(T)$
0.25000%	270	0.998152394
0.27484%	316	0.997623374
0.29000%	365	0.997104201
0.30885%	407	0.996562042
0.41000%	730	0.991833528

The forward rate (for the FRN) is now given by: 0.4269% giving a new price = 1 597 221.

The risk on this node is therefore 1 597 221–1 512 121 = 85 100 SEK. A check in the Risk systems shows 89 329 SEK. (The trader said 88 000 SEK). Finally, we make a shift of 1bp on node 270d. We then get

$r(T)$	**T**	$D(T)$
0.26000%	270	0.998078561
0.27516%	316	0.997620646
0.28000%	365	0.997203916
0.30000%	407	0.996660383
0.41000%	730	0.991833528

The forward rate (for the FRN) is then given by: 0.3863% giving a new price = 1 445 047.

The risk on this node is therefore 1 445 047–1 512 121 = -67 074 SEK. A check in the Risk systems shows 71 393 SEK. (The trader said -70 000 SEK).

The total risk is now summed up to give 37 281 SEK (without discounting 37 500 SEK). The risk system gives 37 307 SEK and the traders said 37 500 SEK. As we can see, the error we made on a notional of 1 500 000 000 SEK by guessing the rates is only -26 SEK!

With this simple analysis we can explain for the trader how the risk is calculated in the risk system.

What is important here is that management have to understand that the risk in buckets can be quite high, but that the sum of the risk is still OK. It is therefore important to set the limits so that the traders can hedge their positions without breaching the limits.

6.1.5.2 Repricing the Instruments

When repricing the aforementioned instruments, we can just invert the formulas. Instead, we will now proceed to use the zero rates and the matching discount factors resulting from the bootstrap. We also use interpolation between the calculated values. How close does such a valuation come to the initially given/quoted market prices? Clearly some of the repricing results should be the same as the used quotes, especially for Deposits and FRAs.

When we invert the PV formula for the O/N and T/N Deposits, we get

$$r_{O/N}^{par} = \left(\frac{1}{D_{O/N}} - 1\right) \cdot \frac{360}{d_{O/N}}$$
$$r_{T/N}^{par} = \left(\frac{D_{T/N}}{D_{O/N}} - 1\right) \cdot \frac{360}{d_{T/N}}$$

The rest of the Deposits are given by

$$r_i^{par} = \left(\frac{D_{T/N}}{D_i} - 1\right) \cdot \frac{360}{d_i}$$

All of the FRAs are given by the same formula. This is

$$r_{FRA}^i = \left(\frac{D_{FRA}^{i-1}}{D_{FRA}^i} - 1\right) \cdot \frac{360}{d_{FRA}^i}$$

Note that now we don't have to use the calculation of the stub rate. If the FRA contracts are quoted in price, we instead use the formula

$$P_{FRA} = 100 - \left(\frac{D_{stub}(t)}{D_{FRA}(t)} - 1\right) \cdot \frac{36000}{91}$$

The prices of the swaps are repriced as

$$r_T^{par} = \frac{D_{T/N} - D_T}{\sum_{t=1}^{T} Y_t \cdot D_t} = \frac{D_{T/N} - D_T}{\sum_{t=1}^{T-1} Y_t \cdot D_t + Y_T \cdot D_T}$$

Here, remember to use the year fraction at time t, given by

$$Y_t = \frac{360 \cdot (y_t - y_{t-1}) + 30 \cdot (m_t - m_{t-1}) + d_t - d_{t-1}}{360}$$

where y_{t-1} is the previous year where a rate exists, m_t the month for the rate and d_t the days.

6.1.6 Nelson-Siegel Parameterization

Nelson and Siegel (1987) proposed a mathematical model for the forward curve

$$f(t,s) = \beta_0 + \beta_1 \cdot \exp\left\{-\frac{s}{\tau_1}\right\} + \beta_2 \cdot \frac{s}{\tau_1} \cdot \exp\left\{-\frac{s}{\tau_1}\right\}$$

Integrating this we can derive the corresponding zero-coupon yield curve/zero rates.

$$r(t) = \frac{1}{t}\int_0^t \left[\beta_0 + \beta_1 \cdot \exp\left\{-\frac{s}{\tau_1}\right\} + \beta_2 \cdot \frac{s}{\tau_1} \cdot \exp\left\{-\frac{s}{\tau_1}\right\}\right] ds$$

Making a change in variables: $x = s/\tau_1$, $ds = \tau_1 dx$ we get

$$\begin{aligned} r(t) &= \beta_0 + \frac{\beta_1 \tau_1}{t}\int_0^{t/\tau_1} e^{-x} dx + \frac{\beta_2 \tau_1}{t}\int_0^{t/\tau_1} x \cdot e^{-x} dx \\ &= \beta_0 + \beta_1 \frac{\tau_1}{t}\left[-e^{-x}\right]_0^{t/\tau_1} + \beta_2 \frac{\tau_1}{t}\left\{\left[-xe^{-x}\right]_0^{t/\tau_1} - \int_0^{t/\tau_1} e^{-x} dx\right\} \\ &= \beta_0 + \beta_1 \left[\frac{1 - e^{t/\tau_1}}{t/\tau_1}\right] + \beta_2 \left[\frac{1 - e^{t/\tau_1}}{t/\tau_1} - e^{t/\tau_1}\right] \end{aligned}$$

This implies the following (spot) zero-coupon yield curve

$$\begin{aligned} r^{NS}\left(t, \Theta^{NS}\right) = \beta_0 &+ \beta_1 \left(\frac{1 - \exp\{-t/\tau_1\}}{t/\tau_1}\right) \\ &+ \beta_2 \left(\frac{1 - \exp\{-t/\tau_1\}}{t/\tau_1} - \exp\{-t/\tau_1\}\right) \end{aligned}$$

6.1.6.1 The Svensson Extension

The Nelson-Siegel model was extended by Svensson (1994) in order to take into account a second possible hump in the zero-coupon yield curve. He added an extra term to the polynomial

$$f(t,s) = \beta_0 + \beta_1 \cdot \exp\left\{-\frac{s}{\tau_1}\right\} + \beta_2 \cdot \frac{s}{\tau_1} \cdot \exp\left\{-\frac{s}{\tau_1}\right\} + \beta_3 \cdot \frac{s}{\tau_2} \cdot \exp\left\{-\frac{s}{\tau_2}\right\}$$

This model is called the Nelson-Siegel-Svensson (NSS) model or the Extended Nelson-Siegel model.

$$\begin{aligned} r^{ENS}\left(t, \Theta^{ENS}\right) = \beta_0 + \beta_1 \left(\frac{1 - \exp\{-t/\tau_1\}}{t/\tau_1}\right) \\ + \beta_2 \left(\frac{1 - \exp\{-t/\tau_1\}}{t/\tau_1} - \exp\{-t/\tau_1\}\right) \\ + \beta_3 \left(\frac{1 - \exp\{-t/\tau_2\}}{t/ - \exp\{-t/\tau_2\}}\right) \end{aligned}$$

where$\Theta^{NS} = (\beta_0, \beta_1, \beta_2, \tau_1)$ and $\Theta^{ENS} = (\beta_0, \beta_1, \beta_2, \beta_3, \tau_1, \tau_2)$ are constants to be estimated, used to fit the models to the bond university. This can be done with some kind of least-squares method, or with the solver in Excel. The same constants are assumed to apply for all maturities, so no splines are involved. The simple functional form ensures a smooth and yet quite flexible curve.

The advantage with the Nelson-Siegel models are

1. It can easily be fitted to empirical data.
2. The fitted yield curve converge to a constant value: $\lim_{t\to\infty} r(t) = \beta_0$.
3. The basis functions, given by:

$$\begin{aligned} \varphi_1 &= 1 \\ \varphi_2 &= \frac{1 - e^{-\lambda_1 t}}{\lambda_1 t} \\ \varphi_3 &= \frac{1 - e^{-\lambda_1 t}}{\lambda_1 t} - e^{-\lambda_1 t} \end{aligned}$$

can be interpreted as a parallel shift (φ_1), tilting (φ_2) and flexing (φ_3).

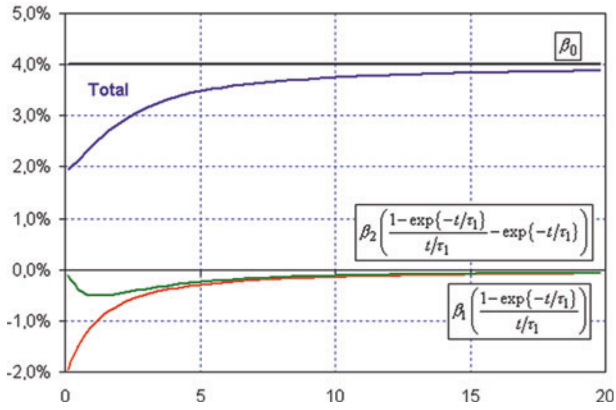

Fig. 6.6 The Nelson-Siegel basis functions

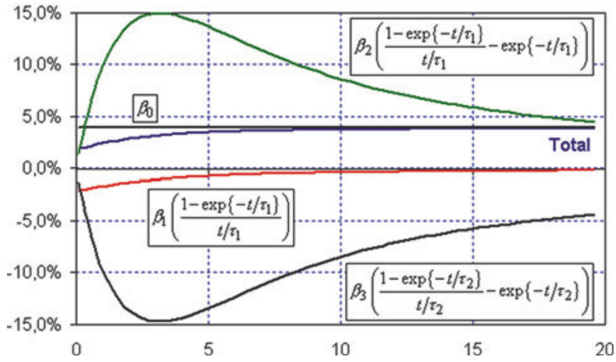

Fig. 6.7 The Extended Nelson-Siegel basis functions

This is illustrated in Fig. 6.6 and Fig. 6.7, which shows the basis functions for the two models when fitted to Swedish Government bonds.

A drawback with the extended (Svensson) model is that there are so many parameters that they cannot always be uniquely identified. Similar numerical problems do not exist in the Nelson-Siegel four parameter model. On the other hand, the extended model can be fitted to a yield curve with two maxima.

The parameters (in the extended model) $\beta_0, \beta_1, \beta_2, \beta_3, \tau_1$ and τ_2 have here been estimated by minimizing the sum of the squared bond price errors weighted by $(1/\Phi_j)$:

$$\underset{\beta_0,\beta_1,\beta_2,\tau_1,\tau_2}{Min} \sum \left\{ \left[P_j - P_j^{ENS}(\beta_0,\beta_1,\beta_2,\tau_1,\tau_2) \right] / \Phi_j \right\}^2$$

where Φ equals the duration * price/(1 + yield to maturity) of the bond. The minimizing problem can easily be solved by using the solver in Excel. Alternatively, the sum of the mean absolute deviations can be minimized.

6.1.7 Interpolation Methods

Several methods for interpolation are available. Here we will discuss

- Linear interpolation
- Logarithmic interpolation
- Polynomial interpolation
- Cubic spline interpolation
- Hermite interpolation

6.1.7.1 Linear Interpolation

A linearly interpolated curve of *ytms* from the following market data consist of tree linear equations $Y_i(t)$, where each of them start at T_i and end at T_{i+1}.

$$Y_i(t) = Y_i + \frac{Y_{i+1} - Y_i}{T_{i+1} - T_i} \cdot (t - T_i)$$

Years	YTM
0.0	4.00%
2.0	5.00%
4.0	6.50%
10.0	6.75%

If we use this market data to calculate the *ytm* for an instrument with maturity in 6 years from now we obtain

$$Y_2(6) = 6.50 + \frac{6.75 - 6.50}{10 - 4} \cdot (6 - 4) = 6.5833\%$$

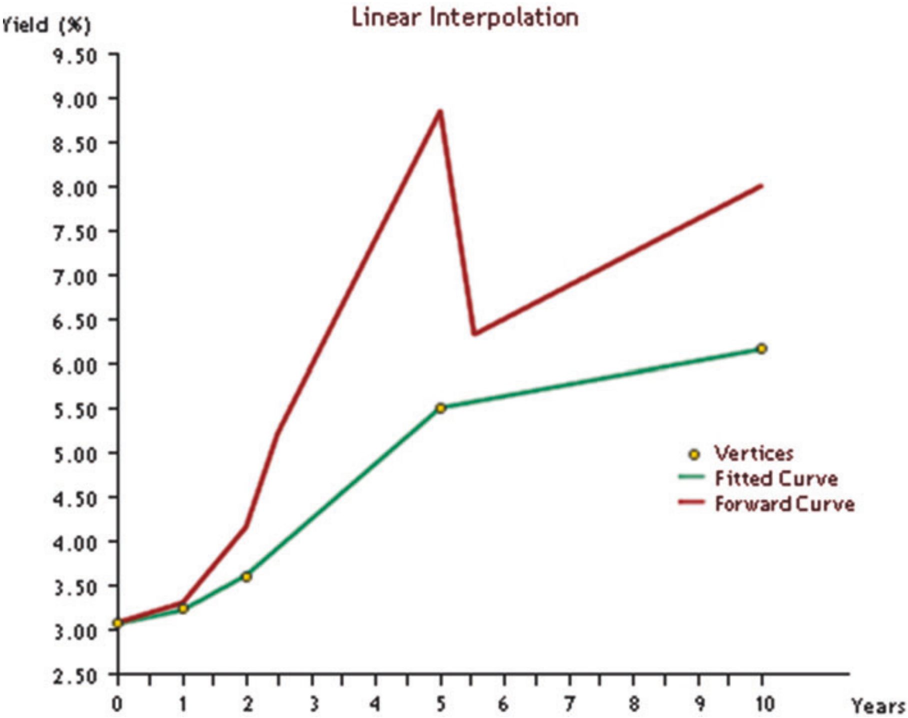

Fig. 6.8 Linear interpolation. Remark the sharp knees in the forward curve

A problem with linear interpolation is that the resulting yield curve can get sharp angles at the intersection points, which gives jumps when we calculate the forward rates, as shown in Fig. 6.8.

For an implementation in C/C++, see the function `double IPOL(double x, double, *pX, Double *pY, int N) above.`

6.1.7.2 Logarithmic Interpolation

In logarithmic interpolation we use the following expression based on the discount function

$$D(t) = D(T_i)^{\frac{T_{i+1}-t}{T_{i+1}-T_i}} \cdot D(T_{i+1})^{\frac{t-T_i}{T_{i+1}-T_i}}$$

Taking the logarithm of both sides, we get

$$\ln\{D(t)\} = \ln\{D(T_i)\} + \frac{\ln\{D(T_{i+1})\} - \ln\{D(T_i)\}}{T_{i+1} - T_i} \cdot [t - T_i]$$

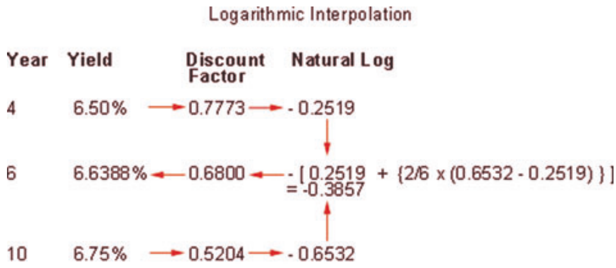

Fig. 6.9 The alculation made in logarithmic interpolation

Using the same market data as before to calculate *ytm* for an instrument with maturity in 6 years we now perform the following steps

1. Calculate the discount factors for the years 4 and 10.
2. Take the logarithm of the values.
3. Make a linear interpolation between the values.
4. Calculate the discount factor for the 6 years interest rate.
5. Translate that into a zero-coupon yield.

This is illustrated in Fig. 6.9.

There are a number of disadvantages with logarithmic interpolation. The calculated interest rate will be higher than with linear interpolation. There will be jumps in the forward curve. The zero-coupon yield/zero rate will become piecewise linear between the interpolated points (see Fig. 6.10). This is however, what we want when we use hazard rates.

6.1.7.3 Polynomial Interpolation

With polynomial fitting the sharp edges are smeared out and we get a smooth/differentiable curve. If we have $n + 1$ given discrete points we can fit the data to a polynomial of degree n through all the points:

$$Y_n(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2 + a_3 \cdot t^3 + \ldots + +a_n \cdot t^n$$

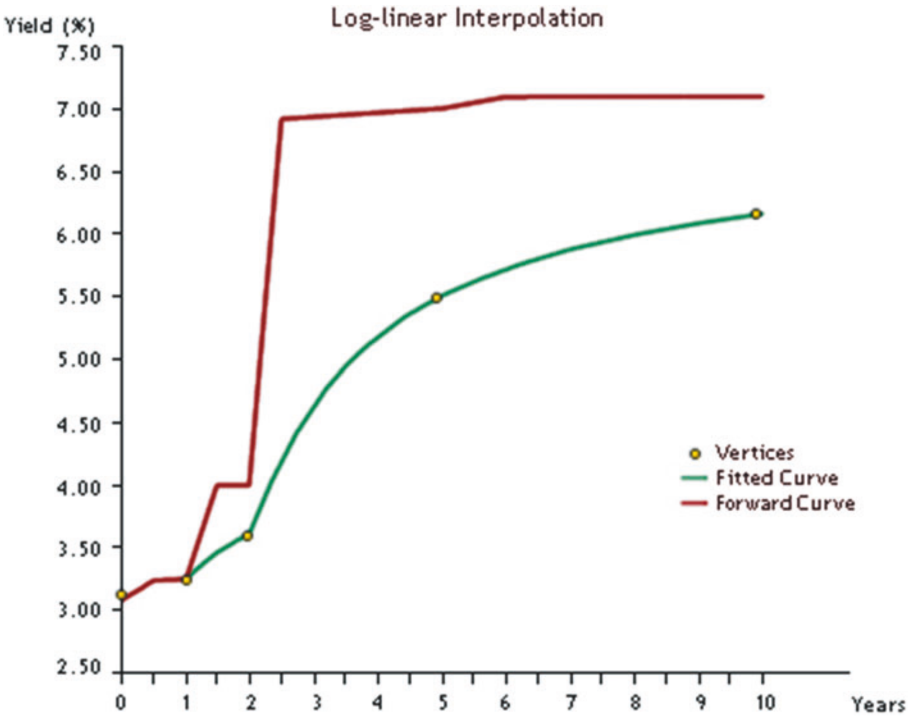

Fig. 6.10 Logarithmic interpolation

using the method of Lagrange

$$\begin{aligned}
Y_n(t) = & \frac{(t-T_1)(t-T_2)(t-T_3)\dots(t-T_n)}{(T_0-T_1)(T_0-T_2)(T_0-T_3)\dots(T_0-T_n)} \cdot Y_0 \\
& + \frac{(t-T_0)(t-T_2)(t-T_3)\dots(t-T_n)}{(T_1-T_0)(T_1-T_2)(T_1-T_3)\dots(T_1-T_n)} \cdot Y_1 \\
& + \frac{(t-T_0)(t-T_1)(t-T_3)\dots(t-T_n)}{(T_2-T_0)(T_2-T_1)(T_2-T_3)\dots(T_2-T_n)} \cdot Y_2 \\
& + \dots \\
& + \frac{(t-T_0)(t-T_1)(t-T_2)\dots(t-T_{n-1})}{(T_n-T_0)(T_n-T_2)(T_n-T_3)\dots(T_n-T_{n-1})} \cdot Y_n
\end{aligned}$$

With numbers from our example, we get

$$\begin{aligned}
Y_3(t) = & \frac{(t-2)(t-4)(t-10)}{(0-2)(0-4)(0-10)} \cdot 4.00 + \frac{(t-0)(t-4)(t-10)}{(2-0)(2-4)(2-10)} \cdot 5.00 \\
& + \frac{(t-0)(t-2)(t-10)}{(4-0)(4-2)(4-10)} \cdot 6.50 + \frac{(t-0)(t-2)(t-4)}{(10-0)(10-2)(10-4)} \cdot 6.75 \\
= & -0.0151 \cdot t^3 + 0.15313 \cdot t^2 + 0.25417 \cdot t + 4.0
\end{aligned}$$

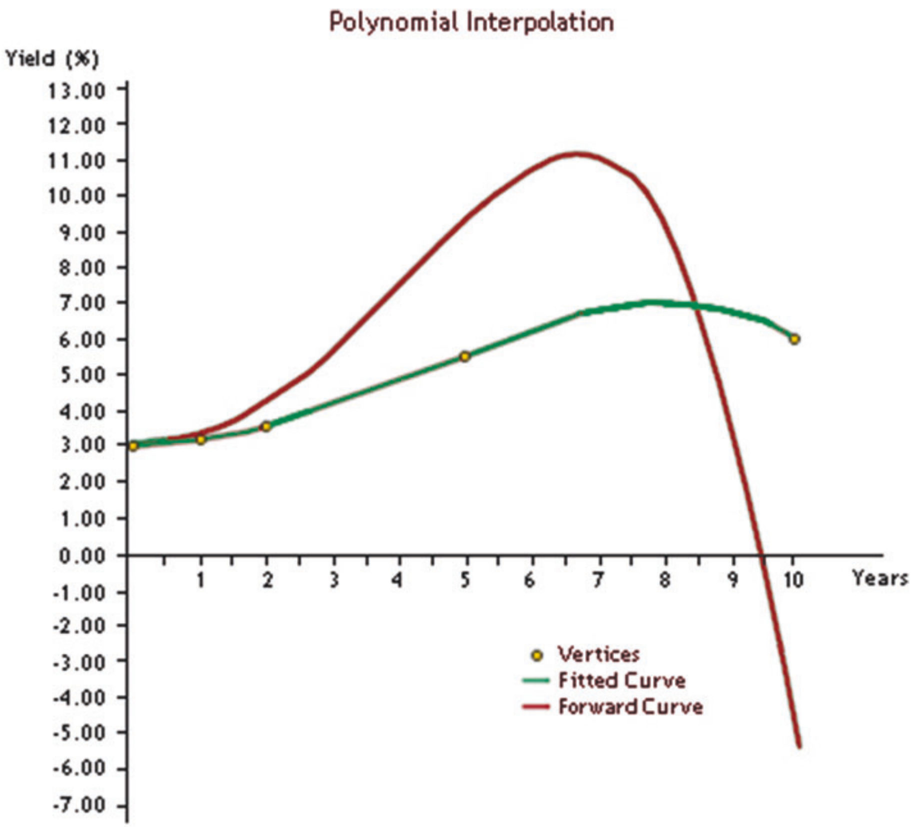

Fig. 6.11 Polynomial interpolation. Here the forward rate might be negative

When $t = 6$ we get $Y_3(6) = 7.775\,\%$.

A disadvantage with this method can be seen in the Fig. 6.11. When a point gets the polynomial slope to change the sign, we can get negative forward rates.

6.1.7.4 Cubic Spline

In this technique, we add certain stiffness to the yield curve. At the same time, the curve will be continuous and differentiable. We fit a third-order polynomial between the points

$$Y_0(t) = a_0 + b_0 \cdot t + c_0 \cdot t^2 + d_0 \cdot t^3 \text{ between } T_0 \text{ and } T_1$$

$$Y_1(t) = a_1 + b_1 \cdot t + c_1 \cdot t^2 + d_1 \cdot t^3 \text{ between } T_1 \text{ and } T_2$$

$$Y_2(t) = a_2 + b_2 \cdot t + c_2 \cdot t^2 + d_2 \cdot t^3 \text{ between } T_2 \text{ and } T_3$$

Each equation has four unknown (the coefficients a – d). With tree equations, we get the following system with 12 unknowns to solve:

$$\begin{bmatrix} 1 & T_0 & T_0^2 & T_0^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & T_1 & T_1^2 & T_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T_1 & T_1^2 & T_1^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T_2 & T_2^2 & T_2^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & T_2 & T_2^2 & T_2^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & T_3 & T_3^2 & T_3^3 \\ 0 & 1 & 2T_1 & 3T_1^2 & 0 & -1 & -2T_1 & -3T_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2T_3 & 3T_3^2 & 0 & -1 & -2T_4 & -3T_4^2 \\ 0 & 0 & 2 & 6T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6T_4 \\ & 0 & 2 & 6T_2 & 0 & 0 & -2 & -6T_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6T_3 & 0 & 0 & -2 & -6T_3 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} 4.00 \\ 5.00 \\ 5.00 \\ 6.50 \\ 6.50 \\ 6.75 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where we have used that, the first and second derivatives in the connections are equal. The second derivatives at T_0 and T_3 are zero. In our example the coefficients is calculated as

$$\{0.022, 0.000, 0.413, 4.000, 0.047, 0.411, -0.410,$$
$$4.548, 0.008, -0.249, 2.230, 1.029\}$$

The 6-year interest rate is then given as

$$Y_2(6) = 0.008 \cdot 6^3 - 0.249 \cdot 6^2 + 2.230 \cdot 6 + 1.029 = 7.173\%$$

The curve is shown in Fig. 6.12.

However, there also exist some disadvantages in this model. When we are studying risk measures by shifting a part of the yield curve, the entire curve will get affected. This effect is small but not desirable.

Bootstrap and Cubic Splines

Bootstrapping can only provide knowledge of the discount factors for (some of) the payment dates of the traded bonds. In many situations, information about market discount factors for other future dates will be valuable. In this section and the next, we will consider methods to estimate the entire discount function $P(t, T)$ (at least up to some large maturity T). To simplify the notation in what follows, let $P(\tau)$ denote the discount factor for the next τ periods (i.e. $P(\tau) = P(t, t+\tau)$). Hence, the function $P(\tau)$ for $\tau \in [0, \infty)$ represents the time t market discount function. In particular, $P(0) = 1$.

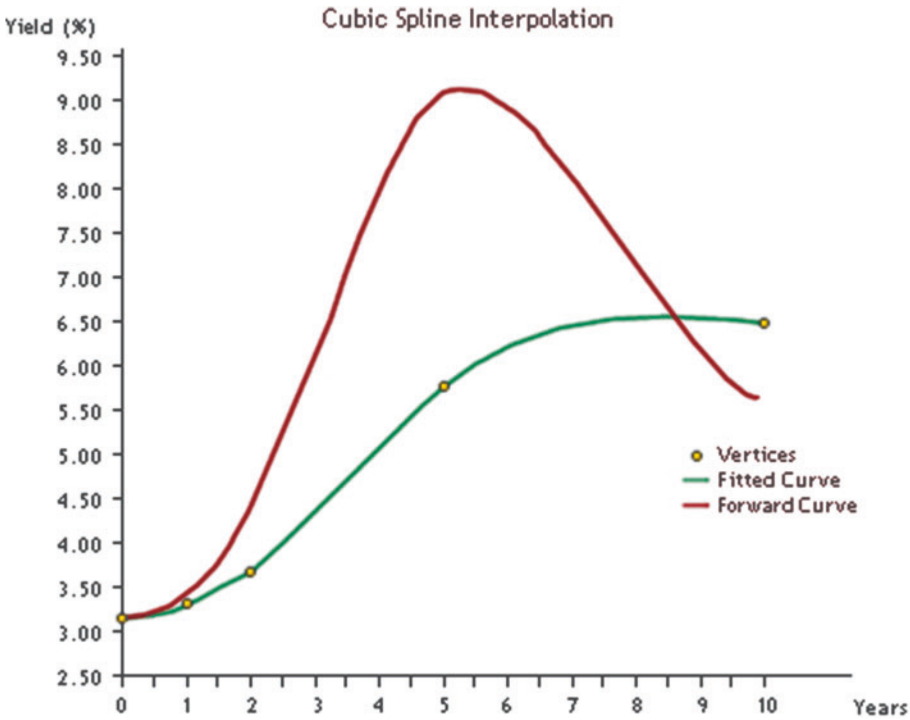

Fig. 6.12 Cubic spline interpolation

We will use a similar notation for zero-coupon rates and forward rates: $y(\tau) = y(t, t + \tau)$ and $f(\tau) = f(t, t + \tau)$. The methods studied in this section are both based on the assumption that the discount function $P(\tau)$ can be described by some functional form involving some unknown parameters. The parameter values are chosen to get a close match between the observed bond prices and the theoretical bond prices computed using the assumed discount function.

The approach studied in this section is a version of the cubic splines approach introduced by McCulloch (1971) and later modified by McCulloch (1975) and Litzenberger and Rolfo (1984).

The word spline indicates that the maturity axis is divided into subintervals and that the separate functions (of the same type) are used to describe the discount function in the different subintervals. The reasoning for doing this is that it can be quite hard to fit a relatively simple functional form to prices of a large number of bonds with very different maturities. To ensure a continuous and smooth term structure of interest rates, one must impose certain conditions for the maturities separating the subintervals.

Given prices for M bonds with time-to-maturities of $T_1 \leq T_2 \leq \cdots \leq T_M$. Divide the maturity axis into subintervals defined by the "knot points" $0 = \tau_0 < \tau_1 < \cdots < \tau_k = T_M$. A spline approximation of the discount function $P(\tau)$ is based on an expression like

$$P(\tau) = \sum_{j=0}^{k-1} G_j(\tau) I_j(\tau),$$

where the G_j's are basis functions, and the I_j's are the step functions

$$I_j(\tau) = \begin{cases} 1, & \text{if} \quad \tau \geqslant \tau_j, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $P(\tau) = G_0(\tau)$ for $\tau \in [\tau_0, \tau_1)$, $P(\tau) = G_0(\tau) + G_1(\tau)$ for $\tau \in [\tau_1, \tau_2)$, etc. We demand that the G_j's are continuous and differentiable and ensure a smooth transition in the knot point's τ_j. A polynomial spline is a spline where the basis functions are polynomials. Let us consider a cubic spline, where

$$G_j(\tau) = \alpha_j + \beta_j(\tau - \tau_j) + \gamma_j(\tau - \tau_j)^2 + \delta_j(\tau - \tau_j)^3,$$

and α_j, β_j, γ_j, and δ_j are constants. For $\tau \in [0, \tau_1)$, we have

$$P(\tau) = \alpha_0 + \beta_0 \tau + \gamma_0 \tau^2 + \delta_0 \tau^3.$$

Since $P(0) = 1$, we must have $\alpha_0 = 1$. For $\tau \in [\tau_1, \tau_2)$, we have

$$\begin{aligned} P(\tau) = {} & \left(\alpha_0 + \beta_0 \tau + \gamma_0 \tau^2 + \delta_0 \tau^3\right) \\ & + \left(\alpha_1 + \beta_1(\tau - \tau_1) + \gamma_1(\tau - \tau_1)^2 + \delta_1(\tau - \tau_1)^3\right) \end{aligned}$$

To get a smooth transition between in the point $\tau = \tau_1$, we demand that

$$\begin{cases} P(\tau_1 + \delta) = P(\tau_1 - \delta), \\ P'(\tau_1 + \delta) = P'(\tau_1 - \delta), \\ P''(\tau_1 + \delta) = P''(\tau_1 - \delta) \end{cases}$$

The conditions ensure that the discount function is continuous and twice differentiable. The first condition implies $\alpha_1 = 0$. Differentiating the two connected polynomials, we find

$$P'(\tau) = \beta_0 + 2\gamma_0 \tau + 3\delta_0 \tau^2 \; 0 \leq \tau < \tau_1$$

and

$$P'(\tau) = \beta_0 + 2\gamma_0 \tau + 3\delta_0 \tau^2 + \beta_1 + 2\gamma_1(\tau - \tau_1) + 3\delta_1(\tau - \tau_1)^2 \tau_1 \leq \tau < \tau_2$$

The second condition now implies $\beta_1 = 0$. Differentiating again, we get

$$P''(\tau) = 2\gamma_0 + 6\delta_0\tau \qquad 0 \le \tau < \tau_1$$

and

$$P''(\tau) = 2\gamma_0 + 6\delta_0\tau + 2\gamma_1 + 6\delta_1(\tau - \tau_1)\tau_1 \le \tau < \tau_2$$

Consequently, the third condition implies $\gamma_1 = 0$. Similarly, it can be shown that $\alpha_j = \beta_j = \gamma_j = 0$ for all $j = 1, \ldots, k-1$. The cubic spline is therefore reduced to

$$P(\tau) = 1 + \beta_0\tau + \gamma_0\tau^2 + \delta_0\tau^3 + \sum_{j=1}^{k-1} \delta_j(\tau - \tau_j)^3 I_j(\tau)$$

Let $t_1, t_2, \ldots, t_N$ denote the time distance from today (date t) to the each of the payment dates in the set of all payment dates of the bonds in the data set. Let Y_{in} denote the payment of bond i in t_n periods. From the no-arbitrage pricing relation (the present value of the coupon paying bond is equal to the sum of the present value of each cash flow), we should have that

$$PV_i = \sum_{n=1}^{N} Y_{in}P(t_n)$$

where PV_i is the current market price of bond i. Since not all the zero-coupon bonds involved in this equation are traded, we will allow for a deviation ε_i so that

$$PV_i = \sum_{n=1}^{N} Y_{in}P(t_n) + \varepsilon_i$$

We assume that ε_i is normally distributed with mean zero and variance σ^2 (assumed to be the same for all bonds) and that the deviations for different bonds are mutually independent. We want to pick parameter values that minimize the sum of squared deviations

$$\sum_{i=1}^{M} \varepsilon_i$$

Substituting the polynomial expression with these yields

$$PVi = \sum_{n=1}^{N} Y_{in} \left\{ 1 + \beta_0 t_n + \gamma_0 t_n^2 + \delta_0 t_n^3 + \sum_{j=1}^{k-1} \delta_j (t_n - \tau_j)^3 I_j(t_n) \right\} + \varepsilon_i$$

which implies that

$$PVi - \sum_{n=1}^{N} Y_{in} = \beta_0 \sum_{n=1}^{N} Y_{in} t_n + \gamma_0 \sum_{n=1}^{N} Y_{in} t_n^2 + \delta_0 \sum_{n=1}^{N} Y_{in} t_n^3 \\ + \sum_{j=1}^{k-1} \delta_j \sum_{n=1}^{N} Y_{in} (t_n - \tau_j)^3 I_j(t_n) + \varepsilon_i$$

Given the prices and payment schemes of the M bonds, the $k + 2$ parameters $\beta_0, \gamma_0, \delta_0, \delta_1, \ldots, \delta_{k-1}$ can now be estimated using ordinary least squares. Substituting the estimated parameters, we get an estimated discount function; from which estimated zero-coupon yield curves and forward rate curves can be derived as explained earlier.

It remains to describe how the number of subintervals k and the knot point's τ_j are to be chosen. McCulloch suggested to let k be the nearest integer to $\sqrt{M}$ and to define the knot points by

$$\tau_j = T_{h_j} + \theta_j \left(T_{h_j+1} - T_{h_j} \right)$$

where $h_j = [j \cdot M/k]$ (here the square brackets mean the integer part) and $\theta_j = j \cdot M/k - h_j$. In particular, $\tau_k = T_M$. Alternatively, the knot points can be placed at for example 1 year, 5 years and 10 years, so that the intervals broadly correspond to the short-term, intermediate-term and long-term segments of the market.

The Fig. 6.13 shows the discount function on the Swedish government bond markets on April 24, 2006 (the same data as in the previous bootstrap) estimated using cubic splines and data with maturities up to 15 years.

The nodes and values are given in Table 6.4.(Fig. 6.14)

Discount functions estimated using cubic splines would usually have a credible form for maturities less than the longest maturity in the data set. Although there is nothing in the approach that ensures that the resulting discount function is positive and decreasing, as it should be, this will usually be the case. As the maturity approaches infinity, the cubic spline discount function will approach either plus or minus

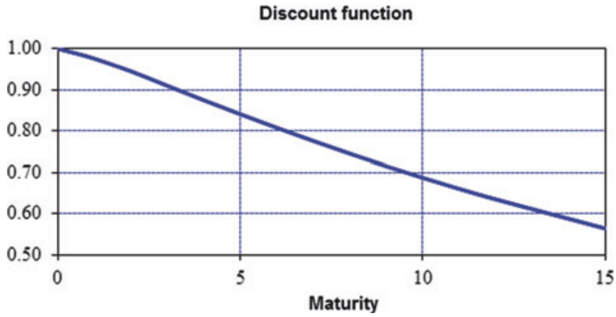

Fig. 6.13 The discount function

Table 6.4 Parameters of fitting the discount function

Parameter	Value	Time
β	−2,27E-02	
γ	−2,64E-04	
δ_0	−1,30E-03	
δ_1	1,84E-03	1,00
δ_2	5,21E-04	2,00
δ_3	−1,08E-03	3,00
δ_4	4,09E-05	5,00
δ_5	−5,17E-05	10,00
δ_6	5,94E-05	14,00

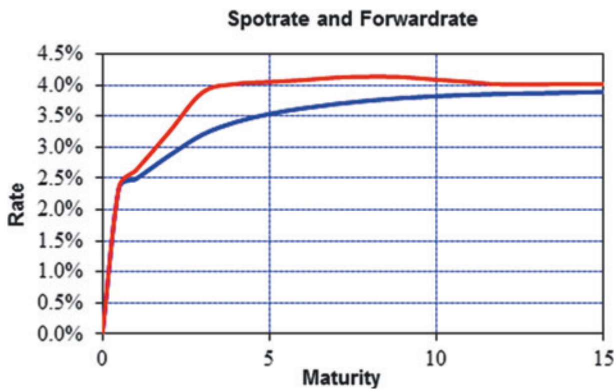

Fig. 6.14 Spot and orward rate with cubic spline

infinity depending on the sign of the coefficient of the third-order term. Of course, both properties are unacceptable, and the method cannot be expected to provide reasonable values beyond the longest maturity T_M, since none of the bonds are affected by that very long end of the term structure.

Two other properties of the cubic splines approach are more disturbing. First, the derived zero-coupon rates will often increase or decrease significantly for maturities approaching T_M,. Second, the derived forward rate curve will typically be quite rugged especially near the knot points, and the curve tends to be very sensitive to the bond prices and the precise location of the knot points. Therefore, forward rate curves estimated using cubic splines should only be applied with great caution.

6.1.7.5 Hermite Interpolation

With this technique, sometimes called **clamped cubic spline**, this effect is eliminated. When a shift is made, only the closest part of the curve will affect. Let r be a vector $Y' = \{y_1, y_2, \dots y_n\}$ and

$$Y(t) = Y_i + m_i(t) \cdot (Y_{i+1} - Y_i) + m_i(t) \cdot (1 - m_i(t)) \cdot g_i + m_i^2(t) \cdot (1 - m_i(t)) \cdot c_i$$

where

$$\begin{aligned} m_i(t) &= \frac{t - T_i}{T_{i+1} - T_i} \\ g_i &= (T_{i+1} - T_i) \cdot y_i - (Y_{i+1} - Y_i) \\ c_i &= 2\,(Y_{i+1} - Y_i) - (T_{i+1} - T_i) \cdot (y_{i+1} + y_i) \end{aligned}$$

The vector Y' is calculated as

$$y_i = \frac{1}{T_{i+1} - T_{i-1}} \left[\frac{(Y_i - Y_{i-1}) \cdot (T_{i+1} - T_i)}{T_i - T_{i-1}} + \frac{(Y_{i+1} - Y_i) \cdot (T_i - T_{i-1})}{T_{i+1} - T_i} \right]$$

With the boundary

$$y_1 = \frac{1}{T_3 - T_1} \left[\frac{(Y_2 - Y_1) \cdot (T_3 + T_2 - 2 \cdot T_1)}{T_2 - T_1} + \frac{(Y_3 - Y_2) \cdot (T_2 - T_1)}{T_3 - T_2} \right]$$

$$y_n = \frac{1}{T_{n-1} - T_{n-2}} \times \left[\frac{(Y_{n-1} - Y_{n-2}) \cdot (T_n - T_{n-1})}{T_{n-1} - T_{n-2}} + \frac{(Y_n - Y_{n-1}) \cdot (2 \cdot T_n - T_{n-1} - T_{n-2})}{T_n - T_{n-1}} \right]$$

A variant of the Hermite interpolation is a method called **Hermite RT**. In this method, we interpolate with Hermite as shown previously but via the logarithm of the discount rates given as

$$D(t) = e^{-r \cdot t}$$

where r is the continuous interest rate at time t. $\ln[D(t)] = -rt$ which has named the method. For the spot rate, we then have

$$(1 + Y)^t = e^{-r \cdot t}$$

This curve is nicer than the ordinary Hermite curve. Especially when the liquidity is low and the number of instruments are limited. In many advanced risk software, there is a possibility to combine the earlier methods with different magnitudes.

6.1.8 Spread and Spread Curves

Spreads are used to value securities with certain properties, such as

- The credit ranking of the issuer
- Liquidity
- Ranking due to a default
- Embedded options

A spread is defined as a number of bps above an underlying yield curve, the base curve. The value of the instruments decreases due to the spread.

Example 6.1.8.1
A spread above the spot rate gives a discount rate as:

$$r_{t_1}^{discount} = \frac{1}{\left(1 + r_{t_1}^{spot} + s\right)^{t_1}}$$

7

The Interbank Market

7.1 Spreads and the Interbank Market

We will now take a look at the **Interbank market** and different kind of spreads. We explain some of the details using the Swedish market (as Riksbanken, the Central bank in Sweden[1]).

Banks can borrow under the marginal lending facility (Swedish: utlåningsfaciliteten) (if they made adequate security) in the National Bank at an interest rate, lending rate, which is a bit above (typically 0.75 %) the **repo rate**.[2] Banks with a surplus can use the National Bank deposit facility (Swedish: inlåningsfaciliteten) that provides a deposit rate a bit (typically 0.75 %) lower than the repo rate.

Since there is a quite large gap between deposit rate and lending rate, this gives a strong incitement for banks, to instead, settle directly with each other to get a better interest rate. This rate is called the **overnight rate** (O/N). The central bank tries to control this rate, via the repo rate. With this rate, the central bank signals where they want the O/N will be a week ahead.

If the banking system as a whole have a deficit or a surplus, the central bank implements a reverse every week. Imbalances may still occur day by day. To create balance and gain greater control over the O/N, the central bank also try on a daily basis, get the banks' total deficit is the same as the total surplus. This is done by fine-tuning operations, which lend money at the repo rate +0.10 % and lend at the repo rate –0.10 % to create a balance.

[1] The Swedish central bank is the first central bank in the world.
[2] The Swedish Repo Rate is the reference (policy) rate decided by the Central bank in Sweden.

J.R.M. Röman, *Analytical Finance: Volume II*,
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Overall, the central bank therefore ensures that balance exists in the single payment system (in Sweden called RIX). Surplus and deficit held by individual banks, can however, been managed by the banks themselves. At the end of each day, banks that need to borrow use a bank with a surplus.

TheO/N for today runs from today until tomorrow. **Tomorrow next** (tom next or T/N) runs from tomorrow until the next day. Next maturity of fixed income market is called the S/N (**spot/next**) which runs from the day after tomorrow and one business ahead, 1W runs from the day after tomorrow for a week. All days above are bank days.

7.1.1 TED-Spread and Other Spreads

XIBOR (the general Interbank Offer Rate) is the rate that banks can borrow from each other's. To assess how the market views the risk of lending to another bank, we put the XIBOR rate in relation to any other interest. The safest player on the market is the government because they can always pay debts by printing new money (debt monetization). Therefore, we compare the XIBOR rates by the interest rate on government securities with the same maturity, to see which risk premium imposed on bank loans. The difference in yield between 2 securities with similar characteristics is called a spread. The difference between a 3-month interbank rate and the rate on 3-month government securities is known as the **Treasury-Euro-Dollar (TED) spread**. TED denotes the spread between the Treasury bill yield and the Libor rate for the same maturity (usually 3 months). In Fig. 7.1 we show the TED spread in USD[3] from the beginning of 2007 to the end of 2009. We observe the very high spread during the period of the financial crisis.

7.1.2 Overnight Indexed Swaps (OIS) and Basis Spread

A 12-month XIBOR rate reflects not only the expectations of the O/N that will prevail in the average for the next 12 months, but also a risk premium, which raises the rate of long-term loans. Therefore, it is easy to understand that the market is interested in interest-rate instruments

[3] Source, FRED, https://fred.stlouisfed.org/ Federal Reserve Economic Data – St. Louis Fed

Fig. 7.1 The USD TED-spread during the financial crises.

for a period longer than O/N, but that keeps the same rate as the average for the O/N during the period. Such instruments are called as the overnight indexed swaps (OIS).

With an OIS, 2 parties may agree that *party 1* receives a fixed rate from *party 2* (the fixed leg of the swap) and *party 2*, a rate equal to the average of the O/N from *party 1* (the floating leg of the swap).

In several areas, currency swaps are based on the central bank policy rate. In Sweden, this rate is the repo rate, but unfortunately, Sweden does not have an OIS based on the repo rate. The closest we have is the STIBOR T/N *Average* (STINA) swaps. STINA is the average rate for the minimum rate on the STIBOR market. A STINA swap gives the holder of the floating leg the average rate of STIBOR T/N over the period of the swap.

Example 7.1.2.1

Given a 4-year to maturity bond with a principal 1000 and an annual coupon *Party 1* signs a 3-month STINA swap with *party 2* and receives a 4.6 % rate from *party 2*. *Party 2* receives STIBOR T/N from *party 1*. No payments are made during the term, but instead, after3 months, the average interest rate for STIBOR T/N, is calculated and the difference against the swap are paid to the party who should have paid the lowest rate over the period. In that way, the contract is pretty riskless, since the maximum loss is the profit from the swap itself.

STIBOR T/N is closer related to the repo rate than the government securities. Therefore the STINA swaps better reflects the expectations of the repo rate than, for example, a Swedish 3-month Treasury bill. It is therefore interesting to complement the comparison between

STIBOR-interest and the interest rate on government securities (TED spread, and so on), with a spread calculation that sets STIBOR rate in relation to the STINA rate. This spread is "cleaner" in the sense of risk premium than the expected average of the repo rate. This spread is sometimes called **Basis spread**. The spread between STINA swap and government bond rate is called the **swap spreads**.

Unfortunately, this Basis spread is not "clean" in the sense that it would consist only of interbank risk premium. Remember that we have seen previously, that there is a risk premium built in STIBOR T/N. Basis spread contains thus the risk premium in the interbank market for the period you study, minus the expectations on the risk premium in STIBOR T/N compared to the repo rate. As long as we in Sweden have no repo-rate swap, we will get to live with this problem. Let us look at the numbers and calculate the Basis spread and TED spread at 2007-12-28. This is shown in Table 7.1.

In Fig.7.2 and Fig.7.3 we show the market interest rates and the spreads in SEK 2007-12-28.

We see in Fig.7.3 that the Basis spread indicates a great concern that the market does not believe the financial turmoil is heading off in the near future. The risk premium on interbank loans against STIBOR T/N market is much higher for longer maturities. This is what we saw from the spread between STIBOR and government securities (the TED spread and the TED spread curve).

In general, basis swap spreads reflect the underlying funding needs of the general banking community. Thus for basis swaps within a single currency the spread reflects the need for banks to preserve their liquidity (i.e. funding for long periods). This results in the fact that 6-month money is generally more expensive than rolled up 3-month money. This spread between swaps with different tenors is called a **Tenor spread**.

Table 7.1 Market rates and their spreads in SEK 2007-12-28

Term	STINA	STIBOR	Treasury-bill	Basis-spread	TED-spread (SEK)	Swap-spread
1M	4.64	4.7	4.22	0.06	0.48	0.42
2M	4.65	4.808		0.16		
3M	4.73	5.103	4.345	0.37	0.758	0.39
6M	4.85	5.42	4.406	057	1.014	0.44
9M	4.885	5.603		0.72		
12M	4.92	5.725	4.561	0.81	1.164	0.36

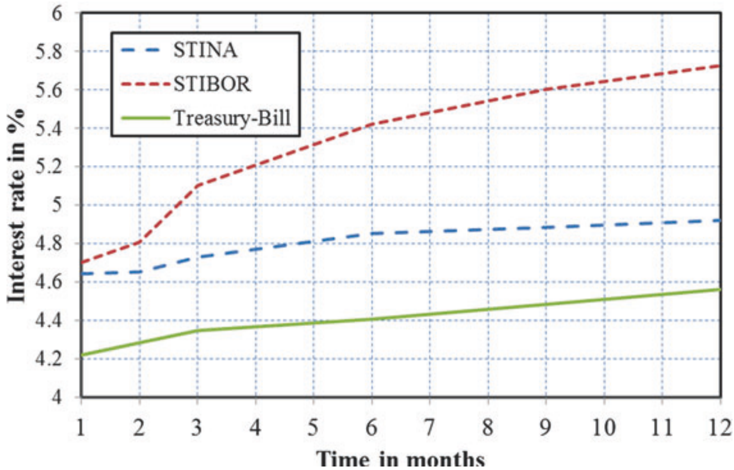

Fig. 7.2 The market rates in SEK 2007-12-28

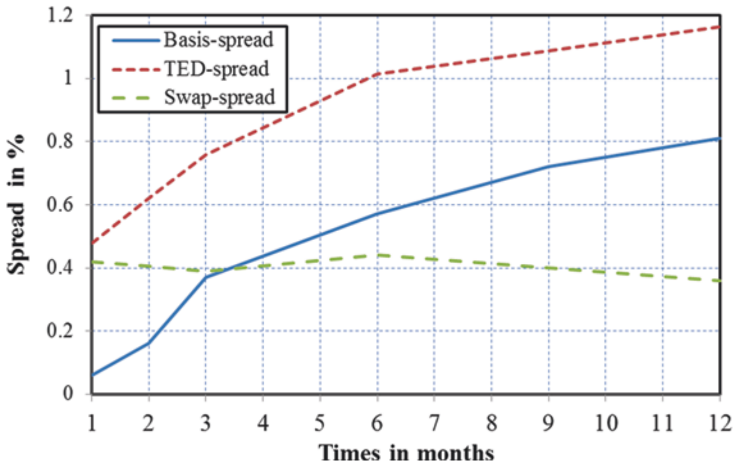

Fig. 7.3 The spreads in Swedish maket rates 2007-12-28

For **cross currency basis spreads** it's similar. There is more demand for funding in one currency and more supply in another currency. For instance many Japanese banks have funding sources in Japanese Yen (JPY) but have commitments in USD. They therefore will swap their JPY for USD. The basis swap spread reflects this supply and demand situation. The same effect is seen in the FX swap market which is the other means of exchanging the funds.

7.1.3 Some Overnight Indices

EONIA (**Euro Overnight Index Average**) is an effective O/N interest rate calculated by the European Central Bank as a *weighted average of all overnight unsecured lending transactions in the interbank market*. It has been initiated within the euro area by the contributing Panel Banks. It is one of the 2 benchmarks for the money and capital markets in the euro zone (the other one being Euribor). The banks contributing to Eonia are the same as the Panel Banks quoting for Euribor. In Fig. 7.4 we show the Eonia rates between 1999-01-04 to 2016-08-12.

SONIA is the acronym for **Sterling Overnight Index Average**. It is the reference rate for O/N unsecured transactions in the Sterling market. Each London business day the Sonia fixing is calculated as the weighted average rate of all unsecured O/N sterling transactions brokered in London by WMBA members. The rate conventions are annualised rate, act/360, and 4 decimal places.

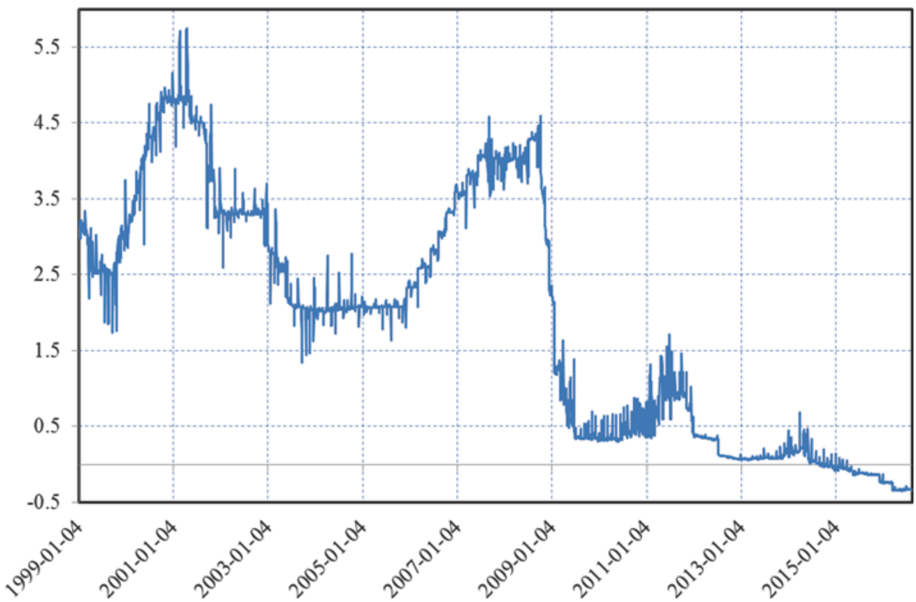

Fig. 7.4 The Eonia (EUR OIS) between 1999 and mid Augist 2016[4]

[4] Source, FRED, https://fred.stlouisfed.org/ Federal Reserve Economic Data - St. Louis Fed

CHOIS (based on **SARON, Swiss Average Rate Overnight)** is an O/N interest rates average referencing the Swiss Franc interbank repo market. It was launched by the Swiss National Bank (SNB) in co-operation with 6 Swiss Exchange. Since August 25 2009, SARON has replaced the previously used repo O/N index. The reference rate is based on CHF repo interbank market data provided by Eurex Zurich Ltd.
TONAR is the acronym for **Tokyo Overnight Index Average**. It is the reference rate for O/N unsecured transactions in the Japanese Yen.

7.1.4 Basis Swaps

Strictly speaking, a basis swap or a floating/floating cross currency basis swap, is a swap in which 2 streams of money market floating rates of 2 different currencies are exchanged.
In contrast to a standard interest rate swap fixed for floating, notional are exchanged at the starting of the swap and exchanged back at termination. Typical example of a basis swap is swapping dollar Libor versus Yen Libor.

By extension, basis swap refers to floating/floating (cross currency or not) swap in which 2 streams of floating rates are exchanged, regardless if these floating rates are in the same currency. Typical example of basis swap in the same currency are swapping dollar Libor for floating commercial paper, Prime Treasure bills or Constant Maturity Treasury rates or even 90 days Dollar Libor for 180 days Dollar Libor. In the case of a swap in the same currency, notional do not change hands as there is no currency exposure.

As far as the cross currency swap market is concerned, basis swap enables traders and investor to swap their interest rate risk exposure in another currency. Basis swap market reflects the global demand for swapping from one currency into another as well as the credit quality of the central bank. This is a huge market with billions of notional transaction every day. One of the most active markets is the Yen–Dollar market.

When an investor wants to swap his currency exposure into another one, he may go to the forward foreign exchange markets. However, this market is only liquid up to 2 to 3 years, after which the basis swap market is taking over. Basis swap market is an important component to build a cross currency swap market used for cross currency swap pricing as well as other cross currency type transaction.

Basis swap should not be confused with:

- General cross currency swaps: the intersection between basis swap and cross currency swap lies in the floating for floating cross currency swap. However, a basis swap is not necessarily based on 2 currencies, while a cross currency swap is not necessarily floating for floating but can be fixed for floating, floating for fixed or fixed for fixed.
- Quanto- or differential-swap, which implies to pay in 2 currencies but with the same notional and no exchange of notional.

Like any standard swap, a basis swap can have tailor made notional such as amortising, accreting, or roller coaster notional. A rollercoaster swap is a swap with a notional principal that differs during various payment periods. In other words, it is a swap agreement in which counterparties agree to flexibility of payments.

7.1.4.1 Pricing Methodology

The basis is more pronounced on the USD/JPY market, hence we will examine the case of the 10 year basis swap paying US Dollar 3-month Libor Flat versus receiving JPY 3-month Libor plus a spread. The market quotes this spreads as being 15 basis points (bps) running. This means that to enter into a swap where one would pay US Dollar 3-month Flat Libor, one would require receiving JPY 3-month Libor plus 15 bps.

At first sight, this may look strange to someone accustomed to plain vanilla interest rates as she has been always taught that a floating leg should always be at par. Hence the 2 legs, USD 3-month Libor and JPY 3-month Libor should be equal. However, one has to bear in mind that interest rate swap Libor are approximate averages of offer rates from different banking institution. Libor rates bare credit and liquidity risk. Hence a USD Libor rate may have a better credit and liquidity quality as the JPY Libor fixing, hence the spread required by USD investors to receive a worst currency. In addition, the basis swap market is very much driven by supply and demand for issuance. A spread of 15 bps means that there is little demand to receive JPY Libor, hence one has to pay a premium to convince investors to swap. To build a consistent methodology for pricing, one can take 2 approaches

- **Single interest rate curve method**: Build an interest curve that uses all the constraint of the forward foreign exchange market and the basis swap market to price consistently basis swaps. Although simple, this method has the disadvantage to oblige the trader to change curve when pricing a JPY leg in the JPY market as opposed to pricing a JPY versus USD swap. Interestingly, one can look at the discount factor difference between the JPY normal bootstrapped interest rate curve and the basis swap interest rate curve to quantify the basis swap market effect.
- **JPY Libor curve and spread curve used to account for the basis swap market**: One has first to create the normal interest rate swap curve by bootstrapping the domestic market. Then using this curve, one can bootstrap another curve called the basis swap spread curve that adjusts for credit quality to get the JPY basis swap leg (JPY plus 15 bps) to be at par. Basically this spread curve says that a 3-month JPY Libor leg is not at par and one need 15 bps to bring it at par. When pricing in the JPY domestic market, one only uses the standard interest swap curve. In this market, a leg paying JPY 3-month Libor is at par. When pricing a cross currency swap JPY versus USD, one has to apply to the JPY curve the basis swap spread curve to price correctly the JPY leg. More generally, one can build an interest rate infrastructure that uses standard Libor curve plus a funding curve or spread curve to account for various market effects like credit and counterparty risk, basis swap market, CMS and CMT adjustment and etc.

The existence of this basis swap curve implies the same swap could be at par for a JPY investor while not at par for a USD investor as they have a different view on credit quality of the Bank of Japan.

8

Measuring the Risk

8.1 Risk Measures

In this section we present some traditional risk measures based on the present value formula used in the markets for the quoting of prices and yields to maturity (*ytms*). These measures are calculated by trading software in order to at least partially manage the risk in instruments and portfolios.

8.1.1 Delta

The delta value of an instrument shows the sensitivity of the price[1] to changes in the main source of risk of an underlying instrument. Examples of sources of risk are yield curves changes and the price of underlying asset and the delta is calculated separately for these.

8.1.1.1 Delta Price

The price delta calculations are only applicable for derivative instruments with an underlying instrument (that have a price[2]), valued on the basis of a non-term structure model. It shows the change in theoretical price given a unit change in the price of the underlying.

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[1] With price, we here refer to the present value sometimes called the fair value of a financial instrument.
[2] Interest rate is not a tradeble instrument. But a bond option have an underlying instrument, the bond.

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_8

The general *Delta Price* formula is

$$\Delta_{price} = \frac{\partial (PV)}{\partial U} = \frac{PV(U+h) - PV(U)}{h} \times scale$$

where U is the current value of the main source of risk and h is the differentiation step. When the market price of the underlying is used, the price shift is a relative shift, that is,

$$h = 0.0001 \times U$$

When a theoretical underlying price is used, the price shift is an absolute shift, that is,

$$h = 0.01$$

Sometime this value is called **delta explicit**.

Example 8.1.1.1

We want to calculate the *Delta Price* for a bond using

a) the market price of the underlying, and
b) the theoretical price of the underlying.

First, assume that the current market price of the underlying asset is $U = 100.17$. The present value will now be calculated twice, the first time using the current price of the underlying and the second time after applying a shift to the price of the underlying with the shift size expressed as $h = U\text{x } 0.0001 = 0.010017$.

We obtain

$$\Delta_{price} = \frac{PV(U+h) - PV(U)}{h} \cdot \frac{100}{Nom}$$

where *Nom* is the nominal amount, typically one million.

Next, assume that the theoretical price of the underlying asset is $U = 100.15$. The shift size is now $h = 0.01$ and $U + h = 100.16$. The present value is calculated twice, using these two different prices for the underlying, which gives the result with the previous formula.

8.1.1.2 Delta Yield

The yield curve delta shows the change in the present value, given a shift of 1 basis point (bp) in all yield curves used. The shift is applied to the annually compounded zero coupon curve, using the day count fraction Act/365.

The yield delta can either refer to an upward or a downward shift of yields. The general **Delta Yield** formula is

$$\Delta_{yield} = \left[PV\left(\underline{r} + h\right) - PV\left(\underline{r}\right)\right] * scale$$

where

$$h = \pm 0.00001 \quad \text{and} \quad scale = 1000.$$

Sometimes, but not always, the shift step used in the calculations is thus actually 1/1000 of a bp to get high accuracy of the slope, but the result is scaled to a 1 bp shift. The delta can also be broken down according to different time buckets, to illustrate the sensitivity to a particular shift in a given time bucket. These time buckets can be defined in the software used to calculate the risk. This can be 1, 2, 3, 7 days followed by 2 and 4 weeks, then 3, 6, 9 and 12 months and 2, 3, 4, 5, 7, 10, 12, 15, 20, 25 and 30 years. In such a way that the time buckets are well defined between the given terms.

Example 8.1.1.2

We will calculate the *Delta Yield* of a bond using the theoretical price of the underlying interest rate, the yield $\underline{y}$.

The calculations are based on the present value using the current yield curves and the present value using yield curves that are shifted with a shift size of 1×10^{-5}. The result is then scaled so that the shift represents a size of one bp. We have

$$\Delta_{yield} = \left[PV(\underline{y} + 0.00001) - PV(\underline{y})\right] \cdot 1000$$

8.1.2 Duration and Convexity

The Macaulay duration (or just duration) is a measure of the price sensitivity of an interest rate instrument with the respect to an absolute change in the *ytm*. This measure can be interpreted as the average life of the bond, when a bond is the financial instrument. It is easy to show that the duration for a zero-coupon bond is the same as its time to maturity.

The *modified duration* measures the percentage bond price change for an absolute yield change. It can also be interpreted as the negative slope of the price-yield relation. In a similar way convexity can be interpreted as the curvature of the relation between the price and *ytm*.

We use the following 4 risk measures

- Macaulay's duration
- Modified duration
- Dollar duration
- Convexity

They will depend on

- Time to maturity
- The coupon rate
- The coupon frequency
- The market rate

For a bond, the duration measure of the weighted average of the times until the fixed cash flows are received and is given in year. Therefore, the duration of a zero-coupon bond is the same as its time to maturity. A coupon-paying bond has duration less than its time to maturity because part of the cash flows, the coupons, are paid before maturity. This will be illustrated later.

Suppose we have n cash flows $c_i, i = 1, 2, \ldots, n$ at times t_i. Then, the quoted bond price, P is given by (using continuous compounding):

$$P = \sum_{i=1}^{n} c_i e^{-yt_i}$$

where y is the *ytm*. The duration is defined as:

$$D = \frac{1}{P} \sum_{i=1}^{n} t_i c_i e^{-yt_i} = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-yt_i}}{P} \right]$$

where the factor in [.] is the present value of each cash flow using the continuously compounded *ytm* for discounting of the cash flows. This can also be expressed as

$$D = -\frac{1+y}{P} \frac{\partial P}{\partial y} = \frac{1}{P} \left[\sum_{i=1}^{n} \frac{t_i \cdot C_i}{(1+y)^{t_i}} + \frac{t_m \cdot N}{(1+y)^{t_m}} \right].$$

where for simplicity we assume there is 1 coupon per year and

P = the present value (which is the quoted price, if this exist),
y = the bond *ytm*,
C = coupon size (the coupon rate times the nominal amount, N)
N = the nominal amount (or the principal)
n = number of years to maturity (if we have 1 coupon per year).

P is given by

$$P = \frac{N}{(1+y)^n} + \sum_{i=1}^{n} \frac{C}{(1+y)^i}$$

The factor $(1 + y)$ comes from the fact that duration is defined as the derivative with respect to the *ytm* in the market quoting formula. For continuously compounded *ytm* we get

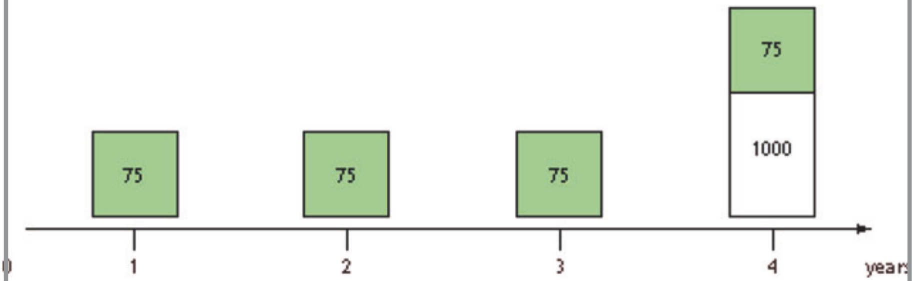
$$D = -\frac{1}{P}\frac{\partial P}{\partial y} = -\frac{1}{P}\frac{\partial P}{\partial y}\left[\sum_{i=1}^{n} C_i \cdot e^{-y \cdot t_i} + N \cdot e^{-y \cdot t_m}\right]$$

$$= \frac{1}{P}\left[\sum_{i=1}^{n} C_i \cdot t_i \cdot e^{-y \cdot t_i} + N \cdot t_m \cdot e^{-y \cdot t_m}\right].$$

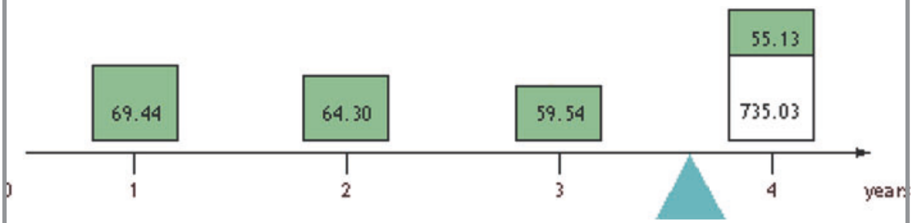
If the coupon frequency is m times per year, the formulas has to be slightly modified.

Example 8.1.2.3

Given a 4-year maturity bond with a principal 1000 and an annual coupon rate of 7.5%. This bond will have the following projected[3] cash flows.



Suppose we have a constant (flat) market rate of 8%. Then the present value of the cash flows will be



We then get duration of 3.6 year.

8.1.2.1 Swap Duration

Duration, as we have seen for aforementioned bonds, can also be defined for other kinds of interest rate instruments. Portfolio managers like to find the duration for their entire portfolio. Therefore we also

[3] Projected cashflows are the coupons payes or received in the future given by the coupon rate.

need also to define a duration for other instruments. One problem is that no *ytm* is defined for other instruments.

Some managers use an approximation for swaps by calculating the duration of the fixed leg as 0.75 times the time to maturity. Similarly, one may calculate the duration of the floating leg as 0.5 times the tenor. This means that for a floating leg with 3-month tenor, the duration should be $0.25 * 0.5 = 0.125$ years, for a 6-month tenor $0.5 * 0.5 = 0.25$ and for a 1-year tenor it will be 0.5.

Example 8.1.2.4

A 10-year receiver swap with a 3-month tenor will have a duration of $7.5 - 0.125 = 7.375$ year. Similarly, the payer swap has -7.375 year. Portfolio managers used payer swaptions to hedge duration from bonds.

A better calculation of the Swap duration would be to use the interest rate sensitivity and use the following formula

$$Dur_{swap} = \frac{MV_0 - MV_1}{N + MV_0} \times 10000$$

Here MV_0 is the market value of the swap and MV_1 the market value we get if we shift the market swap curve 1 bp (up) and N is the nominal amount.

Example 8.1.2.5

In the following table, we show how the duration varies for a semi-annual coupon-paying bond when the *ytm* is 5% and the coupon rate is 1, 2, 5 and 10%, respectively.

	Coupon Rate			
Years to maturity	1%	2%	5%	10%
1	0.997	0,995	0.988	0.977
2	1.984	1.969	1.928	1.868
5	4.875	4.763	4.485	4.156
10	9.416	8.950	7.989	7.107
25	20.164	17.715	14.536	12.754
50	26,666	22.284	18.765	17.384
100	22.572	21.200	20.363	20.067
Infinity	20.500	20.500	20.500	20.500

When time to maturity increases to the limit, we find the value

$$D \xrightarrow[T \to \infty]{} \frac{1 + y/f}{y}$$

Where y is the *ytm* per annum.

Example 8.1.2.6

A 3-year bond with principal 1000 paying an annual coupon of 10%. If the market price of this bond is 951.97 with a yield of 12%, the duration is given by

$$D = \frac{0.10 \cdot 1000/(1+0.12) + 2 \cdot 0.10 \cdot 1000/(1+0.12)^2 + 3 \cdot (1+0.10) \cdot 1000/(1+0.12)^3}{951.97}$$

$$= 2.73 \text{ years}$$

If we have a portfolio of interest rate instruments, the **portfolio duration** is defined by

$$D_{portfolio} = \frac{1}{PV_{portfolio}} \cdot \sum_i PV_i \cdot D_i$$

8.1.3 Modified Duration, Dollar Duration and DV01

In contrast to the Macaulay duration, modified duration (*MD*) is a price sensitivity measure, defined as the percentage derivative of price with respect to yield. MD applies when a bond or other asset price is considered as a function of yield. In this case one can measure the logarithmic derivative with respect to yield. The MD shows the change in price in percentage terms, resulting from a change in the *ytm*. It is defined by

$$MD = -\frac{1}{P}\frac{\partial P}{\partial y} = \{\text{using the simple formula}\} = \frac{D}{1 + y/n}$$

where n is the number of cash flows per year and D is the Macaulay Duration:

$$D = \frac{1}{P}\left\{\sum_{i=1}^{n} \frac{t_i \cdot C_i}{(1+y)^{t_i}} + \frac{t_n \cdot N}{(1+y)^{t_n}}\right\}.$$

The duration gives a value of the risk. Long duration $\Leftrightarrow$ high risk.

Definition 8.1.3.1. *Dollar duration* (DV01) measures the change in price (in money, £, \$, SEK) if the market interest rate increases with 1%.

$$\mathrm{DD} = \mathrm{MD} \cdot \mathrm{N}$$

The **DV01** is defined as the derivative of the value with respect to yield.

$$D_{\$} = \mathrm{DV01} = -\frac{\partial \mathrm{PV}(y)}{\partial y}$$

DV01 is analogous to the delta in derivative pricing since it is the ratio of a price change in output (dollars) to a unit change in input (1 bp of yield). DV01 is called Dollar duration because it is the change in price in *dollars,* not in *percentages.* It gives the dollar variation in a bond's value per unit change in the yield. It is often measured per 1 bp – DV01 is short for "dollar value of a 01" (or 1 bp).

DV01 can be used for instruments with zero upfront value such as interest rate swaps where percentage changes and MD are less useful.

For a par bond and a flat yield curve, the DV01 is the derivative of the price with respect to the yield, and PV01, the value of a one-dollar annuity will actually have the same value.

8.1.3.1 PV01 – Val01 – BPV

The names PV01 (or Val01, present value of a bp) refers to the change in the present value on a shift of 1 bp (1/100 of a %) on the yield curve. Often, this is also referred as a BPV (the bp value). PV01 also refers to the value of a 1 dollar or 1 bp annuity.

Definition 8.1.3.2. The *Base Point Value* measures the change in price if the market rate increases by 1 bp $(1\mathrm{bp} = 0.01\%)$.

$$\mathrm{BPV} = \frac{D_{\mathrm{modified}}(\%)}{100} \cdot \frac{\mathrm{DirtyPrice}}{100}$$

*Val*01 is calculated as

$$\mathrm{Val01} = \mathrm{P}(\mathrm{YTM} - 0.5\mathrm{bp}) - \mathrm{P}(\mathrm{YTM} + 0.5\mathrm{bp})$$

where P represents the dirty price and bp 1 bp. The shifts are added to the yield compounded according to the period of the bond.

In the BPV formula we first divide the MD by 100 to convert it from a percentage into a decimal (i.e. 5% is 0.05). The second divisor of 100 reduces the scale of risk from a 100 bp change in yield (MD) to just 1 bp.

Example 8.1.3.7

Calculation of a Base-Point Value (*BPV*) – its price sensitivity to a 1 bp change in yield?

Security:	5% US Treasury note
Type:	Semi-annual, actual/actual
Price	99.48
Accued interest:	0.84
MD:	1.50 %

$$BPV = \frac{1.50}{100} \cdot \frac{99.48 + 0.84}{100} = 0.015048$$

Thus, a 1 bp rise in the bond's yield will result in:

- A fall in the price from 95.4800 to 95.4654
- A loss of USD 1.46 cents per USD 100 nominal
- A loss of USD 145.98 on a USD 1 million position

BPVs tend to come out as very small figures with many decimal places. For convenience, many bond analysis systems scale the BPV figure by a factor of 100, so in our example the reported **Risk Factor** would be 1.4598. Thus, a 100-point change in the bond's yield would result in:

- A fall in the price from 95.48 to 94.02 (minus 1.46)
- A loss of USD 1.46 per 100 nominal
- A loss of USD 14,598 on a USD 1 million position

8.1.3.2 CV01

CV01 is the sensitivity to a 1bp shift in credit spreads.

8.1.4 Convexity

Convexity measures the percental change in the MD if the market rate increases with 1 bp. This can also be defined as the change in BPV for a change in the yield. The **convexity** can be calculated as the derivative of the duration with respect to the yield or as the second order derivative of the bond price with respect to time. This is the corresponding measure to gamma in option theory.

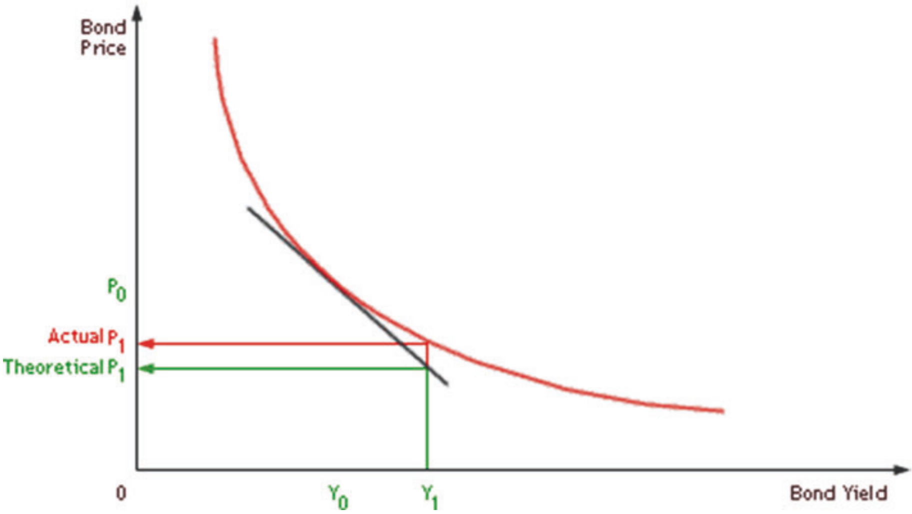


Fig. 8.1 The slope or derivative of the bond price with respect to the yield

The convexity is a nonlinear function, which can be compared by gamma in the option analysis. In Fig. 8.1, we show the error in the theoretical price if we do not consider the convexity on a change in yield.

If we take the derivative of the bond price with respect to the yield (continuously compounded) we get

$$\frac{\partial P}{\partial y} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i} = -PD$$

That is,

$$\frac{\Delta P}{P} = -D\Delta y$$

This can be applied to a portfolio as well. If we express y in terms of annual profit we get

$$\Delta P = -\frac{P \cdot D \cdot \Delta y}{1 + y/m} = -P \cdot MD \cdot \Delta y$$

where m is the number of annual coupons. The convexity can be written as

$$Cnvx = \frac{1}{2P}\sum_{i=1}^{n} t_i^2 c_i e^{-yt_i}$$

or

$$Cnvx = \frac{(1+y)^2}{P}\frac{\partial^2 P}{\partial y^2} = \frac{1}{2P}\left[\sum_{i=1}^{n}\frac{t_i \cdot (t_i+1)\cdot C_i}{(1+y)^{t_i}} + \frac{t_m \cdot (t_m+1)\cdot N}{(1+y)^{t_m}}\right]$$

Example 8.1.4.8

Consider a 7% bond with semi-annual coupons 3 years to maturity. Assume that the bond is selling at a yield of 8%. We then have

Year	Cashflow	Discounting	PV	PV/Price	Yeai*pv/Price
0.5	3.5	0.962	3.365	0.035	0.017
1.0	3.5	0.925	3.236	0.033	0.033
1.5	3.5	0.889	3.111	0.032	0.048
2.0	3.5	0.855	2.992	0.031	0.061
2.5	3.5	0.822	2.877	0.030	0.074
3.0	103.5	0.790	81.798	0.840	2.520

The price is the sum of individual PV, giving 97.379. The duration is 2.753. Suppose the yield changes to 8.2%, then the change in bond price is approximated by:

$$\frac{1}{P}\frac{\Delta P}{\Delta y} \approx -\frac{Dur}{1+y} \quad \Rightarrow \quad \Delta P = -\frac{97.379 \cdot 0.2\% \cdot 2.753}{1+0.04} = -0.5156$$

Using the convexity the change in the bond price on a change in yield is given by:

$$\frac{\Delta P}{P} \cong -\frac{Dur}{1+y}\Delta y + \frac{1}{2}Cnvx \cdot (\Delta y)^2$$

So, if X and Y are 2 different portfolios with the same duration, the difference in convexity can become large for changes in the yield. The convexity, C has its maximum close to a coupon payment day.

8.1.5 Gamma

The gamma value shows the extent of the change in the delta value when the same shift is applied to the delta as was used when the delta was first calculated.

8.1.5.1 Gamma Price

The *Gamma price* differentiation formula is

$$\Gamma_{price} = \frac{\partial\,(U+h) - \partial\,(U)}{h} = \frac{PV\,(U+2h) - 2PV\,(U+h) + PV\,(U)}{h^2} * scale.$$

The *Gamma price* calculates the change in *Delta Price*, given a unit change in the underlying asset price. This value is sometimes called **gamma explicit**.

Example 8.1.5.9

We are going to calculate the *Gamma Price* of a bond call option where the price of the underlying asset is $U = 100.17$, the shift step $h = U * 0.0001 = 0.010017$ and the present values are $PV(U) = 6,419.56, PV(U+h) = 6,434,49$ *and* $PV(U+2h) = 6,449.46$. The *Gamma Price* is then given by:

$$\Gamma_{price} = \frac{6,449.46 - 2 * 6,434,49 + 6,419.56}{0.010017^2} \cdot \frac{100}{1,000,000} = 0.0399.$$

8.1.5.2 Gamma Yield

The *Gamma yield* formula can be represented as

$$\Gamma_{yield} = \left[\partial\left(\underline{y}+h\right) - \partial\left(\underline{y}\right)\right] = \left[PV\left(\underline{y}+2h\right) - 2PV\left(\underline{y}+h\right) + PV\left(\underline{y}\right)\right] * scale$$

The yield curve gamma can, like the delta, be broken down into different time buckets. If this is the case, the gamma value shows the change in delta for the corresponding time bucket given a 1 bp change in the yield curve as a whole and not just in the individual bucket.

Example 8.1.5.10

Calculate the *Gamma yield* of a call option on a bond where the shift size is 0.00001 and the present values are $PV(y) = 7,377.6943211, PV(y+h) = 7,377.69432195$ and $PV(y+2h) = 7,377.6943231$:

$$\begin{aligned}\Gamma_{yield} &= \left[PV(y+2h) - 2PV(y+h) + PV(y)\right] * scale \\ &= [7,377.6943231 - 2 * 7,377.69432195 + 7,377.6943211] \cdot 1,000^2 = 0.3.\end{aligned}$$

8.1.6 Accrued Interest

Accrued interest is defined and calculated as the upcoming coupon payment times the number of days after the previous coupon was paid using the relevant day-count convention. When expressed in percentage points of the nominal amount of the bond it is equal to the difference between the dirty price and the (quoted) clean price.

8.1.7 Rho

Rho represents the change in the present value, given a shift of 1 bp in the repo curve. In the calculations, the yield is normally shifted by 1/1000 of a bp. The result is then scaled to a 1 bp shift by multiplying it by 1000.

Example 8.1.7.11

To calculate the *Rho* value of a put option on a bond we base the calculations on the present value of the option with unchanged conditions and the present value calculated using a repo curve that is shifted 1 bp

$$\rho = \left[PV(r_{repo} + 0.00001) - PV(r_{repo})\right] \cdot 1000$$

8.1.8 Theta

The *Theta* value shows the change in present value (*PV*) from the valuation date until the next calendar date, given unchanged market conditions.

Unchanged market conditions here imply that the yield curve will stay the same on both dates. For generic periods, the zero coupon rates are the same on both days, while all rates for fixed dates will be rolled down the curve by 1 day. Forward rates for fixed periods will also be affected when shifting the zero coupon yield curve.

Volatility values used for option pricing are also affected by a shift in the valuation date. This is only significant when a volatility landscape with a slope in the option expiry dimension is used. When shifting the valuation date, the time to expiration of the option will be 1 day shorter and another volatility will be fetched. When the underlying market price is used in the calculations, the underlying price is not affected by the 1-day forward shift.

Example 8.1.8.12
Consider a bond position in a 5-year government bond.

$$\Theta = PV(2004-11-05) - PV(2004-11-06)$$

where *PV*(2004-11-05) and *PV*(2004-11-06) are calculated after moving the valuation date and the yield curve rates one day forward. In a positive interest rate environment, the *Theta* of a bond position is normally positive.

The theta value has 2 components

- The first is due to the decrease in time to maturity. When valuing the bond 1 day later, the value of the bond will be higher because of the shorter time used when discounting the cash flows.
- The second component is due to the shape of the yield curve. If the yield curve is upwards sloping, each cash flow will be discounted with a slightly lower yield when valued as of tomorrow.

This is often referred to as "rolling down the curve". The exact slope of the curve will decide the size and sign of the contribution to the theta value. If the yield curve has a negative slope, this contribution can make the theta value negative for a bond.

8.1.8.1 Theta Classic

The *Theta Classic* value shows the change in *PV* from the valuation date until the next calendar date, given unchanged market conditions.

Here, an unchanged market condition means that the yield curve for tomorrow will be the one implied by today's forwards. There is no "rolling-down-the-curve" effect.

Volatility values, repo rates and underlying prices used for option valuation are kept constant for the 2 days in the calculations. The underlying price is kept constant, even if the theoretical price of the underlying is used.

For options priced with the Black-Scholes formula, the *Theta Classic* value represents the time value derived from that formula.

8.1.9 Vega

The *Vega* value shows the change in the *PV* from an upward shift in volatility of 1 %. The calculations are normally performed using a shift size of 0.01% and then scaling the result to a 1% shift:

$$\nu = [PV(\sigma + 0.01) - PV(\sigma)] \cdot 100$$

8.1.10 YTM

As we have seen, for bond price quoting, several *ytm* calculation methods are available.

As the ISMA and the Moosmüller methods were presented earlier, we will not repeat the formulas here. When calculating *ytm*, bond coupons are treated as follows

- Coupons are estimated to be the full yearly coupon dividend divided by the number of coupons.
- The time factor used when discounting each cash flow is:
- Time to next coupon according to instruments day count convention + (number of Coupon x number of coupons per year)

A simple version of the *ytm* formula looks like this

$$P = \sum_i \frac{c_i}{(1 + ytm)^{t_i}} + \frac{100}{(1 + ytm)^{t_n}}$$

where c_i are the coupons of the bond, t_i the time for the payouts and P the market price of the bond. With continuously compounding *ytms*, we can write this formula as

$$P = \sum_i c_i \cdot e^{-t_i \cdot ytm} + 100 \cdot e^{-t_n \cdot ytm}$$

In the case of promissory loans, a minor correction has to be made because the accrued interest is not paid on the value date, but deducted from the next coupon. However, this simple pricing formula is not used very frequently in practice, because it is a cumbersome process to incorporate the exact time elements of the coupons. If the adjustment to the coupon payment dates, to account for non-banking days are ignored, the formula can be simplified. Since all bonds pay coupons periodically, time can be measured in coupon periods defined previously.

The main problem with the use of *ytm* as a measure of interest rates is that it is not consistent across instruments. One 5-year bond will typically have a different *ytm* compared with another 5-year bond if they have different coupons. It is therefore impossible to associate a single interest rate with each maturity. One way of overcoming this problem is to use forward-rates.

Forward-rates are interest rates that are assumed to apply over a given periods between 2 future times. This contrasts with yields that are assumed to apply up to maturity, with a different yield for each bond. This is why the forward rate can be calculated by an arbitrage condition and the spot rate. The forward rate will also depend on the method used for the rate, if using continuously compounding rate or using a certain day-count method.

8.1.10.1 Simple Yield Formula

Another formula used for transformations between price and *ytm* in fixed income markets is the *Simple yield-to-maturity formula*, also known as *Japanese yield*. It takes into account the effect of the Capital gain or loss on maturity of the bond, as well as the current yield. Any Capital appreciation/depreciation is assumed to occur uniformly over the bond's life

$$ytm = \frac{c + \frac{Nom - P_{clean}}{L}}{P_{clean}},$$

where c is the annual coupon rate in % and L the life to maturity in years. A special day count fraction is used: L = the number of days to maturity, excluding February 29 in any year divided by 365.

8.1.10.2 The Money Market Formula

The money market formula is given by

$$P_{dirty} = \frac{Nom + c}{1 + ytm \cdot T}$$

where c is the annual coupon rate in % and T the time from spot to maturity in years, using the day count method of the instrument.

This method is relevant for instruments with one remaining coupon and for non-coupon instruments (zero bonds and bills). For the latter the ytm reduces to the simple annualized rate of interest.

8.1.11 Portfolio Immunization Using Duration and Convexity

Let $PV_L(y)$ denote the present value of some liability for a given yield y and $PV_j(y)$ the present value of some bonds, j in a bond portfolio given the yield y. We suppose in the following example that we have 3 bonds (i.e. $j = 1, 2, 3$). The present value of the bond portfolio is then given by:

$$PV_p(y) = PV_1(y) + PV_2(y) + PV_3(y)$$

Furthermore let $D_L(y)$ denote the MD of the liability, $D_j(y)$ the MD of bond j, $C_L(y)$ the convexity of the liability and $C_j(y)$ the convexity of bond j.

The derivative is then given by

$$\frac{d}{dy}PV_L(y) = -D_L(y) \cdot PV_L(y)$$

$$\frac{d}{dy}PV_p(y) = -\left[D_1(y) \cdot PV_1(y) + D_2(y) \cdot PV_2(y) + D_3(y) \cdot PV_3(y)\right]$$

and

$$\frac{d^2}{dy^2}PV_p(y) = \left[C_1(y) \cdot PV_1(y) + C_2(y) \cdot PV_2(y) + C_3(y) \cdot PV_3(y)\right]$$

for all y. Ideally, we would like to have $PV_p(y) = PV_L(y)$ for all y, since that would immunize the liability using the bond portfolio. If y changes, the portfolio can still be used to meet the liability.

Say that y_1 is the present yield value. Certainly we want $PV_p(y_1) = PV_L(y_1)$. We can conclude that PV_p is a better approximation to PV_L at y_1 if also $PV_p'(y_1) = PV_L'(y_1)$ and $PV_p''(y_1) = PV_L''(y_1)$.

Thus, we want to achieve

$$\begin{cases} PV_p(y_1) = PV_L(y_1) \\ PV_p'(y_1) = PV_L'(y_1) \\ PV_p''(y_1) = PV_L''(y_1) \end{cases}$$

If we use a Taylor series expansion

$$PV_L(y) = PV_L(y_1) + PV_L'(y_1)(y - y_1) + \frac{1}{2}PV_L''(y_1)(y - y_1)^2$$
$$+ \text{ higher order terms}$$

where the right hand side converges and represents the function $PV_L(y)$ for those values of y for which all the derivatives exist and for which the higher order terms go to zero. Note that

$$PV_L(y) - PV_L(y_1) = PV_L'(y_1)(y - y_1) + \frac{1}{2}PV_L''(y_1)(y - y_1)^2$$
$$+ \text{ higher order terms}$$

If we let $\Delta y = y - y_1$, we get

$$PV_L(y) - PV_L(y_1) \approx PV_L'(y_1) \cdot \Delta y + \frac{1}{2}PV_L''(y_1)\Delta y^2$$

when we drop the higher order terms.

If we have $PV_p'(y_1 = PV_L'(y_1)$ and $PV_p''(y_1) = PV_L''(y_1)$, we can conclude that $PV_L(y) - PV_L(y_1) \approx PV_p(y) - PV_p(y_1)$. Thus, a change in the value of the portfolio would track the change in the value of the liability if there were a change in the yield.

We are essentially trying to construct $PV_p(y)$ to make it a good approximation to $PV_L(y)$. We can make the approximation perfect for $y = y_1$, and "good" for y "near" y_1. The higher order terms we consider the better the approximation we get. To have $PV_p(y_1) = PV_L(y_1), PV_p'(y_1) = PV_L'(y_1)$ and $PV_p''(y_1) = PV_L''(y_1)$ we need

$$\begin{cases} PV_1(y_1) + PV_2(y_1) + PV_3(y_1) = PV_L(y_1) \\ PV_1(y_1) \cdot D_1(y_1) + PV_2(y_1) \cdot D_2(y_1) + PV_3(y_1) \cdot D_3(y_1) = PV_L(y_1) \cdot D_L(y_1) \\ PV_1(y_1) \cdot C_1(y_1) + PV_2(y_1) \cdot C_2(y_1) + PV_3(y_1) \cdot C_3(y_1) = PV_L(y_1) \cdot C_L(y_1) \end{cases}$$

If these three equations hold then PV_p is a good approximation to PV_L up to second order.

Because $PV_j(y_1)$ is the present value of bond j in the portfolio at y_1 we can choose it. It is just the amount of bond j we purchase. Further, the duration values and convexity values do not depend upon the amounts of the bonds that we purchase; they just depend on the yield. Hence, we have a system of three equations in three unknowns to solve. The unknowns are the $PV_j(y_1)$.

Note we can use durations instead of MDs because all the durations have the same modifier and it cancels out.

8.1.12 The Fisher-Weil Duration and Convexity

The Macaulay duration measures do not provide any information for how the price of a bond is affected by a change in the zero-coupon yield curve. Therefore, they are not useful for comparing the interest rate risk of different bonds. The problem is that the Macaulay measures are defined in terms of the bond's own *ytm*, and a given change in the zero-coupon yield curve will generally result in different changes in the yields of different bonds. It is easy to show that the changes in the yields of all bonds will be the same if and only if the zero-coupon yield curve is always flat. In particular, the yield curve is only allowed to move in parallel shifts. Such an assumption is not only unrealistic, it also conflicts with the no-arbitrage principle.

In 1938 Macaulay defined an alternative duration measure based on the zero-coupon yield curve rather than the bond's own yield. After decades of neglect this duration measure, it was revived by Fisher and Weil in 1971. They demonstrated the relevance of the measure for constructing immunization strategies. We will refer to this duration measure as the *Fisher-Weil duration.* The precise definition is

$$D^{FW} = \frac{1}{P}\sum_{i=1}^{n} t_i c_i e^{-y_i t_i} = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-y_i t_i}}{P}\right]$$

or

$$D^{FW} = \frac{1}{P}\cdot\left[\frac{N\cdot t_n}{(1+y_n)^n} + \sum_{i=1}^{n}\frac{C\cdot t_i}{(1+y_i)^i}\right]$$

where y_i is the zero-coupon yield prevailing at time 0 for the period up to time t_i. If the changes in all the zero-coupon yields are identical, the relative price change is proportional to the Fisher-Weil duration. Consequently, the Fisher-Weil duration represents the price sensitivity towards infinitesimal parallel shifts of the zero-coupon yield curve. Note that an infinitesimal parallel shift of the curve of continuously compounded yields corresponds to an infinitesimal proportional shift in the curve of yearly compounded yields. We can also define the *Fisher-Weil convexity* as

$$Cnvx^{FW} = \frac{1}{2P}\sum_{i=1}^{n} t_i^2 c_i e^{-y_i t_i}$$

or

$$Cnvx^{FW} = \frac{1}{2P} \cdot \left[\frac{N \cdot t_n^2}{(1+y_n)^n} + \sum_{i=1}^{n} \frac{C \cdot t_i^2}{(1+y_i)^i} \right]$$

8.1.13 Hedging with Duration

Suppose we want to use a futures contract to hedge a position in an interest rate instrument. Let F_c be price of a futures contract, D_F the duration of the underlying instrument, P the future value of the portfolio we want to hedge and D_P the duration on the portfolio at the end day of the contract. We then have

$$\Delta P = -P \cdot D_P \cdot \Delta y$$
$$\Delta F_c \approx -F_c \cdot D_F \cdot \Delta y$$

The duration based hedge factor expressed in the number of futures contracts is given by

$$N = \frac{P \cdot D_P}{F_c \cdot D_F}$$

This approach is subject to the following complications

- We have to guess which instrument that is cheapest to deliver (CTD) at expiration.
- The CTD instrument may change in time.

- The convexity.
- We can have non-parallel shifts on the yield curve.

8.1.14 Shifting the Zero-Coupon Yield Curve

The delta and gamma yield values are calculated by shifting the segment of the zero coupon yield curve that corresponds to the time bucket. The bucket shifts are constructed so that their sum always represents a single bp shift in the whole curve.

The calculations used when shifting the yield curve are actually performed using a differentiation step of 1/1000 bps and then scaling to 1 bp by multiplying by 1000. The reason for this is that changes in implied forward rate calculations can be quite substantial when shifting a segment of the curve in which only one of the forward points is located. These substantial changes can then suddenly make out-of-the money options, caps and floors, for example, in the money. This kind of non-linear effect should be avoided in a first order measure such as delta.

The choice of yield shift method affects the distribution of risk figures (such as delta and gamma) between time buckets. Their sum, that is, the total risk figure, is however not affected.

8.1.14.1 Rectangle Shift

The yield curve is shifted 1 bp between the end of the previous bucket specification (exclusive) and the end of the current bucket specification (inclusive) (Fig. 8.2).

8.1.14.2 Triangle Shift

The shift in the yield curve takes the form of a triangle with its apex at the bucket date and ending at the boundaries of the two adjacent buckets. In the first and last bucket, the two triangles are extended indefinitely to ensure that the sum of all the shifts corresponds to a total parallel shift of 1 bp (Fig. 8.3).

Example 8.1.14.13

Suppose there are only four time buckets: 3y, 5y, 10y and the rest bucket. The 5y bucket has its maximum shift (1 bp) at 5y, and linearly decreasing to 0 at 3y and to 10y. The shift at 7y is 0.6 bp.

The 10y bucket has its maximum shift at 10y and decrease linearly to 5 years. The shift at 7y is 0.4 bp. It also has a linearly decreasing shift towards higher maturities; with 0 shifts at 15y (a date determined using the step between 5y and 10y). The shift at 12y is 0.6 bp.

Therefore, the Rest Bucket has a 1 bp shift starting at 15y, running parallel to the end of time. It is linearly decreasing from 15y back to 10y. With this definition, the total of all buckets will add up to a parallel shift of 1 bp, which is important.

It may be argued that a rest bucket should not be used at all in the application, since it introduces the seemingly strange 15y time point. The reason is that if we use rectangular shifts, the rest bucket is needed for including all sensitivities above 10y. The 10y bucket includes every maturity between 5y and 10 y (including 10y).

8.1.14.3 Smooth Shift

The shift in the yield curve takes the form of a smooth shape with its highest point at the bucket date and ending at the boundaries of the two adjacent buckets.

The smooth yield shift method is recommended for contracts that are valued according to finite difference methods. The smooth shape implies a continuous shape of the yield curve, which is essential when using the finite difference solver.

The smooth shift has a cubic formula, where the first half representing the upward slope is defined by

$$3 \cdot t^2 - 2 \cdot t^3$$

where t is the time factor of the bucket ($0 < t < 1$). The second half of the shift is symmetrical to the first half.(Fig. 8.4)

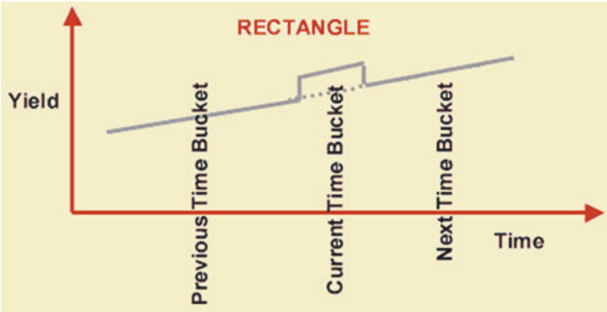


Fig. 8.2 A rectangular shift on the yield curve

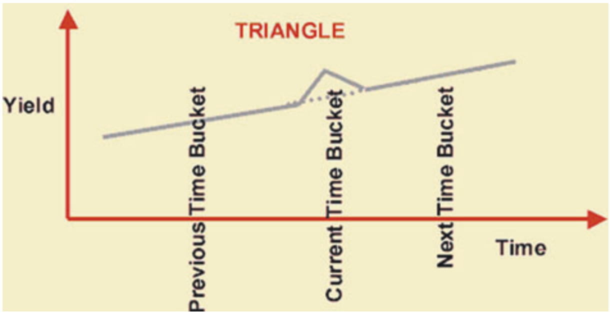


Fig. 8.3 A triangular shift on the yield curve

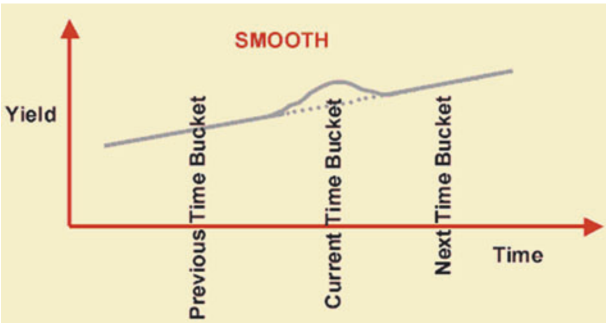


Fig. 8.4 A smooth shift on the yield curve

9

Risk Management

9.1 Introduction to Risk Management

We will now give a short introduction of how to measure risk and how to define limits on risks for a portfolio with many different instruments. Such limits are used by financial institutions to control and minimize risks. There have been more and more focus on risk management, especially after the financial crises in 2007–2008.

Financial risk management is the practice of creating economic value in a firm by using financial instruments to manage exposure to risk, particularly **Counterparty Credit risk** and **Market risk**. Here we will focus on market risk. Market risks includes:

- **Equity risk**, the risk that stock or stock indices prices and/or their implied volatility will change.
- **Interest rate risk**, the risk that interest rates and/or their implied volatility will change.
- **Currency risk**, the risk that foreign exchange rates and/or their implied volatility will change.
- **Commodity risk**, the risk that commodity prices and/or their implied volatility will change.

Some other sources of risk have been discussed in other sections of these lecture notes. Such risks include foreign exchange risk, liquidity risk, inflation risk, model risk, settlement risk, correlation risk, operational risk etc.

In order to ensure the survival of the financial firm and to comply with the provisions of the regulators, firms must have methods in place

J.R.M. Röman, *Analytical Finance: Volume II*,
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to regularly measure and maintain sufficient Capital to cover the nature and level of the risks to which the firm is or may be exposed to. The firm has both an obligation and an opportunity to design appropriate risk management systems that are tailored to their unique business requirements. Financial risk management can be both qualitative and quantitative.

In the banking sector, worldwide regulations are developed, such as the Basel Accords which are generally adopted by internationally active banks for tracking, reporting and exposing operational, credit and market risks.

The companies' systems for managing market risks should fulfil two general purposes. First, from a general risk management perspective, the systems should sufficiently provide the companies with a good understanding of the size of the market risk. Secondly, the systems should allow the companies to take risk mitigation measures that will ensure that their balance sheets are not exhausted. These systems can also form the basis for the companies' Capital requirement calculations.

The recent financial crisis has demonstrated numerous weaknesses in the global regulatory framework and in banks' risk management practices. In response, regulatory authorities have considered various measures to increase the stability of the financial markets and to prevent future negative impact on the economy. One major focus is on strengthening global Capital and liquidity rules.

Perhaps, the most important risk to measure is the potential loss due to the market and credit risk. As with all forms of risk, the loss amount may be measured in a number of ways or conventions. Traditionally, one convention is to use **Value-at-Risk** (VaR). The use of VaR is well established and accepted in the short-term risk management practice.

However, traditional VaR contains a number of limiting assumptions that constrain its accuracy. The first assumption is that the composition of the measured portfolio remains unchanged over the specified period in time. Over a short time horizon, this limiting assumption is often reasonable. However, over longer time horizons, many of the positions in the portfolio may have been changed and the VaR of the initially measured portfolio is no longer relevant. Another assumption is that changes are normally distributed which excludes fat tails and black swans. This has caused extensions of traditional CaR such as stressed VaR or credit VaR.

The Variance-Covariance and Historical Simulation approach to calculating VaR also assumes that historical correlations are stable and will not change in the future or breakdown under times of market stress.

In addition, care has to be taken regarding the intervening cash flows, embedded options, changes in floating interest rates etc. in the portfolio. These features cannot be ignored since their impact might be large.

9.1.1 Capital Requirement

One critical question is how much, and what type of, Capital a bank needs to hold so that it has adequate protection from losses due to future events.

In its simplest form, Capital represents the portion of the bank's liabilities, which does not have to be repaid, and therefore is available as a buffer in case the value of the bank's assets decline. If banks always made profits, there would be no need for Capital.

Unfortunately, such an ideal world does not exist, so Capital is necessary to act as a cushion when banks are impacted by large losses. In the event that the bank's asset value is lower than its total liabilities, the bank becomes insolvent and equity holders are likely to choose to default on the bank's obligations.

Naturally, regulators would hold the view that banks should hold as much Capital as possible, in order to ensure that insolvency risk and the consequent system disruptions are minimised. On the other hand, banks would wish to hold the minimum level of Capital that supplies adequate protection, since Capital is an expensive form of funding, and it dilutes earnings.

There are three views on what a bank's minimum Capital requirement should be.

First, in the **regulatory view**, the minimum Capital requirement as demanded by the regulators; it is the amount a bank must hold in order to operate. A regulator's primary concern is to ensure that there is sufficient Capital in order to buffer a bank against large losses. Regulatory Capital could be seen as the minimum Capital requirement in a "liquidation" view, whereby all liabilities can be paid. Recently regulators have introduced the concept of survival horizon for banks whereby its capital should be enough to sustain its business during a time period of 30 days in case markets break down.

Regulatory Capital is a standardised calculation for all banks, although, there are differences to various regulatory regimes and countries.

Second, in the **economic view** there is a theoretical view on minimum Capital requirement based on the underlying risks of the bank's assets and operations. Economic Capital could be seen as the minimum Capital requirement so that the bank is in continual operation. Here we are only concerned to hold enough Capital to survive.

Economic Capital was originally developed by banks as a tool for Capital allocation and performance assessment. For these purposes, it did not need to measure risk in an absolute, but only in the relative sense.

Over time, with advances in risk quantification methodologies, the concept of economic Capital has been extended to applications that require accuracy in the measurement of risk. This is evident in the ICAAP, whereby banks are required to quantify the absolute level of internal Capital.

Finally, in the **rating agency view** the minimum Capital a bank needs to hold is the amount it needs in order to meet a certain credit rating. The amount and type of Capital a bank holds in relation to its total risk weighted assets (RWA) is a crucial input to the reviewing mechanisms used to determine its credit rating. In addition, since credit ratings provide important signals to the market about the financial strength of the bank, they can have significant downstream impact on a bank's ability to raise funds, and the cost at which the funds can be raised. Therefore, having sufficient Capital to meet the requirements of the rating agencies becomes an important consideration for senior management.

A key part of bank regulations is to make sure that firms operating in the industry are **prudently managed**. The aim is to protect the firms themselves, their customers and the economy, by establishing rules to make sure that these institutions hold enough Capital to ensure continuation of a safe and efficient market and that the banks are able to withstand any foreseeable problems.

The main international effort to establish rules around Capital requirements has been the Basel Accords, published by the **Basel Committee on Banking Supervision** housed at the **Bank for International Settlements**. This sets a framework for how banks and depository institutions must calculate their Capital. In 1988, the Committee decided to introduce a Capital measurement system commonly referred to as Basel I. This framework has been replaced by a significantly more complex Capital adequacy framework commonly known as Basel II. This is currently being replaced by Basel III.

The Capital ratio, the percentage of a bank's Capital to its risk-weighted assets is dictated under the relevant Accord, Basel II. It requires that the total Capital ratio must be no lower than 8 per cent.

Each national regulator normally has a slightly different way of calculating bank Capital, designed to meet the common requirements within their individual national legal framework.

In the European Union, member states have enacted Capital requirements based on the Capital Adequacy Directive CAD1 issued in 1993 and CAD2 issued in 1998. In the United States, depository institutions are subject to risk-based Capital guidelines issued by the Board of Governors of the Federal Reserve System (FRB).

9.1.1.1 Regulatory Capital

According to Basel II, the bank Capital was divided into two "tiers", each with some subdivisions.

Tier 1 Capital

Tier 1 Capital, the more important of the two, consists largely of shareholders' equity and disclosed reserves. This is the amount paid up to originally purchase the stock (or shares) of the Bank, retained profits subtracting accumulated losses, and other qualifiable Tier 1 Capital securities. In simple terms, if the original stockholders contributed $100 to buy their stock and the Bank has made $10 in retained earnings each year since, paid out no dividends, had no other forms of Capital and made no losses, after 10 years the Bank's Tier 1 Capital would be $200. Shareholders equity and retained earnings are now commonly referred to as "Core" Tier 1 Capital, whereas Tier 1 is core Tier 1 together with other qualifying Tier 1 Capital securities.

Tier 2 (Supplementary) Capital

Tier 2 Capital, or supplementary Capital, comprises undisclosed reserves, revaluation reserves, general provisions, hybrid debt capital instruments and subordinated term debt. Undisclosed reserves are not common, but are accepted by some regulators where a Bank has made a profit but this has not appeared in normal retained profits or in general reserves.

A *revaluation reserve* is a reserve created when a company has an asset revalued and an increase in value is brought to account. A simple example may be where a bank owns the land and building of its headquarters and bought them for $100 a century ago. A current revaluation is very likely to show a large increase in value. The increase would be added to a revaluation reserve.

A *general provision* is created when a company is aware that a loss has occurred, but is not certain of the exact nature of that loss. Under pre-IFRS accounting standards, general provisions were commonly created to provide for losses that were expected in the future. As these did not represent incurred losses, regulators tended to allow them to be counted as Capital.

Hybrid debt Capital instruments consist of instruments, which combine certain characteristics of equity as well as debt. They can be included in supplementary Capital if they are able to support losses on an ongoing basis without triggering liquidation. Sometimes, it includes instruments, which were initially issued with interest obligation (e.g. debentures), but the same can later be converted into Capital.

Subordinated debt is classed as Lower Tier 2 debt, it usually has a maturity of at least 10 years and ranks senior to Tier 1 debt, but is subordinate to senior debt.

9.1.2 Risk Measurement and Risk Limits

To measure and control the risk, financial institutions in general use risk matrices, VaR calculations and other methods. The risk calculated using such models is then analysed each day and compared with the limits. The management and the board of directors decide these limits. We will next give a description of the most common models.

9.1.2.1 Risk Matrices

Risk matrices are used to measure, control and report risks. A risk matrix is an outcome analysis of a scenario in which two risk factors are stressed at different intensities. The factors that are altered are usually the price of the underlying asset (delta and gamma risk) and expected volatilities (Vega risk). An example of a risk matrix is presented in Table 9.1.

The aforementioned matrix shows gains or losses for different prespecified scenarios in which volatilities and underlying prices fluctuate within an interval of +/–30 per cent and +/–20 per cent, respectively.

Table 9.1 An example of a risk matrix

	–20 %	–10 %	–5 %	0 %	5 %	10 %	20 %
–30 %	–33 480 000	–29 430 000	–26 955 000	–24 180 000	–21 105 000	–17 730 000	–10 080 000
–20 %	–25 420 000	–21 370 000	–18 895 000	–16 120 000	–13 045 000	–9 670 000	–2 020 000
–10 %	–17 360 000	–13 310 000	–10 835 000	–8 060 000	–4 985 000	–1 610 000	6 040 000
0 %	–9 300 000	–5 250 000	–2 775 000	–	3 075 000	6 450 000	14 100 000
10 %	–1 240 000	2 810 000	5 285 000	8 060 000	11 135 000	14 510 000	22 160 000
20 %	6 820 000	10 870 000	13 345 000	16 120 000	19 195 000	22 570 000	30 220 000
30 %	14 880 000	18 930 000	21 405 000	24 180 000	27 255 000	30 630 000	38 280 000

Many companies use risk matrices to set limits for the level of the largest acceptable loss. Another alternative is to define a subsection within the matrix and set the limit where the greatest loss may occur within this subsection.

Risk matrices have both strengths and weaknesses due to their simplicity. A strong point is that they offer an extremely clear and comprehensible method in which to place potential outcomes in direct relation to changes in relevant market variables, which is attractive for traders as well as risk functions and senior management.

Their weaknesses are that they do not Capture **basis risks**, (see next) between different maturities, exercise prices and underlying assets and they completely ignore risk factors other than price and volatility. Many times, the degree of stress that is tested, in particular with regard to volatilities, is not high enough. We will show an example of this subsequently (see the HQ Bank section).

Basis Risk

Risk matrices ignore basis risks between different underlying assets, maturities and the exercise prices of options. This basis risk is the risk that opposite positions in a hedging strategy do not move as expected in relation to one another. This risk does not appear in the risk matrix because the risk matrix calculations simply adds all positions in the trading portfolio without any regard for correlation of prices or volatilities. Note here, exercise prices refer exclusively to option exercise prices.

Many financial firms and fund managers are not sufficiently aware of the consequences of this weakness. By calculating the sum of their positions, two strong assumptions are implicitly made:

- The market prices of all assets and liabilities in the portfolio are assumed to be perfectly correlated. In other words, for example, it is assumed that if the price of an asset increases 1 per cent, all other underlying assets in the portfolio will also increase by 1 per cent. This means that a negative position and a positive position in two assets would cancel one another out and as a result the risk (expressed as delta and gamma) can appear to be very small or non-existent.
- With regard to options, it is assumed that implied volatilities for different maturities and different exercise prices are perfectly correlated. For example, if the volatility of an option maturing in three months increases by 5 per cent, it is assumed that volatilities of all

other options in the portfolio with different maturities and exercise prices will also rise by 5 per cent. This implies that negative and positive positions in different maturities could completely cancel out one another in which case the risk (expressed as Vega) could appear to be small or non-existent in the risk matrix.

It is worth noting that it was this exclusion of basis risks between both different underlying assets and different maturities that to a large extent contributed to the failure of the Swedish HQ Bank. It did not identify the enormous risks in its trading portfolio. The HQ Bank case therefore represents a good example of the danger of risk matrices.[1]

Example 9.1.2.1

HQ Bank in Sweden

HQ Bank's main market risk measure for its trading portfolio was a risk matrix as described earlier. The bank simulated a worst-case scenario within the matrix and set its limits based on this measure.

The absolute largest exposures in HQ Bank's trading portfolio were index linked options on the German DAX index and the Swedish OMX index. The following table shows the exposures expressed as delta and Vega at 18 May 2010. The table also shows the exposures broken down by underlying asset and maturity (which the risk matrix calculations do not include).

Position date	2010-05-18		
		Data	
Underlying	Exp.date	Sum of Vega	Sum of Delta
ODAX	2010-05-21	–259 069	–4 317 526
	2010-06-18	11 136 970	–37 881 767
	2010-12-17	51 855 354	82 996 030
	2010-12-17	–53 655 171	–67 476 377
ODAX Total		9 078 084	–26 679 640
OMXS30	2010-05-21	742 152	15 725 210
	2010-06-18	–581 817	–3 214 468
	2010-07-16	–3 272 317	–10 597 947
	2010-10-15	–4 174 850	–1 880 670
	2011-01-21	–6 019 420	133 047
OMXS30 Total		–13 306 252	–165 172
Total		–4 228 168	–26 514 468

[1] See the report by the Swedish FSA (Finansinspektionen).

Vega in the table is expressed as the change given an increase in implicit volatilities of one percentage point.

Delta is expressed as the change given an increase in the underlying asset price of one percent.

Several important events are evident from the previous table.

The bank had a negative Vega exposure in OMX and DAX for long maturities (primarily December 2010) and an opposite exposure for short maturities (primarily June and September 2010).

The total exposure in DAX was positive in terms of Vega and negative in terms of delta, while the total exposure in OMX had the opposite signs. The aggregate exposure (Swedish Krona (SEK) -4.2 million in Vega and SEK -26.5 million in delta) appears to be relatively small compared to the sub exposures per maturity and underlying asset.

When the risk in these exposures is transferred to a risk matrix, the implicit assumption is made, as mentioned before, that all maturities and underlying assets are perfectly correlated. Given this assumption, the exposures in HQ Bank, at least in terms of Vega, undeniably appeared to be relatively small. In Vega, the December outcome in DAX seemed to eliminate the September outcome and the total exposures in both Vega and delta in OMX seemed to compensate for the opposite exposures in DAX. As a result, only the total exposure of SEK -4.2 million in Vega and SEK -26.5 million in delta were visible in the matrix. This was naturally a gross simplification of the risk profile.

The table shows that if all underlying prices would remain unchanged and if the volatility would increase by 1 percentage point in the DAX December outcome at the same time as the volatility would remain unchanged for all other maturities and underlying prices (which was not an improbable scenario), HQ Bank would have lost SEK 53.6 million at the same time as the risk matrix would have indicated a loss of SEK 4.2 million.

This example illustrates how the risk matrix's underlying simplified assumptions regarding basis risks can lead to a gross underestimation of risks.

One conclusion that can be drawn from the HQ example is that if a portfolio contains significant positions which cannot be assumed to have a particularly strong correlation and/or significant option positions with different maturities and exercise prices that are not proven to be strongly correlated, the basis risks are likely to be significant t. These risks, therefore, must be measured and controlled, which an aggregate risk matrix does not do.

There are several ways to improve risk matrices so that they Capture basis risks:

1. Lower aggregate levels: For example, it is conceivable that the correlation between two shares in one country is greater than

the correlation between two shares in different countries. It is therefore possible to form several groups, one for each country, and then construct a risk matrix for each of these groups. This will ensure that correlation within the groups is actually high in order to ensure that significant basis risks are not underestimated.

2. Other correlation assumptions: The basic (implicit) assumption in the risk matrix is that there is perfect correlation between underlying assets, maturities and exercise prices, which may be viewed as an extreme assumption. In order to examine what would happen in the presence of imperfect correlation, it is helpful to simulate the matrix under other assumptions. The most common alternative is to test the opposite extreme scenario – the total absence of any correlation – but other correlation assumptions may also need to be tested. Simulating two extreme cases can be a good exercise since the results provide an interval of outcomes for comparison. This type of simulation also illustrates what could happen if correlations drastically change, which is important information since correlation patterns are not constant over time.
3. Combinations with other risk measures: This is the absolute most common and the most robust way to manage basis risks. For maturities, it is common to measure Vega in time buckets, which are often also subject to limits. For basis risks between underlying assets, scenario analyses in which the largest positions in individual assets are stressed under an assumption that the correlation is zero are often used. For financial firms that use VaR models, these models usually function as a good complement, provided that the same correlation assumptions are not made between the risk factors in the VaR model as in the risk matrix.

Exclusion of Risks

As described previously, risk matrices measure the exposure to two types of risk:

- Change in price of underlying assets (delta and gamma risks).
- Change in expected volatility in underlying assets (Vega).

These two definitely qualify as significant risks for, for example, an equity portfolio with optionality. However, the portfolio may be exposed to other significant risks which might need to be analysed and measured outside of the matrices like:

- Sensitivity to changes in maturity (theta) is one factor that is often excluded from risk measurements. One possible explanation for why theta is often excluded is that it is questionable if it is a "risk" in the true meaning of the word since it is not directly affected by market risk factors. Because theta is the gain/loss arising due to the additional passage of time, it is relatively predictable. Theta may still need to be measured in order to be able to derive the origin of the results.
- If the portfolio contains optionality, the interest rate sensitivity (rho) can be a significant risk factor.
- For some asset classes there are also other types of risks that are difficult to measure with risk matrices, for example credit spread risks and twist risks in bond portfolios or dividend risks for equity derivatives.

Volatility Stress in the Risk Matrices

Asset prices and volatility might be stressed differently. For share portfolios, the price dimension is stressed by +/–10 – 15 per cent while the volatility dimension is stressed by +/–20 – 30 per cent. A 10 – 15 per cent fluctuation in a share portfolio is an extreme stress scenario over a short period of time, particularly since the stress is often applied to a diversified portfolio and not to individual share. However, it is not particularly unusual for implicit volatilities to fluctuate in considerable excess of 20 – 30 per cent. In other words, the model to a stress scenarios can be too weak. For example, the VIX index rose by more than 50 per cent in one day during the Lehman crash in 2008. In 2011 alone there were two trading days during which the volatility in VIX fluctuated by more than 30 per cent in just one day. The stress on volatility is therefore not proportionate to the stress on price for many companies. Similar differences can also be observed for asset classes other than equities.

9.1.2.2 VaR Models

VaR in general is a probability-based risk measure that is statistically created by a model. The measure should be interpreted as a loss that with a specified probability is not expected to be exceeded during a certain period of time. Companies normally use a VaR with a probability of 99 per cent or 95 per cent and a 10-day time horizon. For example, a VaR measure of -200 million, 99 per cent and 10 days would mean that, at the date of measurement, a company could expect with 99 per cent probability not to exceed losses of more than 200 million over a period of 10 days. However, VaR does not say anything about what the losses could be in extreme cases. VaR is also almost always based on historic market fluctuations, and forward-looking hypothetical market fluctuations and correlation patterns are not captured in the model.

VaR models are a globally accepted method for measuring and controlling risk. This method is primarily used by the larger companies, but also by some of the smaller companies. VaR models are good supplements to risk matrices and other sensitivity measures since they contain a probability aspect that is not found in these methods. The VaR measure is also more comprehensive than, for example, risk matrices since it takes into account many more risk factors than only price and volatility. A VaR model is relatively intuitive and easy to understand as a concept and it also enables comparisons of risk-taking between different parts of a company's business.

It is important to understand the function of the VaR model in order to be able to understand its limitations. A VaR model rapidly becomes more complex as more asset classes and types of instruments are included. A number of assumptions and simplifications must be made in the model to simulate the risk of loss. The most important, is to ensure that these simplifications are not so significant as to render the VaR measure unrealistic. Therefore back-testing of the VaR model on historic data is important in order to verify that losses are neither over- nor underestimated.

There exist a number of common methods for simulating a distribution of losses. These methods can be divided into three groups, **Parametric VaR**, **Monte-Carlo Simulated VaR** and **Historically Simulated VaR**.

Parametric VaR – This method is the least robust of the three. The reason for this is that an assumption is made about the underlying probability distribution and a full revaluation of the financial instruments is not carried out. This type of model can be used for areas with low complexity (e.g. isolated parts of an organization that handle simpler instruments). If this model is to be used for risk control with well-defined limits, it should be supplemented with additional limits on well-planned and robust risk measurement methodologies, such as stress tests and scenario analyses that reflect extreme fluctuations in risk factors.

According to this method, an assumption is made about the probability distribution of the daily returns. Input data required by the model includes standard deviations, means and correlations between the various risk factors. Instruments are not fully revalued individually, rather the model's calculations are based on sensitivity measures. The most common assumption is normal distribution, even if other distribution assumptions could be possible.

For linear instruments (i.e. instruments where the prices do not depend non-linear on another underlying instrument), a method based on delta should be sufficient but if optionality or convexity is present in the portfolio, a delta/gamma method should be used. The obvious disadvantage of this type of VaR model is the distribution assumption. After a number of financial crises, it has become generally accepted that few financial markets are characterised by normally distributed prices. Extreme fluctuations are much more common than what is indicated by a normal distribution. The true probability of observing a loss greater than the one predicted by the VaR model is therefore greater than the chosen degree of confidence.

Monte-Carlo Simulated VaR – Provided that a full revaluation of financial instruments is carried out, this model is better suited for complex, non-linear instruments than parametric VaR. Therefore, this model can be used for risk control and risk measurement when such instruments are included. However, since a distribution assumption is made in the simulation of risk factors (often normal distribution), extreme fluctuations, just like for parametric VaR, should be taken into account separately via scenario analyses and stress tests.

This method simulates time series for various risk factors via a stochastic process. The "Geometric Brownian Motion" or a similar process is often used in the simulation. For each simulated outcome, every instrument in the portfolio is fully revaluated in order to identify effects on profit. The advantage of this method is that its simulation of exotic financial instruments is more accurate and therefore also more appropriate for portfolios containing many complex instruments. The disadvantage of this model is that it also makes an assumption about distribution since the stochastic process must follow a probability distribution.

Historically Simulated VaR – The advantage of this VaR model is that it does not require any explicit distribution assumption while, at the same time, a full revaluation of the instruments is carried out. The disadvantage of this model is that the simulation is strongly dependent on the model being based on a "representative" historical period of time (more on this later).

This method uses actual historical time series to identify changes in risk factors to which the portfolio is sensitive. The exposures are simulated by actual historical scenarios, which occurred within the historical period. A result is simulated for each day during the historical period and thus builds the distribution. The loss amount that corresponds to the degree of confidence chosen by the company is then sorted out and represents the VaR measure. This method does not require any explicit assumptions about the distribution, which is a clear advantage. Full revaluation means that the model can also be used to simulate very complex instruments. The disadvantages are that this method requires considerable computer power to simulate large portfolios with complex instruments and it can be difficult or impossible to obtain sufficient historical data for certain instruments. This method is also particularly sensitive to the span of the historical period. The most common period of time consists of the most recent one-year period. Some banks uses a two-year period and other shorter periods was 17 days.

Irrespective of the analytical structure, a number of assumptions must be made to estimate the parameters used to construct the probability distribution. The parameters are usually estimated using actual data from a past period in time. The VaR models have proved to be extremely sensitive to the period that was chosen, which in particular applies to historically simulated models. For example, a company's VaR value more than doubled if the historical period included the financial crisis in 2008.

Companies using VaR models to calculate Capital requirements for market risks need historical data for a period of at least one year. A shorter historical period of data makes the VaR value more sensitive to new data, while a longer historical period of data makes the value less sensitive.

Choice of Risk Factors for the VaR Model

A VaR model selected as a risk measurement method for specific operations should include all significant risk factors associated with such operations. However, if the firm's operations in a specific market are very small, or if the risks are negligible, some risk factors may be estimated or even completely excluded from the model.

It is relatively common for risk factors to be excluded or for risk factors that cannot be assumed to have a reasonable correlation with the actual risk to be included in the estimations. Many companies do not have any ongoing validation of this process in order to regularly monitor how accurate the estimates are.

9.1.3 Risk Control in Treasury Operations

The main responsibility of treasury operations is to manage the company's borrowing and lending transactions and to identify any differences in maturity and currency between cash flows. Also the treasury operations usually include management of the financial firm's liquidity reserves. In general, the treasury operations often represent a significant portion of the firms' total market risk, primarily in the form of interest rate risks, credit spread risks and cross currency basis swap spread risks.

In general, the treasury operations are separate from other areas of the company that generate market risk. This is evident in that both methods for and of reporting market risk often differ significantly from other areas of the company. Treasury operations can have a considerable less transparency in terms of risk than other areas of the companies. Less sophisticated methods and fewer risk measures are used, and as a result, several significant risks are generally not identified.

9.1.3.1 Cross Currency Basis Swap Risk

These risks arise in companies, which borrow funds in a different currency than they lend funds. The interest rate risk that arises is normally hedged with an interest rate swap and the currency risk with a cross currency basis swap. The Fig. 9.1 shows such a typical arrangement

Explanation of the model

- Bank A issues a five-year fixed interest rate bond in EUR. However, the bank's lending is primarily in SEK with three-month maturities.
- To neutralise this discrepancy in maturity between borrowing and lending, Bank A enters into an interest rate swap in EUR where the fixed interest rate for the issued bond is transformed to a variable EURIBOR-based interest rate.

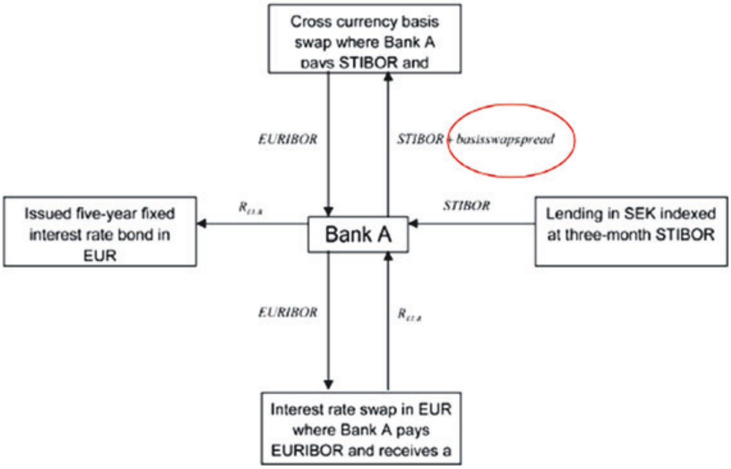

Fig. 9.1 A trade where the risks are hedged in another currency

- The currency discrepancy is neutralised via a cross currency basis swap where Bank A pays STIBOR and receives EURIBOR. In the previous example, Bank A pays a spread in addition to STIBOR in the basis swap (encircled in previous instance).

In the example, all transactions are made to maturity (i.e. to five years). From a risk perspective, it therefore appears that borrowing and lending are fully hedged. At maturity, Bank A will receive the interest rate margin it locked in via swaps related to its borrowing and lending. However, changes in basis swap spreads result in gain/loss effects during the term of the hedge. According to applicable accounting rules, changes in market value attributable to changes in basis swap spreads have a direct effect on Bank A's profit/loss and often on its Capital adequacy as well. The investigation demonstrated that the incurred profit/loss risk is often significant.

9.1.3.2 Credit Spread Risk

Credit spread from a specific bond is defined as the difference between the bond's market rate and the rate of a risk-free bond with the same maturity. Credit spread risk is the risk of loss in the form of a change in value of the bond when the credit spread changes. The credit spread is, as implied by its name, primarily attributable to the creditworthiness of the issuer. Credit spread risks are not unique to treasury operations; they are also found in many other areas of the companies. However, they are often of considerable size in the treasury operations, particularly in liquidity portfolios which often hold large amounts of bonds, notes and bills.

A normal procedure within treasury operations is to use interest rate swaps for liquidity portfolios in order to lower the maturity of the portfolio (often under three months) and thereby decrease the sensitivity to changes in interest rates. Credit spread risk, however, remains unchanged after such a procedure. The treasury operations focus in most cases specifically on the interest rate risk in the portfolio, which is often sharply reduced after hedging with swaps. Companies might place considerably less importance on the credit risk spreads, which in some cases actually are larger than the interest rate risks.

10

Option-Adjusted Spread

10.1 The OAS Model

A common method to value bonds, zero bonds and promissory loans with embedded options (that is, callable and putable instruments) is the use of option-adjusted spread (OAS). This model will use a spread on a benchmark curve to calculate bond prices for risky bonds, due to embedded options and since they are so called corporate bonds.

The model we will use is based on a Black-Derman-Toy (BDT) (see next) interest rate binomial tree approach and adjusts for the cost of the embedded option and the difference between model price and market price due to other risks, for example credit and liquidity risks.

The BDT model is a single-factor short-rate model matching the observed term structure of forward rate volatilities, as well as the term structure of the interest rate. A binomial tree is constructed for the short rate in such a way that the tree automatically returns the observed yield function and the volatility of different yields. The model is described by a SDE where the rates are log-normally distributed. Therefore, the interest rates cannot be negative.

To adjust the theoretical price on the binomial tree to the actual price, a spread (called option-adjusted spread since the context of OAS started with trying to correct for miss-pricing in option embedded securities) is added to all short rates on the binomial tree **such that the new model price after adding this spread makes the model price equal the market price** (this is the defining purpose of OAS).

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_10

The value of OAS is that it enables investors to directly compare fixed income instruments, which have similar characteristics, but traded at significantly different yields because of embedded options.

The OAS model has three dependent variables:

- Option-adjusted spread
- Underlying price
- Volatility

The model can calculate the following model specific risk measures (except for the risk measures discussed earlier):

- Effective duration
- Effective modified duration
- Effective convexity
- Option-adjusted spread

10.1.1 Some Definitions

As we will see, the bullet bond can be used to find the "value" of the embedded option. For example, a callable bond of the option value is given by the price difference between the bullet bond and the callable bond.

There are six steps associated with the OAS analysis. The following assumption is that the method is being applied to a callable bond:

1. A binomial tree is built on dates where we have cash flows. Also create nodes in the tree where the instrument is callable or putable. Therefore extra nodes are added at the beginning and/or the ends of call periods (if they not coincide with cash flows).
2. Build a binomial tree using these rates, with equal probabilities $(= {}^{1}/_{2})$.
3. Calibrate the tree to market data by adjusting the nodes until the tree can replicate any cash flow as the discount function given by the benchmark yield curve.
4. Calibrate the model by adding the same number of basis points (the spread factor) to all rates in the tree until the model replicate the actual market price (if this price is known) of the callable bond. The result is the bond's OAS.

5. Apply the same OAS to value a bullet bond with terms identical to the callable/putable bond.
6. Take the difference between the value obtained for the callable bond and the value obtained for the bullet bond. This difference is the value of the embedded option.

The model creates nodes in the binomial tree on the following events:

1. Cash flows
2. Single call or put events
3. Start or end of call or put periods

No intermediate nodes are created, since there are no dynamical changes between the nodes. (For better accuracy in the interval where the bond can be called (or putted) back we can build intermediate nodes in other (all) parts of the tree.)

10.1.2 Building the Binomial Tree

The stochastic process for the short rate in the BDT model is given by

$$dr = a(t) \cdot r \cdot dt + \sigma(t) \cdot r \cdot dz$$

where $z(t)$ is a Brownian motion. In some literature this SDE is written as:

$$d\ln(r) = \{\theta(t) + \rho(t)\ln(r)\}\, dt + \sigma(t)dz$$

where $\theta(t)$ will be shown to be the drift of the short-term rate and $\rho(t)$ the mean reversing term to an equilibrium short-term rate that depends on the interest rate local volatility as follows

$$\rho(t) = \frac{d}{dt}\ln[\sigma(t)] = \frac{\dot{\sigma}(t)}{\sigma(t)}.$$

That is,

$$d\ln(r) = \left\{\theta(t) + \frac{\dot{\sigma}(t)}{\sigma(t)}\ln(r)\right\} dt + \sigma(t)dz$$

Since the volatility is time dependent, there are two independent functions of time, $\theta(t)$ and $\sigma(t)$, chosen so that the model fits the term structure of spot interest rates and the structure of the spot rate volatilities.

Jamshidan (1991) shows that the level of the short rate at time t in the BDT model is given by

$$r(t) = U(t) \exp\{\sigma(t) z(t)\}$$

where $U(t)$ is the median of the lognormal distribution of r at time t, $\sigma(t)$ the level of the short rate volatility and $z(t)$ the level of the Brownian motion, a normal distributed Wiener process that Captures the randomness of future changes in the short-term rate. One of the main advantages of the BDT model is that it is a lognormal model that is able to Capture a realistic term structure of the interest rate volatilities. To accomplish this feature, the short-term rate volatility is allowed to vary over time, and the drift in interest rate movements depends on the level of rates. Due to the property of Brownian motions, we have

$$z(t) = \varepsilon \cdot \sqrt{t}$$

where the values

$$\varepsilon = \begin{cases} +1 & \text{or} \\ -1 & \end{cases}$$

is used to build the tree. From the previous discussion, a fixed spacing, Z_i between the nodes in the binomial tree is defined as ($\varepsilon_{max} - \varepsilon_{min} = 2$):

$$Z_i = e^{2\sigma_i \cdot \sqrt{t_i - t_{i-1}}} \tag{10.1}$$

where σ_i is the volatility at time t. The risk-neutral probabilities of the binomial branches of this model are assumed equal to $^1/_2$. (It by no means implies that the actual probability for an interest rate increase or decrease is equal to $^1/_2$.) The tree uses the short-rate annual volatility, σ, of the benchmark rates which should be given in the Black-Scholes framework. The process can be illustrated using the following four short rates (all expressed with semi-annual compounding):

$$f_1 = 6.000\,\%$$
$$f_2 = 7.200\,\%$$
$$f_3 = 8.150\,\%$$
$$f_4 = 8.836\,\%$$

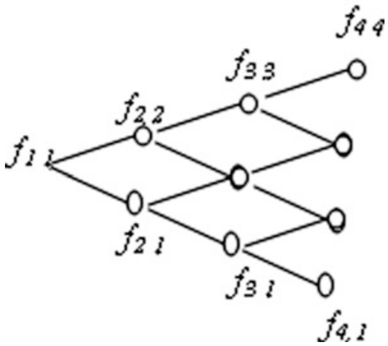

Fig. 10.1 The forward rates in a OAS tree

Assume for simplicity that annual volatility of the short rates is constant, and given by 15%. When the tree is built, the volatility spread factors, Z_i are kept constant and the tree is built with the following relation between the nodes:

$$f_{i,j} = Z_i^{j-1} \cdot f_{i,1} \tag{10.2}$$

where $f_{1,1} = f_1$. This results in the tree in Fig. 10.1 where the rates is given by

$$\begin{cases} f_{2,2} = Z_2 \cdot f_{2,1} \\ \frac{1}{2} f_{2,1} + \frac{1}{2} f_{2,2} = f_2 \end{cases} \Rightarrow \quad f_{2,1} = \frac{2 \cdot f_2}{1 + Z_2} \quad \Rightarrow \quad f_{2,2}$$

$$\begin{cases} f_{3,3} = Z_3^2 \cdot f_{3,1} \\ f_{3,2} = Z_3 \cdot f_{3,1} \\ \frac{1}{4} f_{3,1} + \frac{1}{2} f_{3,2} + \frac{1}{4} f_{3,3} = f_3 \end{cases} \Rightarrow \quad f_{3,1} = \frac{4 \cdot f_3}{1 + 2 \cdot Z_3 + Z_3^2} \quad \Rightarrow \quad f_{3,2} \quad \Rightarrow \quad f_{3,3}$$

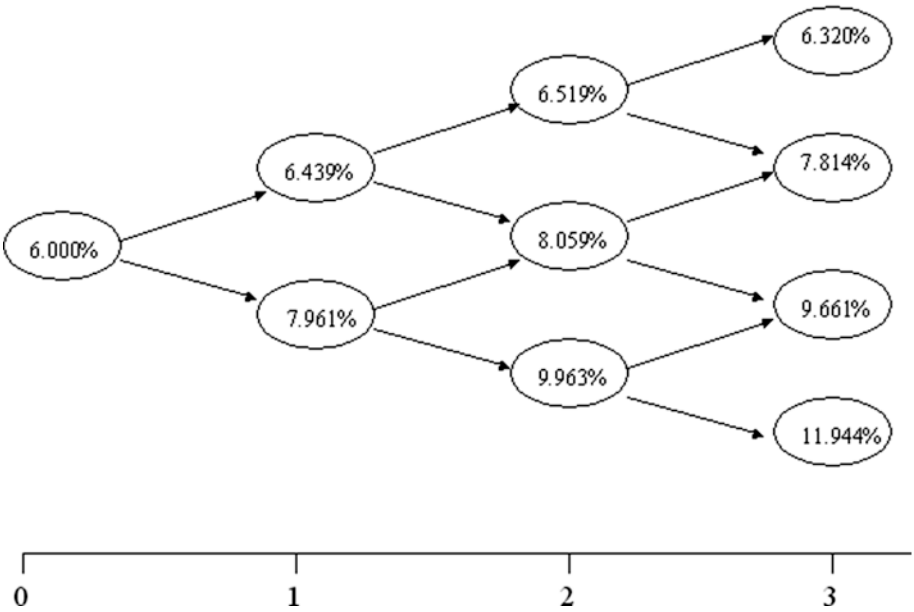

Fig. 10.2 The uncalibrated tree in the OAS model

and so on. This results in a tree with the following values
Time
Generally the rates are expressed as:

$$f_{n,1} \cdot \sum_{i=0}^{n-1} \binom{n-1}{i} \cdot Z_n^i = 2^{n-1} \cdot f_n \quad \Rightarrow \quad f_{n,1} \quad \Rightarrow \quad f_{n,2}, \ldots, f_{n,n}$$

In this example the volatility is constant for simplicity. Generally, the volatility will change by time.

10.1.3 Calibrate the Binomial Tree

Before the tree is used it will be calibrated with the market data. This calibration process involves raising (or lowering) the estimates of the rates in the tree by an amount just sufficient so that the value for the cash flows given by the tree exactly equals the values given by the discount function. As this is done, the relationship (equation 10.2) between the different nodes must be simultaneously preserved. First, the nodes are calibrated at time 1. Once this is finished, the nodes at time 2 are calibrated, and so on. At time 1 the following must hold

$$\left(\frac{1/2}{1+f_{2,1}\cdot(t_2-t_1)}+\frac{1/2}{1+Z_2\cdot f_{2,1}\cdot(t_2-t_1)}\right)\cdot\frac{1}{1+f_{1,1}\cdot(t_1-t_0)}=P(t_0,t_2)$$

The left side of this equation is the price of a cash flow equal 1 (with equal probabilities $1/2$ given by the tree, and the right side is the price of the same cash flow given by the discount function $P(t,T)$. The discount function discount any value from $t = t_2$ to $t = t_0$, where t_0 = valuation time. This equation is solved numerically by a Van Winjgaarden-Decker-Brent method. In the previous equation, the following relationship is used:

$$f_{2,2}=Z_2\cdot f_{2,1}$$

Therefore $f_{2,2}$ can be calculated as soon as $f_{2,1}$ is known.

At the next level, the following equation needs to be solved (note, it is not necessary to know the size of the cash flow).

$$\begin{aligned}&\frac{1}{2}\left\{\left(\frac{1/2}{1+Z_3^2\cdot f_{3,1}\cdot(t_3-t_2)}+\frac{1/2}{1+Z_3\cdot f_{3,1}\cdot(t_3-t_2)}\right)\cdot\frac{1}{1+f_{2,2}\cdot(t_2-t_1)}\right.\\&\quad\left.+\left(\frac{1/2}{1+Z_3\cdot f_{3,1}\cdot(t_3-t_2)}+\frac{1/2}{1+f_{3,1}\cdot(t_3-t_2)}\right)\cdot\frac{1}{1+f_{1,2}\cdot(t_2-t_1)}\right\}\\&\quad\cdot\frac{1}{1+f_{1,1}\,(t_1-t_0)}\\&=P(t_0,t_3)\end{aligned}$$

Solving this equation for $f_{3,1}$ also gives $f_{3,2}$ and $f_{3,3}$ from the relations $f_{3,2}=Z_3\cdot f_{3,1}$ and $f_{3,3}=Z_3\cdot f_{3,2}$. Using the same method for cash flows, at all times in the tree, the tree will be fully calibrated to produce the same value as the forward rates. The new calibrated tree is now given in Fig. 10.3.

The rates in the calibrated tree are compared with the rates from the un-calibrated. The reason for the previous calibration is shown in Fig. 10.4, where the error is caused by the bond's convexity.

Notice that the present value curve is not linear. The curvature represents *convexity*. The value of the cash flow, labelled the "calculated value" as mentioned earlier, is an average of the two values V_1 and V_2. Note that this average is higher than the actual value. After the calibration, the situation is described in Fig. 10.5.

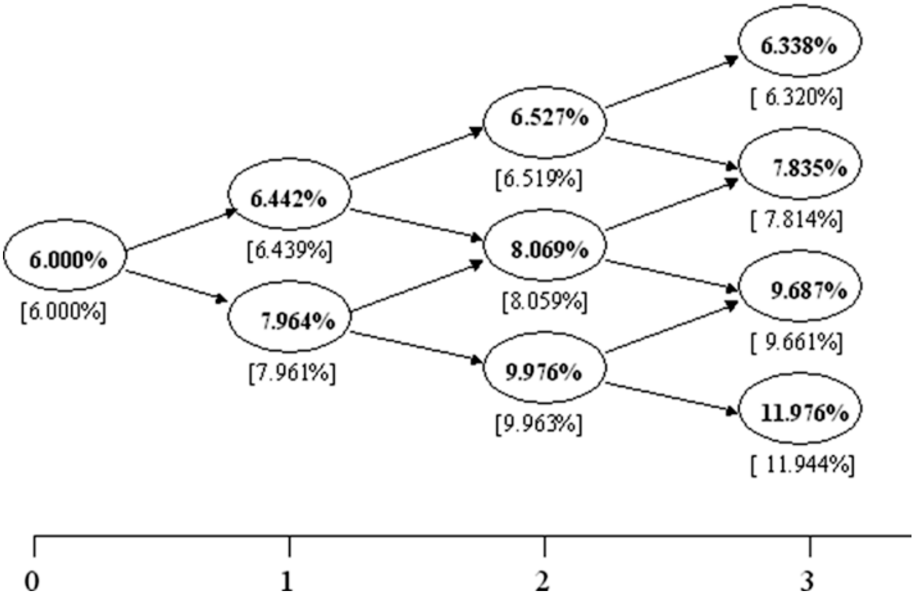

Fig. 10.3 The calibrated tree in the OAS model

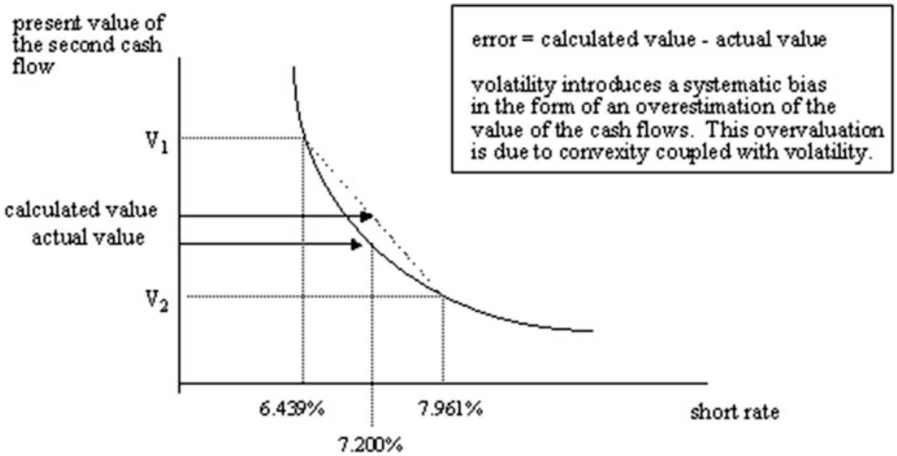

Fig. 10.4 Explanation of the reason to calibrate the OAS model

10.1.4 Calibrate the Tree With a Spread

The calibrated binomial tree just derived is applicable to valuing a benchmark bullet instrument. Now, consider how the same, calibrated tree could be adapted to value a non-benchmark (corporate) callable bond. To simplify the analysis, it is assumed that a corporation incurs

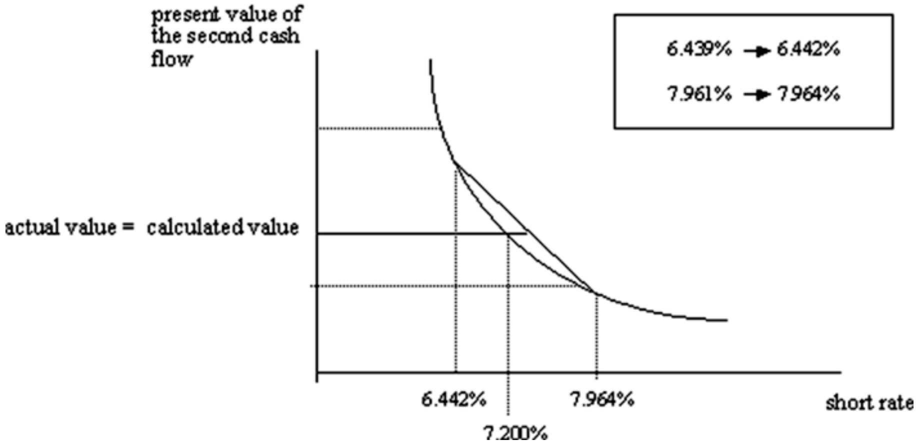

Fig. 10.5 The values after the calibration in the OAS model

no transaction costs either when it calls a bond or when it issues a new bond, and that it will always call a bond if it is rational to do so.

A general pricing formula at zero spread paying cash flows C_1, $C_2, \ldots C_n$ at time $T_1, T_2, \ldots T_n$ is given by

$$\Pi(0,0) = \sum_{i=1}^{n} C_i \left\{ \prod_{j=1}^{i} \frac{1}{\left(1 + f_j\right)^{T_j - T_{j-1}}} \right\}$$

With a shift $s (s \neq 0)$ in the rate f_j, the price is given as:

$$\Pi(s,0) = \sum_{i=1}^{n} C_i \left\{ \prod_{j=1}^{i} \frac{1}{\left(1 + f_j + s\right)^{T_j - T_{j-1}}} \right\}$$

If the market price Π is given, the aforementioned formula can be applied with different spreads s, until the spread that equals the market price is found. This spread is called the implied spread. When using tree models, the same spread is applied at all nodes.

Consider a 24-month corporate bond paying an annual coupon of 10.50% in two semi-annual instalments (each coupon is therefore \$5.25). The bond is callable in 18 months (period 3) at \$101.00. Suppose that the bond's offer price is \$103.75 –this is the price at which you could buy this bond. The goal is to derive this same value with the model. To get this value, a constant spread is added to all of the rates in the tree until the value of the bond cash flows equal the price of the callable corporate bond. In the calibration procedure we replace the values of the bond with the call value if the bond can be called back

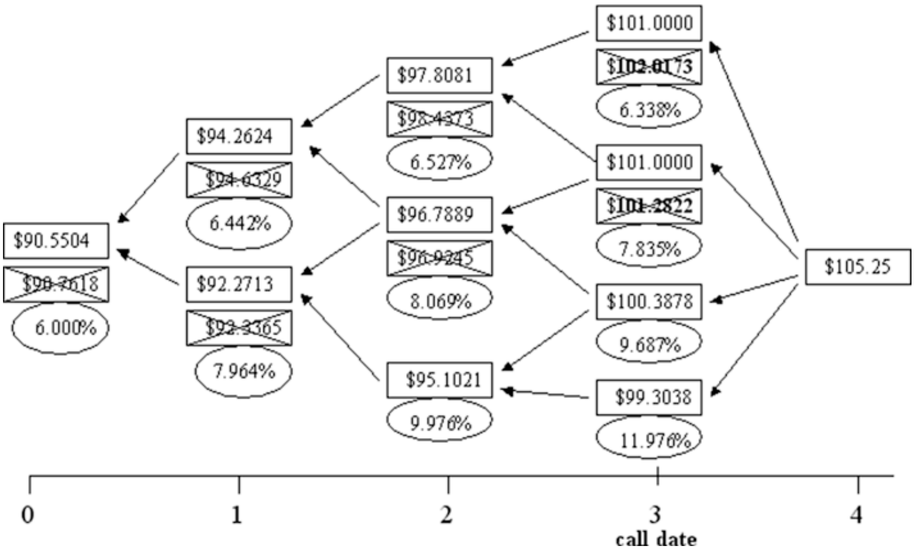

Fig. 10.6 The calculation of the final OAS cash flow

at this time, and the value at this point exceeds the call value – this is shown for the final cash flow in Fig. 10.6.

The same is done for all cash flows, and the sum of these is taken. Then the tree is adjusted to find a new shifted tree. The correct value for the callable corporate bond gives a spread of 90.465 basis points. This spread is called the bond's *option-adjusted spread* (OAS). Essentially, interpret the OAS is interpreted as the number of basis points that must be added to each and every rate in the calibrated binomial tree of risk-free short rates to obtain a model predicted price that precisely equals the observed market value of the bond. These basis points represent the risk premium for bearing the credit risk associated with the bond. The same sort of analysis could have been performed if the bond had contained an embedded put option.

If the market price is unknown, but the size of the spread is known, this spread can be used to find a reasonable price of the callable bond. It is also possible to simulate a price to find the corresponding OAS.

10.1.5 Using the OAS Model to Value the Embedded Option

Now, the OAS can be used to determine the value of the option that is embedded in a callable bond. To accomplish this task we ask, "what

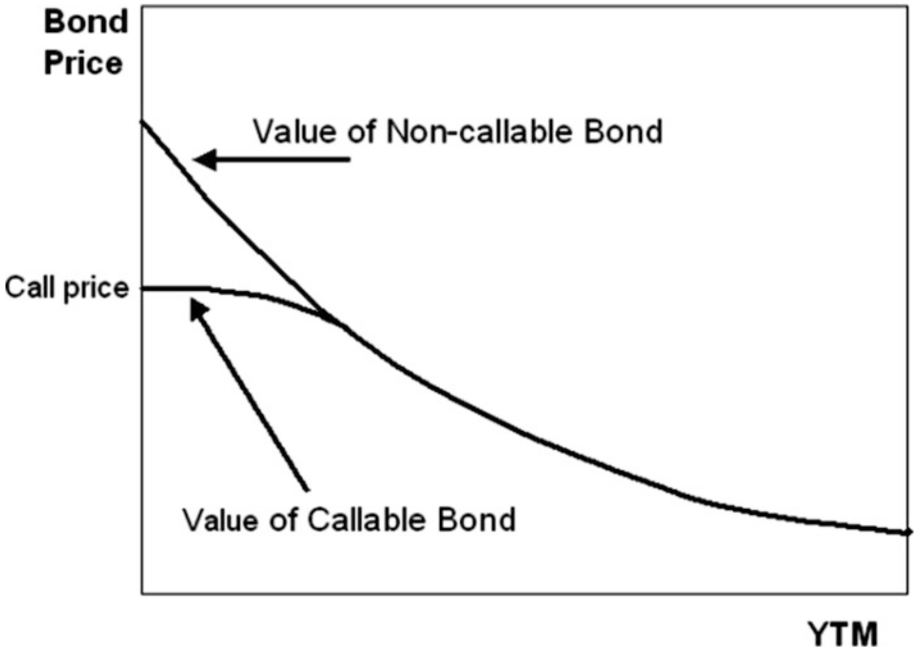

Fig. 10.7 The difference in price of a callable and a non-callable bond

would the value of the bond be *at the same OAS* if the bond had *not* been callable". In this case, the answer is \$103.8143.

A callable bond may be viewed as a portfolio consisting of a long position in a bullet bond and a short position in a call option on a bullet bond that begins on the option's call date. Therefore,

$$B_{callable} = B_{bullet} - C_{bullet}$$
$$103.7500 = 103.8143 - C_{bullet}$$

This implies that $C_{bullet} = 0.0643$

Therefore, the option is worth \$0.0643 for every \$100 of par. Because of the embedded option in a callable bond, the curve, bond price as function of YTM, will differ from the curve for a non-callable (bullet) bond. This is shown in Fig. 10.7.

10.1.6 Effective Duration and Convexity

Modified duration measures the percentage bond price change for an absolute yield change. It can also be interpreted as the negative slope of the price-yield relationship. The convexity can similarly be interpreted as the curvature of the price-yield relationship. Since duration and convexity do not consider that cash flows of an interest rate security with embedded option may change due to the exercise events, they do not provide satisfactory results for instruments with embedded options. Since a callable (or putable) instrument has cash flows that differ under different interest rate scenarios, it follows that the duration is a poor measure for these instruments. The OAS approach makes it possible to get a better measure of interest rate risk. These measures are called the bond's *effective duration* or *option-adjusted duration* and the bond's *effective convexity.*

The most intuitive way to calculate an effective duration is to first calculate the callable bond's fair value using the OAS approach (as done previously). Next, it is assumed that the benchmark yield curve shifts upward by exactly one basis point. The benchmark forward rates are then re-derived, as is the calibrated binomial tree of interest rates. With the new binomial tree the upward shifted value of the callable bond is calculated. Similarly, it is then assumed that the benchmark yield curve shifts downward by exactly one basis point, and the same values are recalculated as shown before. With this tree we calculate the downward shifted value of the bond.

The effective Macaulay duration and convexity is then given by

$$\mathit{Effective\,Duration} = \frac{P_- - P_+}{2P_0\,(\Delta y)}$$

and

$$\mathit{Effective\,Convexity} = \frac{P_+ + P_- - 2P_0}{P_0\,(\Delta y)^2}$$

where

P_- is the down shifted price
P_+ the up shifted price
P_0 the un-shifted price and
Δy the shift in the yield curve

If this technique is used for the corporate bond for which we calculated an OAS of 90.465 basis points, the effective duration will be 1.745 and the effective convexity 4.045. Without the embedded option the values are 1.782 and 4.166 respectively. In this case the differences are small, but for bonds with long maturity the difference between Modified and Effective Duration can be significant. (Fig. 10.2)

11

Stochastic Processes

11.1 Pricing Theory

Modern pricing models generally use one of two powerful approaches; *equilibrium pricing* or *relative pricing*. In an equilibrium framework, certain market characteristics, such as a price risk, are estimated and the model can be used to predict prices for securities in the market. There is no guarantee that the model will price any security at its observed market price. In the relative pricing framework, some observed market prices are used as a starting point, and then other securities are priced relative these.

We will now start to consider the particular problems that appear when we try to apply arbitrage theory to the bond market. The primary objects of investigation are zero coupon bonds, also known as pure discount bonds, with various maturities. All payments are assumed to be made in a fixed currency (e.g. US dollars). Previously the short interest rates have been considered to be deterministic. In reality the interest rates are stochastic. This makes the theory of interest rate difficult and interesting.

We will begin with some definitions and then discuss the stochastic processes concerning the theory of interest rates.

Definition 11.1.0.1. A *zero coupon bond* with maturity date T, also called a T-bond, is a contract which guarantees the holder 1 (dollar, sterling, kronor ...) to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $\mathrm{p}(\mathrm{t}, \mathrm{T})$ or $\mathrm{p}^{\mathrm{T}}(\mathrm{t})$.

The convention that the payment at the time of maturity, known as the **principal value**, **face value** or **nominal amount**, equals one,

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is made for computational convenience. Coupon bonds, which give the owner a payment stream during the interval $[0, T]$ are treated subsequently. These instruments have the common property that they provide the owner with a deterministic cash flow, and for this reason they are also known as **fixed income instruments**. The graph of $p(t, T)$ is called the **term structure of bond prices** at time t.

We assume the following:

- There exists a fix income market of T-bonds for all $T > 0$.
- $p(t, t) = 1$ for all times t.
- For a given $t, p(t, T)$ is differentiable with respect to T.
- At the bond market, $p(t, T)$ there exist an infinite number of securities.

We also define the derivative of the bond price $p(t, T)$ with respect to T as

$$p_T(t, T) = \frac{\partial p(t, T)}{\partial T}$$

A typical problem

We want to write a contract at time t that gives a deterministic interest rate in the interval $[S, T]$. We do this as:

1. At time t we sell one S-bond. This will give us $p(t, S)$ dollars.
2. We use this income to buy exactly $p(t, S)/p(t, T)T$-bonds. Thus our net
3. investment at time t equals zero.
4. At time S the S-bond matures, so we are obliged to pay out one dollar.
5. At time T the T-bonds mature at one dollar a piece, so we will receive the amount $p(t, S)/p(t, T)$ dollars.
6. The net effect of all this is that, based on a contract at t, an investment of one dollar at time S has yielded $p(t, S)/p(t, T)$ dollars at time T.
7. Thus, at time t, we have made a contract guaranteeing a risk-less rate of interest over the future interval $[S, T]$. Such an interest rate is called a **forward rate**.

11.1.1 Interest Rates

We will calculate relevant interest rates on the construction as shown earlier. We will use the **simple forward rate** ***L*** and the **continuous forward rate** ***R*** that solves:

$$1 + L(T - S) = \frac{p(t, S)}{p(t, T)}$$

and

$$e^{R(T-S)} = \frac{e^{-RS}}{e^{-RT}} = \frac{p(t, S)}{p(t, T)}$$

Definition 11.1.1.2. The *simple forward rate* for the period $[S, T]$ contracted at time t is defined by

$$L(t, S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)\, p(t, T)}$$

Definition 11.1.1.3. The *simple spot rate* for $[S, T]$ is defined by:

$$L(S, T) = -\frac{p(S, T) - p(S, S)}{(T - S)\, p(S, T)} = -\frac{p(S, T) - 1}{(T - S)\, p(S, T)}$$

Definition 11.1.1.4. For $t \leqslant S \leqslant T$ we define the *continuously compounded forward rate* for $[S, T]$ contracted at time t as

$$R(t, S, T) = -\frac{\ln [p(t, T)] - \ln [p(t, S)]}{T - S}$$

Definition 11.1.1.5. We define the *continuously compounded spot rate* for the period $[S, T]$ as

$$R(S, T) = -\frac{\ln [p(S, T)] - \ln [p(S, S)]}{T - S} = -\frac{\ln [p(S, T)]}{T - S}$$

Definition 11.1.1.6. Especial we define the *forward rate* for the period $[t, T]$ as

$$R(t, T) = R(t, t, T) = -\frac{\ln[p(t, T)]}{T - t}$$

Definition 11.1.1.7. The *instantaneous forward rate* with maturity at T contracted at t is defined as

$$\mathrm{f}^{\mathrm{T}}(\mathrm{t}) = \mathrm{f}(\mathrm{t},\mathrm{T}) = \lim_{\mathrm{S}\to\mathrm{T}} \mathrm{R}(\mathrm{t},\mathrm{S},\mathrm{T}) = -\frac{\partial\,[\ln \mathrm{p}(\mathrm{t},\mathrm{T})]}{\partial \mathrm{T}}$$

giving

$$\mathrm{p}(\mathrm{t},\mathrm{T}) = \exp\left\{-\int_{\mathrm{t}}^{\mathrm{T}} \mathrm{f}(\mathrm{t},\mathrm{u})\mathrm{du}\right\}$$

This is equivalent as, at time contracted at time t as agree to pay \$1 at time T and then *receive* $e^{f(t,T)} \cdot \Delta T$. In terms of FRA, this is at time t agree to pay \$1 at time T_0 and then at time T receive

$$\exp\left\{\int_{T_0}^{T} f(t,u)du\right\}$$

or to pay

$$\exp\left\{-\int_{T_0}^{T} f(t,u)du\right\}$$

at time T_0 and then receive \$1 at time T.

Definition 11.1.1.8. The *instantaneous short rate* at time t is then defined by

$$\mathrm{r}(\mathrm{t}) = \mathrm{f}(\mathrm{t},\mathrm{t})$$

We then have

$$\mathrm{p}(\mathrm{t},\mathrm{T}) = \exp\{-\mathrm{R}(\mathrm{t},\mathrm{T})\cdot(\mathrm{T}-\mathrm{t})\}$$

Before, we defined the **money account** by the process

$$\begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1 \end{cases}$$

giving

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}$$

Lemma 11.1.9. *The following holds for $t \leqslant s \leqslant T$:*

$$p(t,T) = p(t,s)\exp\left\{-\int_s^T f(t,u)du\right\} = \exp\left\{-\int_t^T f(t,u)du\right\}$$

When we study the interest rate market we have to start with something we know and depending on what choice we make, calculate what is unknown. Therefore we formulate the following questions:

1. If we let the dynamic of the short rate be given. Which bond prices $p(t,T)$ is consistent with the choice of r? Will the bond prices be uniquely given by r? Will these be free of arbitrage?
2. Which internal conditions do the bond prices $\{p^T; T \geqslant 0\}$ have to satisfy to have an arbitrage free money market?
3. Which internal conditions do the family of forward rates $\{f^T; T \geqslant 0\}$ have to satisfy to have an arbitrage free money market?
4. What can we say about the prices of different derivatives on an arbitrage-free money market?

To summarize what we have defined before, if we plot the interest rates, they form the term structure of interest rates or yield curve. We can represent the yield curve in three different but equivalent ways.

1. The first representation is by the prices of pure discount bonds (sometimes called zero-coupon bonds) that give the holder a single unit cash flow (e.g. one dollar) at maturity with no intermediate cash flows. We defined previously the function $p(t,T)$ to be the price, at time t, of a discount bond which matures at time T, with $t \leqslant T, (p(T,T) = 1)$. Remark! This is equivalent to the discount function defined earlier.
2. We can also represent the term structure by associating the continuously compounded spot rate $R(t,T)$ (sometimes called par yield)

with the pure discount bond price $p(t, T)$:

$$p(t, T) = e^{-R(t,T)(T-t)}$$

Inverting this equation we obtain

$$R(t, T) = -\frac{\ln[p(t, T)]}{T - t}$$

3. The third formulation is in terms of the forward rate curve, $f(t, T)$. This function represents at time t, the instantaneously maturing interest rate at time T and is derived from the discount bond function by applying the following transformation:

$$f(t, T) = -\frac{\partial\left[\ln p(t, T)\right]}{\partial T}$$

Combining the aforementioned equations, we can write the price of a pure discount bond as the final cash flow discounted by the instantaneous forward rates

$$p(t, T) = \exp\left\{-\int_t^T f(t, u)du\right\}$$

and the spot rate as the continuous average of forward rates:

$$R(t, T) = \frac{1}{T - t}\left(\int_t^T f(t, u)du\right)$$

For each of these rates, or prices, we associate a volatility. The function that describes these volatilities we call the term structure of interest rate volatilities. In terms of spot rates, a typical volatility structure exhibits short-term interest rates that are more volatile than longer-term interest rates an empirical feature of most markets. The effect is illustrated in Fig. 11.1.

The discount function is related to the bond prices as $D(T) = p(0, T)$, which is the value of $1 paid at time T.

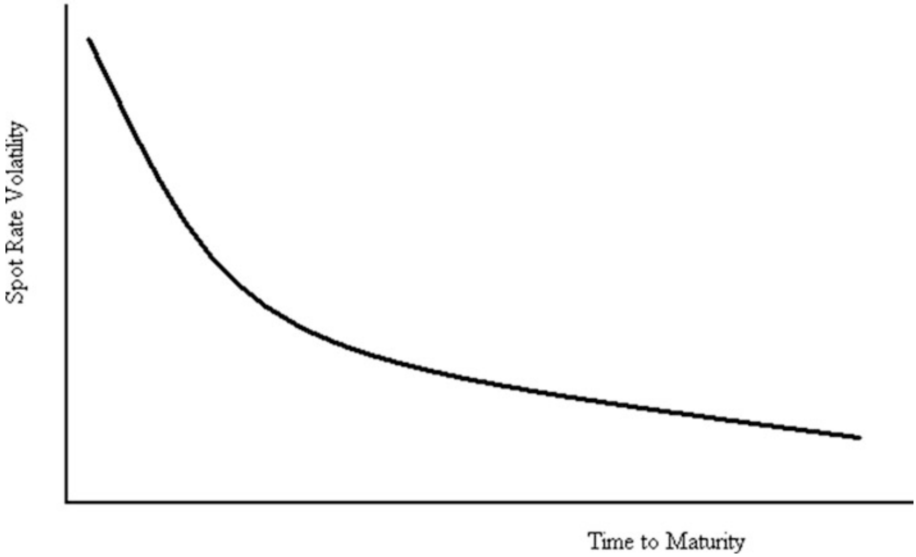

Fig. 11.1 The volatility as function of time-to-maturity

11.1.2 Stochastic Processes for Interest Rates

From now on, we will think of a Wiener process W on a filtrated probability space $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}},)$ to generate the uncertainty. It will then be naturally to specify objects via Itô equations. $\underline{\mathcal{F}}$ is the natural filtration generated by the Wiener process and F the σ-algebra containing all the information on the sample space Ω.

We want to consider the stochastic processes for the short rate, the forward rate and the bond prices as follows

$$dr(t) = \mu(t)dt + \sigma(t)dW(t),$$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \text{ and}$$

$$dp(t,T) = m(t,T)p(t,T)dt + \nu(t,T)p(t,T)dW(t)$$

Here, μ and σ are adapted processes, defined for all times $t \geqslant 0$. For each fixed $T, m(t,T), \nu(t,T), \alpha(t,T)$ and $\sigma(t,T)$ are adapted processes for $0 \leqslant t \leqslant T$. We will also suppose that all the previous processes are continuous in t and two times differentiable. Further, we suppose that $\nu(T,T) = 0$ for all T. This seems to be OK since $p(T,T) = 1$ by definition.

We then have three choices. We can start with

- the dynamics of the short rate dr,
- the dynamics of the forward rates df^T or by
- the dynamics of the bond prices dp^T

We are now ready to study how these are related to each other.

11.1.2.1 The Relation from Bond Prices (dp^T) to Forward Rates (df^T)

We will start with the dynamics of the bond prices to see how this process is related to the process of the forward rates.

Therefore we start with

$$dp(t,T) = m(t,T)p(t,T)dt + \nu(t,T)p(t,T)dW(t)$$

If we integrate this process we get

$$p(t,T) = p(0,T) + \int_0^t p(u,T)m(u,T)du + \int_0^t p(u,T)\nu(u,T)dW(u)$$

Now, we take the derivative of this, and believe we can take the derivatives inside both of the integrals.

$$\begin{aligned} p_T(t,T) = p_T(0,T) &+ \int_0^t \{p_T(u,T)m(u,T) + p(u,T)m_T(u,T)\}\, du \\ &+ \int_0^t \{p_T(u,T)\nu(u,T) + p(u,T)\nu_T(u,T)\}\, dW(u) \end{aligned}$$

We then see that the stochastic differential of $p_T(t,T)$ is given by:

$$dp_T^T(t) = \left\{p_T^T(t)m^T(t) + p^T(t)m_T^T(t)\right\} dt + \left\{p_T^T(t)\nu^T(t) + p^T(t)\nu_T^T(t)\right\} dW(t)$$

We now use the definition of the instantaneous forward rate with maturity at T:

$$f^T(t) = -\frac{\partial\left[\ln p(t,T)\right]}{\partial T} \equiv -\frac{p_T(t,T)}{p(t,T)}$$

Set $p^T = p$ and so on, and use the Itô formula on $f = f^T$, we then get

$$df = \frac{\partial f}{\partial p_T} dp_T + \frac{\partial f}{\partial p} dp + \frac{1}{2}\frac{\partial^2 f}{\partial p_T^2}(dp_T)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial p^2}(dp)^2 + \frac{\partial^2 f}{\partial p \partial p_T} dp dp_T$$

The derivatives are given as

$$\frac{\partial f}{\partial p_T} = -\frac{1}{p}, \quad \frac{\partial^2 f}{\partial p_T^2} = 0, \quad \frac{\partial f}{\partial p} = \frac{p_T}{p^2}, \quad \frac{\partial^2 f}{\partial p^2} = -2\frac{p_T}{p^3} \quad \text{and} \quad \frac{\partial^2 f}{\partial p \partial p_T} = \frac{1}{p^2}$$

That is,

$$\begin{aligned} df &= -\frac{1}{p} dp_T + \frac{p_T}{p^2} dp - \frac{1}{2} 2\frac{p_T}{p^3}(dp)^2 + \frac{1}{p^2} dp dp_T \\ &= -\frac{1}{p} dp_T + \frac{p_T}{p^2} dp - \nu^2 p^2 \frac{p_T}{p^3} dt + \frac{1}{p^2}\left\{p_T p \nu^2 + p^2 \nu_T \nu\right\} dt \\ &= \nu_T \nu dt - \frac{1}{p} dp_T + \frac{p_T}{p^2} dp \end{aligned}$$

where we have used

$$(dp)^2 = \nu^2 p^2 dt.$$

Since we just calculated dp_T and we know dp, we can just multiply the two expressions. To the lowest order we get:

$$dp_T dp = \nu p \{p_T \nu + p \nu_T\} dt = \left\{p_T p \nu^2 + p^2 \nu_T \nu\right\} dt$$

By putting these into the expression of df we find

$$\begin{aligned} df &= \nu_T \nu dt - \frac{1}{p}\{\{p_T m + p m_T\} dt + \{p_T \nu + p \nu_T\} dW\} + \frac{p_T}{p^2}\{m p dt + \nu p dW\} \\ &= \left\{\nu_T \nu - \frac{p_T}{p} m - m_T + \frac{p_T}{p} m\right\} dt + \left\{-\frac{p_T}{p}\nu - \nu_T + \frac{p_T}{p}\nu\right\} dW \\ &= \{\nu_T \nu - m_T\} dt - \nu_T dW = \alpha dt + \sigma dW \end{aligned}$$

where

$$\alpha(t, T) = \nu_T(t, T)\nu(t, T) - m_T(t, T)$$

and

$$\sigma(t,T) = -\nu_T(t,T)$$

To summarize with a known stochastic process for the bond prices:

$$dp(t,T) = m(t,T)p(t,T)dt + \nu(t,T)p(t,T)dW(t)$$

We can find the stochastic process for the forward prices $df(t,T)$:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)rdW(t)$$

where

$$\begin{aligned}\alpha(t,T) &= \nu_T(t,T)\nu(t,T) - m_T(t,T)\\ \sigma(t,T) &= -\nu_T(t,T)\end{aligned}$$

11.1.2.2 The Relation from Forward Rates (df^T) to Short Rates (dr)

We will now start with the dynamics of the forward rates to see how this process is related to the process of the short rates.

Therefore we start with the stochastic process

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

If we integrate this process we get

$$f(t,T) = f(u,T) + \int_u^t \alpha(s,T)ds + \int_u^t \sigma(s,T)dW(s)$$

By using the definition $r(t) = f(t,t)$ and set $T = t$ and $u = 0$ we get:

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \int_0^t \sigma(s,t)dW(s)$$

But, remember that this is not a standard form of a stochastic differential since the processes depends on the integration limits. To overcome this difficulty, we write

$$\alpha(s,t) = \alpha(s,s) + \int_s^t \alpha_T(s,u)du$$

and

$$\sigma(s,t) = \sigma(s,s) + \int_s^t \sigma_T(s,u)du$$

To see this, imagine the integral

$$\int_s^t \alpha_T(s,u)du = \alpha(s,t) - \alpha(s,s)$$

Put these expressions into the integral for $r(t)$

$$r(t) = r(0) + \int_0^t f_T(0,s)ds + \int_0^t \alpha(s,s)ds$$
$$+ \int_0^t \int_s^t \alpha_T(s,u)duds + \int_0^t \sigma(s,s)dW(s) + \int_0^t \int_s^t \sigma_T(s,u)dudW(s)$$

Change the order of integration

$$r(t) = r(0) + \int_0^t f_T(0,s)ds + \int_0^t \alpha(s,s)ds + \int_0^t \int_0^u \alpha_T(s,u)dsdu +$$
$$+ \int_0^t \sigma(s,s)dW(s) + \int_0^t \int_0^u \sigma_T(s,u)dW(s)du$$

We can illustrate the change in the order of integration in Fig. 11.2. This explains the change in the integration limits. Before we change the order, u goes from 0 to t. Then s starts at u on the line $u = s$. In the next graph we have changed the order and then, when s goes from 0 to t, u does the same (i.e. from 0 to t).

At last, we use the process of the short rate

$$dr(t) = \mu(t)dt + \sigma(t)dW(t)$$

and integrate to find

$$r(t) = r(0) + \int_0^t \mu(u)du + \int_0^t \sigma(u)dW(u)$$

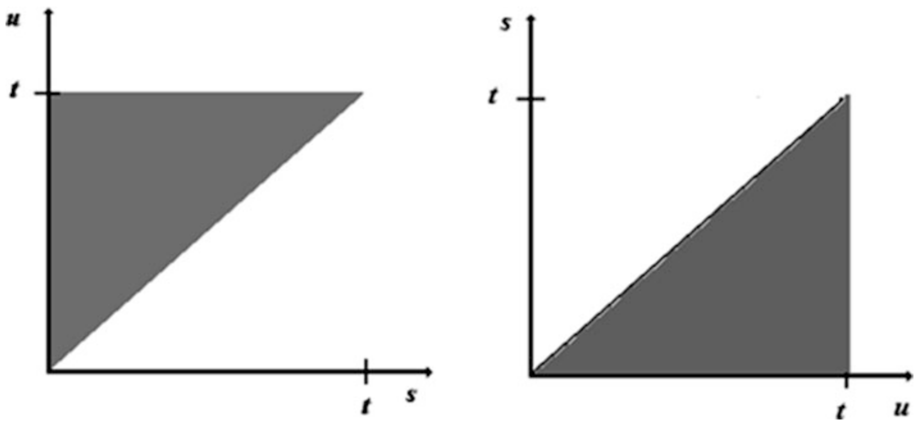

Fig. 11.2 The change in order of integration

Comparing with the previous expression we see that

$$\begin{cases} \mu(t) = f_T(0,t) + \alpha(t,t) + \int_0^t \alpha_T(s,t)ds + \int_0^t \sigma_T(s,t)dW(s) \\ \sigma(t) = \sigma(t,t) \end{cases}$$

If we use:

$$\begin{aligned} & f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s) \\ \Rightarrow & \\ & f_T(t,T) = f_T(0,T) + \int_0^t \alpha_T(s,T)ds + \int_0^t \sigma_T(s,T)dW(s) \end{aligned}$$

this can be simplified to

$$\begin{cases} \mu(t) = f_T(t,t) + \alpha(t,t) \\ \sigma(t) = \sigma(t,t) \end{cases}$$

11.1.2.3 The Relation From Forward Rates (df^T) to Bond Prices (dp^T)

We will start with the dynamics of the forward rates to see how this process is related to the process of the bond prices.

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

If we use the definition of the instantaneous forward rate with maturity at T

$$f^T(t) = -\frac{\partial\left[\ln p(t,T)\right]}{\partial T}$$

and write the bond prices as

$$p(t, T) = \exp\{Z(t, T)\}$$

where

$$Z(t,T) = -\int_t^T f(t,s)ds$$

Compare with the definition of the aforementioned bond price! We also know that

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)du + \int_0^t \sigma(u,s)dW(u)$$

We start by calculating $dZ(t,T)$

$$\begin{aligned}
dZ(t,T) &= f(t,t)dt - \int_t^T df(t,u)du = r(t)dt - \int_t^T \left[\alpha(t,u)dt + \sigma(t,u)dW(t)\right] du \\
&= r(t)dt - \int_t^T \alpha(t,u)dudt - \int_t^T \sigma(t,u)dudW(t) \\
&= \left\{ r(t) - \int_t^T \alpha(t,u)du \right\} dt - \left\{ \int_t^T \sigma(t,u)du \right\} dW(t)
\end{aligned}$$

where we have used

$$\frac{dZ(t,T)}{dt} = -\frac{\partial}{\partial t}\int_t^T f(\tau,u)du\,|_{\tau=t} - \int_t^T \frac{\partial}{\partial t} f(t,u)du = f(t,t) - \int_t^T df(t,u)$$

and

$$df(t,T) = \frac{\partial f(t,T)}{\partial t}dt + \frac{\partial f(t,T)}{\partial T}dT = \frac{\partial f(t,T)}{\partial t}dt$$

since we are studying one family of forward rates with maturity T. By using the Itô formula on $p(t,T) = \exp\{Z(t,T)\}$ we get:

$$\begin{aligned}
dp(t,T) &= \frac{\partial p}{\partial Z}dZ(t,T) + \frac{1}{2}\frac{\partial^2 p}{\partial Z^2}\left(dZ(t,T)\right)^2 \\
&= p(t,T)dZ(t,T) + \frac{1}{2}p(t,T)\left(dZ(t,T)\right)^2 \\
&= p(t,T)\left\{\left[r(t) - \int_t^T \alpha(t,u)du\right] + \frac{1}{2}\left(\int_t^T \sigma(t,u)du\right)^2\right\}dt \\
&\quad - p(t,T)\left\{\int_t^T \sigma(t,u)du\right\}dW(t)
\end{aligned}$$

This can be rewritten as

$$dp(t,T) = p(t,T)\{r(t) + b(t,T)\}\,dt + p(t,T)a(t,T)dW(t)$$

where we can identify $a(t,T)$ and $b(t,T)$

$$\begin{cases} a(t,T) = -\int_t^T \sigma(t,u)du \\ b(t,T) = -\int_t^T \alpha(t,u)du + \frac{1}{2}a(t,T)^2 \end{cases}$$

At last we have:

$$\begin{cases} m(t,T) = r(t) + b(t,T) \\ \nu(t,T) = a(t,T) \end{cases}$$

To summarize, we have that

- The forward rate $R(t,S,T)$ gives the average yield in the interval $[S,T]$ contracted at time t.
- The forward rate $f(t,T)$ gives the instantaneous yield at T contracted at time t.

- The short rate $r(t)$ gives the instantaneous yield of a T–bond contracted at time t. This is the yield of a portfolio strategy where we at each time t invest all of our Capital in the bond with immediately expiration.

The last strategy is called a **rollover-strategy** and its value process is given by:

$$dV(t) = V(t)u(t)\frac{dp(t,t)}{p(t,t)}$$

where $u(t)$ is the Capital at time t invested in the bond p^t. Per definition we have $u(t) = 1$ for all t and:

$$\frac{dp(t,t)}{p(t,t)} = \{r(t) + b(t,t)\}\,dt + a(t,t)dW(t)$$

But, since $a(t,t) = b(t,t) = 0$ we get

$$dV(t) = r(t)V(t)dt$$

The possibility to make a rollover-strategy on the bond market implies the existence of a local risk-free security with the stochastic yield given by $r(t)$

12

Term Structures

12.1 The Term Structure of Interest Rates

We will now consider the problem where we will model price processes on an arbitrage-free market of zero coupon bonds. On this market we will model the short rate, $r(t)$ under the real probability measure P. The process of the short rate will be given as

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

The only possibility to invest Capital is via roll-over:

$$dB(t) = r(t)B(t)dt$$

Therefore we can say that, to make rollover on the bank is equivalent by holding the security for which the price process is given by dB.
We now make the following assumptions:

- The interest rate $r(t)$ is a stochastic process.
- There exists only one security B with the aforementioned dynamic.
- All other securities are considered as derivatives of this ($r(t)$).

This means that we will consider a bond as an interest derivative where the value of the bond depends on the expectation of the future development of the short interest rate $r(t)$. We want to use arbitrage arguments to say something about the bond prices. It will be more difficult to analyse the market of interest rate derivatives than the simple Black-Scholes market we have been studying so far.

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Remark! According to the **Meta Theorem,** we have only one known security, B and one random source. Therefore **the market of interest rates is free of arbitrage but not complete.**

When we were studying the Black-Scholes market we did also know the price of the underlying stock. Therefore we can guess that, as soon we **know** the price of at least one bond, then we can price all other bonds relative this one, and the known security B. This is also true according to the Meta theorem.
We now suppose that we have one T-bond with a price given at t as:

$$p(t,T) = F(r(t), t, T) = F^T(r(t), t)$$

where F is a real function with tree real variables. Sometimes we will consider T as a parameter. We ask ourselves about the properties of the function F so that the Capital market is free of arbitrage. As we can see, we have a simple boundary condition

$$F(r, T, T) = 1 \text{ for all } r.$$

We will now create portfolios of bonds with different time to maturity T. Therefore we need the dynamics of the function F. By using the Itô formula we get

$$\begin{aligned} dF^T &= \frac{\partial F^T}{\partial t} dt + \frac{\partial F^T}{\partial r} dr + \frac{1}{2} \frac{\partial^2 F^T}{\partial r^2} (dr)^2 \\ &= F_t^T dt + F_r^T \{\mu dt + \sigma dW\} + \frac{1}{2} \sigma^2 F_{rr}^T dt \\ &= \left\{ F_t^T + \mu F_r^T + \frac{1}{2} \sigma^2 F_{rr}^T \right\} dt + \sigma F_r^T dW = F^T \alpha_T dt + F^T \sigma_T dW \end{aligned}$$

where

$$\begin{cases} \alpha_T = \dfrac{F_t^T + \mu F_r^T + \frac{1}{2} \sigma^2 F_{rr}^T}{F^T}, \\ \sigma_T = \dfrac{\sigma F_r^T}{F^T} \end{cases}$$

Let us now fix two times S and T and study self-financing portfolios based on bonds with maturities S and T. As usually the Capital of such a portfolio will be described by a value process given by

$$V = h^T F^T + h^S F^S$$

To become self-financing, we must have

$$dV = h^T \cdot dF^T + h^S \cdot dF^S$$

If we use relative portfolios we have

$$dV = V\left\{u^T\frac{dF^T}{F^T} + u^S\frac{dF^S}{F^S}\right\} = V\left\{u^T\alpha_T + u^S\alpha_S\right\}dt + V\left\{u^T\sigma_T + u^S\sigma_S\right\}dW$$

where

$$u^T + u^S = 1$$

Since we only have one random source (one Wiener process), we can make the following choice to eliminate the last bracket in dV in the previous equation, that is, we have

$$\begin{cases} u^T\sigma_T + u^S\sigma_S = 0 \\ u^T + u^S = 1 \end{cases}$$

then, after some algebra we get

$$dV = V\left\{\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S}\right\}dt$$

In a market free of arbitrage, we must have

$$dV = rVdt$$

That is,

$$\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S} = r$$

With some algebra

$$\begin{aligned} &\alpha_S\sigma_T - \alpha_T\sigma_S = r\left(\sigma_T - \sigma_S\right) \\ &\alpha_S\sigma_T - r\sigma_T = \alpha_T\sigma_S - r\sigma_S \\ &\sigma_T\left(\alpha_S - r\right) = \sigma_S\left(\alpha_T - r\right) \end{aligned}$$

We can also write this as

$$(*) \quad \frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t, r)$$

so, we do not have any dependencies between S and T. The function $\lambda(t)$ is called **the market price of risk**. We can see this from

$$dF^T = F^T \alpha_T dt + F^T \sigma_T dW = F^T \{r + \lambda\sigma_T\} dt + F^T \sigma_T dW$$

So, $\lambda(t, r)$ is a risk premium per unit of volatility. We measure the risk in volatility. If we insert the definitions of α_T and σ_T in $(*)$ we get

$$\alpha_T(t) - r(t) - \lambda(t, r)\sigma_T(t) = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T} - r(t) - \lambda(t, r)\frac{\sigma F_r^T}{F^T} = 0$$

$$\Rightarrow$$

$$F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - F^T r(t) - \lambda(t, r)\sigma F_r^T = 0$$

So, we get the partial differential equation

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + \{\mu(t, r) - \lambda(t, r)\sigma(t)\} \dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 F^T}{\partial r^2} - r(t)F^T = 0 \\ F(r, T, T) = 1 \end{cases}$$

This equation is, in the literature called the **equation of the term structure**[1] or the term structure equation (TSE). Remark! This is a Black-Scholes equation where we replaced μ with $\mu - \lambda\sigma$. However, this PDE is more complex since λ is a unknown function: $\lambda = \lambda(r(t), t)$.

12.1.1 Yield- and Price Volatility

In fixed income it is very important to distinguish between yield-volatility and price volatility. In the process for the short rate we have the volatility for the yield, and the process for the bond prices we have the volatility for the prices.

As we saw earlier, if we did start with a process for the rate as

$$dr(t) = \mu dt + \sigma_r dW(t)$$

---

[1] A better name is the Bond Pricing PDE, since the "Term" T is a fixed time and not a variable.

and let the zero coupon price be given by $F = F(t, T, r)$ and use Itô on F we get

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial r}dr + \frac{1}{2}\frac{\partial^2 F}{\partial r^2}(dr)^2 = F_t dt + F_r \{\mu dt + \sigma_r dW\} + \frac{1}{2}\sigma_r^2 F_{rr} dt$$
$$= \left\{F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr}\right\} dt + \sigma_r F_r dW = F\alpha dt + F\sigma_p dW$$

We saw before that the relationship between these volatilities follows

$$\sigma_p = \frac{\sigma_r F_r}{F} \equiv \frac{\sigma_r}{p(t,T)}\frac{\partial p(t,T)}{\partial r} \equiv \sigma_r \cdot D_{\text{mod}}$$

where D_{mod} is the modified duration. If the interest short rate is log-normal distributed, that is,

$$dr(t) = \mu \cdot r \cdot dt + \sigma_r \cdot r \cdot dW(t)$$

we would be given the following relationship

$$\sigma_p = r \cdot \frac{\sigma_r F_r}{F} \equiv r \cdot \frac{\sigma_r}{p(t,T)}\frac{\partial p(t,T)}{\partial r} \equiv r \cdot \sigma_r \cdot D_{\text{mod}}$$

This formula is often used by traders. The true relationship seems to be somewhere in between these results and depends on the time to maturity for the zero coupon bond.

12.1.1.1 Measuring Historical Yield Volatility

We know that volatility is measured in terms of the standard deviation or variance. To find the historical yield volatility we start with the daily data on yields. This can be from bonds quoted in *ytm* or other data of similar kind. We denote an interest rate on day t as y_t. It is important to choose the right number of days T that the volatility measure is going to be based on. Different number of observations would result in a different volatility estimate. Typically, portfolio managers with the longer investment horizon use a greater number of observations when calculating the volatility of interest rates.

We start by computing the daily relative yield change, X_t, assuming continuous compounding

$$X_t = 100 \cdot \ln \frac{y_t}{y_{t-1}}$$

Then we compute the daily standard deviation of yields

$$\sigma_{day} = \sqrt{Var\,(X_t)} = \sqrt{\frac{\sum_t \left(X_t - \bar{X}\right)^2}{T-1}}$$

where $\bar{X}$ is the mean value of X_t and T the number of measurements (maturity). Some market practitioners argue that in forecasting volatility the expected value or mean that should be used in the formula for variance is zero.

Next, we annualize the standard deviation of yields

$$\sigma_{annual} = \sigma_{day} \cdot \sqrt{D}$$

where D is the number of trading-days per year. Analysts can use different number of days in a year in this step, but the usual practice is to exclude holidays ($\sim$10 days a year) from calculations, so that the number of trading days is $5 \times 52 - 10$ holidays = 250 trading days.

How do we interpret yield volatility? Let us assume, for example, that the annualized interest rate volatility of a 5-year note is 10%. Further, let us assume that currently the yield on this note is 2%. The standard deviation of interest rates on this bond would then equal 10% x 2% = 0.2% (20 basis points). Having calculated this standard deviation, an analyst would be able to estimate the confidence interval for interest rates. For example, a 95% confidence interval can be estimated as $2\% + -1.96 \times 0.2\%$.

12.1.1.2 Historical Versus Implied Yield Volatility

The procedure for calculation of yield volatility described previously is based on historical yield data. Another approach is to derive volatility from the valuation of options, in which case it is called the implied volatility. We assume that the options are currently trading near their fair value and derive the yield volatility estimate from the option pricing model. Swaptions are usually quoted on the market as Black volatility.

There are several problems with using implied volatility.

- It is based on the assumption that the option pricing model is correct.
- Models make the simplifying assumption that volatility is constant.
- Options may not be fairly priced by the market, which results in a misleading estimate of implied volatility.

12.1.1.3 Forecasting Yield Volatility

There are three different approaches to forecasting the volatility of interest rates

- Yield volatility forecast equals the variance based on the last T days with the mean yield change assumed to be zero.
- Similar to the first approach, but the formula gives more weight to the more recent interest rate changes. More specifically, observations further in the past should be given less weight.
- Statistical models of time series, such as autoregressive conditional heteroskedasticity (ARCH) model, may also be employed to forecast yield volatility. The ARCH model can incorporate trends in volatility, such as the observation that periods of low volatility are followed by periods of high volatility and vice versa.

12.1.2 The Market Price of Risk

The TSE contains references to the functions $\mu - \lambda\sigma$ and σ. The former is the coefficient of the first-order derivative with respect to the spot rate, and the latter appears in the coefficient of the diffusive, second-order derivative. The four terms in the equation represent, in order as written, time decay, drift, diffusion and discounting. The equation is similar to the backward equation for a probability density function, except for the final discounting term. As such we can interpret the solution of the bond pricing equation as the expected present value of all cash flows. As with equity options, this expectation is not with respect to the real random variable, but instead with respect to the risk-neutral variable. There is this difference because the drift term in the equation is not the drift of the real spot rate μ, but the drift of another rate, called the risk-neutral spot rate. This rate has a drift of

$\mu-\lambda\sigma$. When pricing interest rate derivatives (including bonds of finite maturity) it is important to model, and price, using the risk-neutral rate. This rate satisfies

$$dr = (\mu - \lambda\sigma)dt + \sigma dW.$$

We need the new market-price-of-risk term because our modelled variable, r, is not traded. If we set λ to zero then any results we find are applicable to the real world. If, for example, if we want to find the distribution of the spot interest rate at some time in the future then we would solve a Fokker-Planck equation with the real, and not the risk-neutral, drift. Because, we cannot observe the function λ, except possibly via the whole yield curve.

12.1.3 Solutions to the TSE

The solution to a SDE as the TSE can be represented in an integral form in terms of the underlying stochastic process.

$$F(t,s) = E_t\left[\exp\left(-\int_t^s r(u)du - \frac{1}{2}\int_t^s \lambda^2(u,r(u))du - \int_t^s \lambda(u,r(u))dW(u)\right)\right]$$

To prove this, we define

$$V(s) = \exp\left(-\int_t^s r(u)du - \frac{1}{2}\int_t^s \lambda^2(u,r(u))du - \int_t^s \lambda(u,r(u))dW(u)\right)$$

Now, let us differentiate the process $F(t,s)V(t)$. Let $f = FV$ and then $df = d(FV)$

$$\begin{aligned} d(FV) &= \frac{\partial(FV)}{\partial F}dF + \frac{\partial(FV)}{\partial V}dV + \frac{\partial^2(FV)}{\partial V\partial F}dVdF = VdF + FdV + dFdV \\ &= V\left(F\left\{r + \lambda\sigma_T\right\}dt + F\sigma_T dW\right) + FdV \\ &\quad + \left(F\left\{r + \lambda\sigma_T\right\}dt + F\sigma_T dW\right)dV \end{aligned}$$

with

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial W}dW + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}(dW)^2 \\
&= V\left(-r - \frac{1}{2}\lambda^2\right)dt - V\lambda dW + \frac{1}{2}\lambda^2 V dt \\
&= -rVdt - \lambda V dW
\end{aligned}
$$

we get

$$
\begin{aligned}
d(FV) &= FV\left(\{r + \lambda\sigma_T\}dt + \sigma_T dW\right) + FdV \\
&\quad - FV\left(\{r + \lambda\sigma_T\}dt + \sigma_T dW\right)(rdt + \lambda dW) \\
&= FV\left(\{r + \lambda\sigma_T\}dt + \sigma_T dW\right) - FV(rdt + \lambda dW) - FV\sigma_T\lambda dt \\
&= FV(\sigma_T - \lambda)\,dW
\end{aligned}
$$

By integrating from t to s and taking expectation value[2] yields (the term dW will be zero)

$$E_t\left[F(s,s)V(s) - F(t,s)V(t)\right] = 0$$

Since $F(s,s) = 1$, $V(t) = 1$ $E_t\left[V(s) - F(t,s)\right] = 0$ and $F(t,s) = E[V(s)]$. TSE can also be solved using standard numerical methods such as finite difference methods. In several cases, analytical solutions also exist for the discount functions and European options. The only distinction between instruments is the boundary conditions. The equation is linear, so the superposition principle holds, that is, when all instruments in a portfolio fulfil the equation, the value of the portfolio also fulfils the equation.

In the first term structure models, the form of the functions $\mu(r,t)$ and $\sigma(r,t)$ was specified, containing several parameters that had to be estimated from historical data, or implied from market prices. Also, the market risk price parameter $\lambda(r,t)$ was specified as a single number.

This is a preference-dependent parameter, that is, it may be different from trader to trader. This means that there is no guarantee that the model generates a term structure that agrees with the observed term structure. This is a serious limitation for pricing option elements, since

[2] With expectation value, we always refer to the conditional expectation value, the conditional information known up to a certain time. This time is usual today, since we donot know anything about the future!

any small mispricing of the underlying instruments could mean major mispricing of the options.

Some authors still proposed trading strategies where all model parameters were derived from statistical analysis of historical prices. The strategy tries to find mispriced bonds, where the mispricing is likely to disappear.

12.1.4 Relative Pricing

To overcome these serious limitations of the early pricing models, Ho & Lee (1987) took a new approach. Their model assumed that the whole term structure followed a random evolution.

The model is still one-dimensional, since there is only one stochastic influence. They developed their model in a discrete time, binomial framework. A few years ago many said that this model had a serious disadvantage, since the Ho & Lee model are based on a stochastic evolution of the term structure that generate negative interest rates with positive probability. Now days we know that negative interest rates can occur.

Two papers, Jamshidan (1990) and Hull & White (1990) describe how to adjust the market price of the risk parameter $\lambda(r, t)$ in order to obtain consistency between model prices and the observed term structure of interest rates.

Their approach makes it possible to use the flexibility of the equilibrium models to specify stochastic processes together with the adjusted risk parameters, which generates a term structure consistent with what is observed.

The models resulted in a PDE, which can be solved by standard numerical techniques such as finite differences. Some of the models considered use normally distributed interest rates. In many cases, these models have analytical solutions for the discount function and for European options.

As usual, we can use the **Feynman-Kač representation** on the TSE:

$$(*) \quad F(r, t, T) = E_{t,r}^{Q}\left[\exp\left\{-\int_{t}^{T} r(s)ds\right\}\right]$$

where the Q-dynamics of r is given by

$$\begin{cases} dr(s) = \{\mu(s) - \lambda(s)\sigma(s)\}\, ds + \sigma(s)dV(s) \\ r(t) = r \end{cases}$$

<u>Conclusions</u>: The equation (∗) is incredible simple. If we write it as

$$F(r,t,T) = E^{Q}_{t,r}\left[\exp\left\{-\int_t^T r(s)ds\right\} \times 1\right]$$

we see that the bond price is given as the expectation value of \$1 (£1, 1 Kr. . .) paid at maturity T, discounted to a per cent value. The expectation value is calculated, not with respect to the objective probability measure P, but using the risk adjusted martingale measure Q that depends on $\lambda(t)$. That is, we get a new martingale measure for each $\lambda(t)$, so that the measure Q is not unique. This is because the model is not complete. In Black-Scholes world on the other hand, the martingale measure is unique and the model is complete. The interest market is **not** complete because we only have one given security.

The reason of having different martingale measures for different market prices of risk, $\lambda(t)$ is because of the reason that we can have many different markets, free of arbitrage and consistent with the short rate r. The bond prices on each market will depend on the liquidity and the traders will to enter risky positions. When we have a given market price of one bond, we know the market price of risk. Then we also know the prices of all other bonds.

The bond prices are therefore determined, partly of the P-dynamics of the short interest rate r and partly by the market. A general contingent claim $X = \Phi(r(t))$ is priced as

$$\Pi\,(t,X) = F\,(t,r\,(t)\,,T)$$

where

$$F(t,r,T) = E^{Q}_{t,r}\left[\exp\left\{-\int_t^T r(s)ds\right\} \Phi(r(T))\right]$$

13

Martingale Measures

13.1 Introduction to Martingale Measures

From now on, we will consider the filtrated probability space $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ as given where W is a $\underline{\mathcal{F}}$-Wiener process on $[0, T]$. The interpretation is that we consider an economy on $[0, T]$ where all randomness is generated by W. The time horizon is needed to perform a number of Girsanov transformations in the interval $[0, T]$

We start with the following assumptions:

1. For each $T \geq 0$ there exist an adapted price process $p(t, T)$ for T-bonds.
2. There exists a local risk-free security with the price process B given by:

$$dB(t) = r(t)B(t)dtB(0) = 1$$

   where the short rate is given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

3. There exist a probability measure $Q \sim P$ such as each Z^T-process is a Q-martingale on $[0, T]$, where the discounted bond prices Z^T is defined as

$$Z^T(t) = \frac{p(t, T)}{B(t)}$$

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_13

With the aforementioned assumptions, it easy to show that the bond prices have stochastic differentials. It is also possible to find the relations between the bond price and the short rate. First we notice that:

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}$$

Theorem 13.1.1. *With the previous assumption, we have for each fixed T*:

(i) *The bond prices for* $t \leq s \leq T$ *is given by*:

$$p(t,T) = E^Q\left[p(s,T)\exp\left\{-\int_t^s r(u)du\right\}|F_t\right]$$

Especial, with $s = T$, by

$$p(t,T) = E^Q\left[\exp\left\{-\int_t^T r(u)du\right\}|F_t\right]$$

(ii) *There exist adapted processes m(t, T) and v(t, T) such as:*

$$dp(t,T) = m(t,T)p(t,T)dt + v(t,T)p(t,T)dW(t)$$

(iii) *The Q-dynamics of p(t, T) is given by:*

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dV(t)$$

where V is a Q-Wiener process.

(iv) *The Q-dynamics of the forward rates f(t, T) is given by:*

$$df(t,T) = v(t,T)v_T(t,T)dt - v_T(t,T)dV(t)$$

Proof (i): Since Z^T is a Q-martingale, we have:

$$\frac{p(t,T)}{B(t)} = Z^T(t) = E^Q\left[Z^T(s)\,|F_t\right] = E^Q\left[\frac{p(s,T)}{B(s)}\,|F_t\right]$$

so

$$p(t,T) = E^Q\left[p(s,T)\frac{B(t)}{B(s)}\,|F_t\right] = E^Q\left[p(s,T)\exp\left\{-\int_t^s r(u)du\right\}|F_t\right]$$

Proof (iii): To prove this we will use the reverse of the Girsanov theorem, which says that Q has arisen from P via a Girsanov transformation:

$$dQ = L(T^*)dP \quad \text{on } \mathcal{F}_{T^*}$$

where

$$\begin{cases} dL(t) = \phi(t)L(t)dW(t) \\ L(0) = 1 \end{cases}$$

for some process $\phi(t)$. From Girsanov theorem we get

$$dW(t) = \phi(t)\boldsymbol{dt} + dV(t)$$

where V is a Q-Wiener process. By taking Itô on Z^T we get

$$\begin{aligned} dZ^T(t) &= \frac{\partial Z^T(t)}{\partial p(t,T)}dp + \frac{\partial Z^T(t)}{\partial B(t)}dB \\ &= \frac{1}{B(t)}\left\{p(t,T)m(t,T)\boldsymbol{dt} + p(t,T)\nu(t,T)dW(t)\right\} \\ &\quad - \frac{1}{B^2(t)}p(t,T)B(t)r(r)\boldsymbol{dt} \\ &= Z^T(t)\left\{m(t,T) - r(t)\right\}\boldsymbol{dt} + Z^T(t)\nu(t,T)dW(t) \\ &= Z^T(t)\left\{m(t,T) - r(t)\right\}\boldsymbol{dt} + Z^T(t)\nu(t,T)\left\{\phi(t)\boldsymbol{dt} + dV(t)\right\} \\ &= Z^T(t)\left\{m(t,T) - r(t) + \nu(t,T)\phi(t)\right\}\boldsymbol{dt} + Z^T(t)\nu(t,T)dV(t) \end{aligned}$$

With a choice of $\phi(t) = (r(t) - m(t,T))/\nu(t,T)$ we have the Q-dynamics

$$dZ^T(t) = \nu^T(t)Z^T(t)dV(t)$$

By definition, we have

$$p(t,T) = B(t)Z(t,T)$$

where $B(t)$ and $Z(t,T)$ have stochastic differentials under Q. Therefore we use Itô on the previous expression, and get (since the second order derivatives are zero)

$$
\begin{aligned}
dp(t,T) &= \frac{\partial p(t,T)}{\partial Z^T(t)} dZ^T(t) + \frac{\partial p(t,T)}{\partial B(t)} dB(t) \\
&= B(t)dZ^T(t) + Z^T(t)dB(t) \\
&= B(t)\nu^T(t)Z^T(t)dV(t) + Z^T(t)r(t)B(t)\boldsymbol{dt} \\
&= r(t)p(t,T)\boldsymbol{dt} + \nu^T(t)p(t,T)dV(t)
\end{aligned}
$$

This proves (iii). If we insert $dV(t)$ we get

$$
\begin{aligned}
dp(t,T) &= r(t)p(t,T)\boldsymbol{dt} + \nu^T(t)p(t,T)\left\{dW(t) - \phi(t)\boldsymbol{dt}\right\} \\
&= \left\{r(t) - \phi(t)\nu^T(t)\right\} p(t,T)\boldsymbol{dt} + \nu^T(t)p(t,T)dW(t)
\end{aligned}
$$

Therefore, under P, we have $m^P(t,T) = r(t) - \phi(t)\nu(t,T)$.

To **prove (iv)** we use (iii) which say that under $Q : m(t,T) = r(t)$. This gives

$$
\frac{\partial m(t,T)}{\partial T} = \frac{\partial r(t)}{\partial T} = 0
$$

The relation between dp^T and df^T was given via

$$
\begin{aligned}
dp(t,T) &= m(t,T)p(t,T)\boldsymbol{dt} + \nu(t,T)p(t,T)dW(t) \\
df(t,T) &= \alpha(t,T)\boldsymbol{dt} + \sigma(t,T)dW(t)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(t,T) &= \nu_T(t,T)\nu(t,T) - m_T(t,T) \\
\sigma(t,T) &= -\nu_T(t,T)
\end{aligned}
$$

which gives

$$
df(t,T) = \left\{\nu_T(t,T)\nu(t,T) - m_T(t,T)\right\} \boldsymbol{dt} - \nu_T(t,T)dW(t)
$$

$$\Rightarrow$$

$$
df(t,T) = \nu(t,T)\nu_T(t,T)\boldsymbol{dt} - \nu_T(t,T)dV(t)
$$

Here we have also used that $m(t,T) = r(t) \Rightarrow \phi(t) = 0 \Rightarrow dW(t) = dV(t)$ Therefore we have proved (iv).

The results (i) – (iii) are pretty expected. But (iv) is a little bit of surprise since this shows that under Q there must exist a relationship between the drift and diffusion for the forward rates. In other word: The dynamic of the forward rates under Q is uniquely determined by the diffusion coefficient. This will be essential in a later section where we will study the Heath-Jarrow-Morton framework.

Since we have

$$dF^T(t) = \{r(t) - \lambda(t)\sigma_T(t)\}\, F^T(t)dt + \sigma_T(t)F^T(t)dW(t)$$

where $F^T = p(t,T)$, we have that $\phi(t) = -\lambda(t)$.

Before we will show that the model is free of arbitrage, we will give some **definitions**.

Definition 13.1.0.2. A *portfolio strategy* is a finite adapted process h:

$$\mathrm{h(t)} = \left\{\mathrm{h}^0(\mathrm{t}), \mathrm{h}(\mathrm{t}, \mathrm{T}_1), \ldots, \mathrm{h}(\mathrm{t}, \mathrm{T}_\mathrm{n})\right\}$$

where by definition $\mathrm{h}(\mathrm{t}, \mathrm{T}_\mathrm{k}) = 0$ for $\mathrm{t} > \mathrm{T}_\mathrm{k}$. Furthermore:

$$\int_0^{\mathrm{T}^*} \left|\mathrm{h}^0(\mathrm{t})\right| \mathrm{dt} < \infty \qquad \mathrm{P\ a.s.}$$

$$\mathrm{E}^{\mathrm{Q}}\left[\int_0^{\mathrm{T}^*} \{\mathrm{h}(\mathrm{t}, \mathrm{T}_\mathrm{k})\mathrm{Z}(\mathrm{t}, \mathrm{T}_\mathrm{k})\}^2\, \mathrm{dt}\right] < \infty \quad \mathrm{k} = 1, \ldots \mathrm{n}$$

Definition 13.1.0.3. Given a portfolio strategy h, the *value process* $\mathrm{V(h)}$ is defined by:

$$\mathrm{V}_\mathrm{t}(\mathrm{h}) = \mathrm{h}^0(\mathrm{t})\mathrm{B}(\mathrm{t}) + \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{h}(\mathrm{t}, \mathrm{T}_\mathrm{k})\mathrm{p}(\mathrm{t}, \mathrm{T}_\mathrm{k})$$

Definition 13.1.0.4. A portfolio h is said to be *self-financing* if

$$\mathrm{dV}_\mathrm{t}(\mathrm{h}) = \mathrm{h}^0(\mathrm{t})\mathrm{dB}(\mathrm{t}) + \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{h}(\mathrm{t}, \mathrm{T}_\mathrm{k})\mathrm{dp}(\mathrm{t}, \mathrm{T}_\mathrm{k})$$

Definition 13.1.0.5. The class of self-financing portfolios is denoted by $\boldsymbol{H}$. A *contingent claim* is a stochastic variable X such as

$$\begin{aligned}&\mathrm{X} \text{ is } \mathcal{F}_{\mathrm{T}^*}\text{–measurable.}\\&\mathrm{E}^{\mathrm{Q}}[\mathrm{X}^2] < \infty\end{aligned}$$

The class of contingent claims is denoted by $\boldsymbol{K}$. With $\boldsymbol{K}^+$ we refer to those $\mathrm{X} \in \boldsymbol{K}$ such as

$$\mathrm{P}(\mathrm{X} \geq 0) = 1, \quad \text{and} \quad \mathrm{P}(\mathrm{X} > 0) > 0$$

Definition 13.1.0.6. A contingent claim is said to be *reachable* on $[0, \mathrm{T}]$ if there exist a self-financing strategy h such as

$$\mathrm{V}^{\mathrm{T}}(\mathrm{h}) = \mathrm{X}, \qquad \mathrm{P}\text{–a.s.}$$

Definition 13.1.0.7. A self-financing strategy h is said to be an *arbitrage strategy* if there exist a time T such as

$$\mathrm{V}^{\mathrm{T}}(\mathrm{h}) \in \boldsymbol{K}^+, \text{ and } \mathrm{V}^0(\mathrm{h}) = 0$$

With the earlier definitions, the number of T-bonds in the portfolio at the time t is given by $\mathrm{h}(\mathrm{t}, \mathrm{T})$ and $\mathrm{h}^0(\mathrm{t})$ the number of the risk-free security. Due to our definition, we have at $\mathrm{t} = 0$ decide the number of possible bonds in our portfolio. The rollover strategy discussed in an earlier section is not allowed in the previous portfolio strategy. But we will still consider the short rate r in terms of the rollover. As before we can move to the discounted Z-economy to show that the model is free of arbitrage.

Lemma 13.1.8. *For a given portfolio strategy h, we define $V^Z(h)$ as*

$$V_t^Z(h) = h^0(t) + \sum_{k=1}^{n} h(t, T_k)Z(t, T_k)$$

Then

$$V_t(h) = B(t)V_t^z(h)$$

The strategy is self-financed if and only if

$$dV_t^Z(h) = \sum_{k=1}^{n} h(t, T_k)dZ(t, T_k)$$

If h is self-financed, then V^Z becomes a Q-martingale.

Theorem 13.1.9. *With the assumptions* 1, 2 *and* 3*, the model is free of arbitrage.*

Theorem 13.1.10. *With the assumptions* 1, 2 *and* 3*, $v(t, T) \neq 0$ for all (t, T) with* $0 \leq t \leq T$*. Then*:

(i) *The money market is complete, that is, each contingent claim is reachable via an self-financing portfolio. More precise, if X is a contingent claim $X \in \mathcal{F}^T$, then it is possible to replicate X with a portfolio of T-bonds only and the risk-free security.*

(ii) *For X as given earlier, the arbitrage free price is given by:*

$$\pi_t[X] = E^Q\left[X \cdot \exp\left\{-\int_t^T r(u)du\right\} |\mathcal{F}_t\right]$$

Proof: If X is reachable via a portfolio h we know that V^T is a Q-martingale. Then:

$$V_t^Z(h) = E^Q\left[V_T^Z(h)\,|\mathcal{F}_t\right] = E^Q\left[\frac{V_T(h)}{B(T)}\,|\mathcal{F}_t\right] = E^Q\left[\frac{X}{B(T)}\,|\mathcal{F}_t\right]$$

so

$$\begin{aligned} V_t(h) &= B(t)V_t^Z(h) = B(t)E^Q\left[\frac{X}{B(T)}\,|\mathcal{F}_t\right] \\ &= E^Q\left[X\frac{B(t)}{B(T)}\,|\mathcal{F}_t\right] = E^Q\left[X \cdot \exp\left\{-\int_t^T r(u)du\right\} |\mathcal{F}_t\right] \end{aligned}$$

Theorem 13.1.11. *Suppose r given on $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$. Then, there exist an infinite number of arbitrage-free term structures for this r. More precisely, for each Girsanov kernel ϕ, the bond prices can be defined by:*

$$p(t, T) = E^Q\left[\exp\left\{-\int_t^T r(u)du\right\} |\mathcal{F}_t\right]$$

where Q is defined by

$$df(t, T) = v(t, T)v_T(t, T)dt - v_T(t, T)dV(t)$$

and

$$\begin{cases} dL(t) = \phi(t)L(t)dW(t) \\ L(0) = 1 \end{cases}$$

The term structure is then free of arbitrage. Furthermore, the Girsanov kernel ϕ is related to the market price of risk, λ such as $\lambda = -\phi$. That is, λ is the Girsanov kernel for the transformation from Q to P.

Theorem 13.1.12. *Suppose the short rate r on the martingale measure Q solves the SDE*

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dV(t)$$

and let X be a contingent claim: $X = \Phi[r(T)]$. *Then, the price of X on Q is*

$$\pi_t[X] = F[t, r(t)]$$

where F is a solution to the PDE

$$\begin{cases} \dfrac{\partial F(t,x)}{\partial t} + \mu(t,x)\dfrac{\partial F(t,x)}{\partial x} + \dfrac{1}{2}\sigma^2(t,x)\dfrac{\partial^2 F(t,x)}{\partial x^2} - xF(t,x) = 0 \\ F(T,x) = \Phi(x) \end{cases}$$

To calculate arbitrage prices via a PDE, r has to be a Markov process on the martingale measure Q. r is a Markov process from the suggestions:

(i) We supposed that r on P was a solution to a SDE.

(ii) We supposed that the market price of risk was a function of time and interest rate.

14

Pricing of Bonds

14.1 Bond Pricing

As we have seen the price of a zero coupon bond at $t = 0$ and time to maturity T is given by

$$p(0,T) = E_{t,r}^{Q}\left[\exp\left\{-\int_{0}^{T} r(s)ds\right\}\right]$$

Therefore we can write the price of a coupon-bearing bond as

$$\begin{aligned}
B(0,T) &= \sum_{n:t_n \geqslant 0}^{M} \frac{C \cdot N}{\omega} E_{t,r}^{Q}\left[\exp\left\{-\int_{0}^{t_n} r(s)ds\right\}\right] \\
&\quad + N \cdot E_{t,r}^{Q}\left[\exp\left\{-\int_{0}^{T} r(s)ds\right\}\right] \\
&= \frac{C \cdot N}{\omega} \sum_{n:t_n \geqslant 0}^{M} p(0,t_n) + N \cdot p(0,T)
\end{aligned}$$

where N is the nominal amount, C the coupon rate, ω the coupon frequency, M the number of coupons and t_n the cash-flow dates. We use $p(0, t_n)$ as discount factors so the value of a bond on a particular date is completely determined by the discount curve at that date. We notice that at each time t_n, the bond price has a jump of size CN/ω.

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Thus, the value of the bond changes discontinuously. We can make the "price" continuous if we subtract the accrued interest rate AI. The usual convention is to let the coupon accrues linearly between the payouts. This accrued interest rate is the earned rate by the bondholder.

$$AI(t, t_n) = \frac{C \cdot N}{\omega} \frac{t - t_n}{t_{n+1} - t_n}$$

By definition, the **clean price** of a bond corresponds to the price at which the transaction takes place without including accrued interest. The **dirty price** is the price including accrued interest, that is, how much money trades hands (so to speak). Hence,

$$Dirty\ price = Clean\ price + AI\,(t,\ t_n)\,.$$

In an arbitrage-free economy, the dirty price should be equal to the theoretical value. In particular, the theoretical clean price can be expressed in terms of the term structure of interest rates as

$$B(0, T) = \frac{C \cdot N}{\omega} \sum_{n:t_n \geqslant 0}^{M} p(0, t_n) + N \cdot p(0, T) - AI(0, t_n)$$

The clean and dirty prices coincide on the coupon date after the coupon is paid (since $AI(t_n, t_n) = 0$. Bond quotes in the US Treasury, international and corporate markets are usually in terms of clean prices.

The yield of a bond (or *ytm*) is the effective constant interest rate that makes the bond price equal to the future cash flows discounted at this rate. The *ytm* is usually computed using the same frequency as the bond's interest payments (e.g. semi-annual), rather than the continuously compounded yield used for zeros.

Assume that the current date coincides with a coupon payment date, so that $t = t_m$. In this case, we define the *ytm* of the bond (after the coupon was paid) to be the value of Y such that

$$B(0, T) = \frac{C \cdot N}{\omega} \sum_{n=m+1}^{M} \left(\frac{1}{1 + Y/\omega}\right)^{n-m} + N \cdot \left(\frac{1}{1 + Y/\omega}\right)^{N-m}$$

If the current date does not coincide with a coupon date, we should take into account the fraction of year corresponding to the period between now and the next coupon date.

Accordingly, assume that $t_m < t < t_{m+1}$ and that f represents the ratio of the number of days in remaining until the next coupon date and the number of days in the coupon period, using the appropriate day count convention. (Hence, $0 < f < 1$). The bond yield Y is defined by the relation

$$B(0,T) = \frac{C \cdot N}{\omega} \sum_{n=m+1}^{M} \left(\frac{1}{1 + Y/\omega} \right)^{f+n-m-1} + N \cdot \left(\frac{1}{1 + Y/\omega} \right)^{f+N-m-1}$$

The previous equations define Y implicitly in terms of the bond value. It is easy to see that B is a decreasing function of Y. Moreover, B is convex in Y. To obtain the yield from the bond value, the equations must be solved numerically. Nevertheless, the yield of a bond is a well-defined function of its theoretical value B (the dirty price) and thus of the discount factors.

Notice that if $t = t_m$ we can use the summation formula for a geometric series to obtain

$$B(0,T) = \frac{C \cdot N}{\omega} \left(1 - \left(\frac{1}{1 + Y/\omega} \right)^{N-m} \right) + N \cdot \left(\frac{1}{1 + Y/\omega} \right)^{N-m}$$

This formula shows that if the yield is equal to the coupon rate, the value of the bond is equal to its face value. From this fact and the monotonicity of the price/yield relationship, we can derive some elementary relationships between price, yield and coupon.

If, immediately after a coupon payment, a bond trades at 100% of the principal, we say that the bond trades at par. In this case, its yield is exactly equal to the coupon rate. If the bond price is less than 100% of face value, we say that the bond trades at a discount. In this case, its yield is higher than the coupon rate. If the bond trades above 100% of face value, we say that bond trades at a premium. In this case, the yield is lower than the coupon rate.

In an arbitrage-free market, two bonds with same price and same cash-flow dates cannot have different coupons (otherwise, we can short the one with the smaller coupon and buy the one with the larger one). Similarly, two bonds with the same price and payment dates cannot have different yields. The notion of **par yield** – the yield of a par bond –is sometimes used to represent the term structure of interest rates implied by the bond market. In this case, one speaks of the par yield curve.

14.1.1 Duration

The price-yield relation gives rise to several quantities that are commonly used in bond risk-management. The first notion is that of duration (or average duration, or McCauley duration) which is defined as

$$D = \frac{1}{B}\left(\frac{C \cdot N}{\omega}\sum_{n:t_n \geqslant 0}^{M} t_n \cdot p(0, t_n) + N \cdot T \cdot p(0, T)\right)$$
$$= \frac{\frac{C \cdot N}{\omega}\sum_{n:t_n \geqslant 0}^{M} t_n \cdot p(0, t_n) + N \cdot T \cdot p(0, T)}{\frac{C \cdot N}{\omega}\sum_{n:t_n \geqslant 0}^{M} p(0, t_n) + N \cdot p(0, T)}$$

Thus, the duration represents a weighted average of the cash-flow dates, weighted by the cash flows measured in constant dollars. Mathematically, it is the "barycentre" of the cash-flow dates.

A closely related quantity is obtained by differentiating the bond price with respect to the *ytm*

$$\frac{\partial B}{\partial Y} = -\frac{C \cdot N}{\omega}\sum_{n=m+1}^{M}\left(\frac{f+n-m-1}{\omega}\right)\left(\frac{1}{1+Y/\omega}\right)^{f+n-m} - N \cdot \left(\frac{f+N-m-1}{\omega}\right)\left(\frac{1}{1+Y/\omega}\right)^{f+N-m}$$

It follows from the definition of f, that $(f+n-m-1)$ represents the time between t and t_n measured in coupon periods ($1/\omega$ years). Therefore, the number $(f+n-m-1)/\omega$ represents the time interval between t and the n^{th} coupon date measured in years. We conclude that

$$\frac{1}{B}\frac{\partial B}{\partial Y} = -\frac{D}{1+Y/\omega}$$

Thus, the per cent sensitivity of the bond (dirty) price with respect to yield is of opposite sign and proportional to the average duration. The quantity

$$D_{\text{mod}} = \frac{D}{1+Y/\omega}$$

which represents the exact magnitude of the percentage change which is known as modified duration. These equations express the fact that the longer the duration, the greater the sensitivity of a bond to a change in yield, in percentage terms.

Clearly, a zero-coupon bond has duration equal to the time to maturity. The duration of a coupon-bearing bond trading at par (face value) immediately after the coupon date is

$$D = \frac{1}{\omega} \sum_{n=0}^{M-1} \frac{1}{(1 + Y/\omega)^n} = \frac{1}{Y} \cdot (1 + Y/\omega) \cdot \left(1 - \frac{1}{(1 + Y/\omega)^N}\right)$$

(The derivation of this formula is left as an exercise to the reader.) The formula shows that duration decreases with frequency. In fact, if the bond matures in T years and makes only a single payment, we have $N = 1, \omega = 1/T$. Substituting these values into the previous equation, we find $D = T$, the result for zeros. In the limit $\omega >> 1$, setting $N = \omega T$, we have $D = (1 - e^{-YT})/Y$.

The duration of a coupon-bearing bond is always smaller than the time-to-maturity, because far-away cash-flow dates are "discounted" more than nearby dates. We also get a formula for the modified duration of a par bond, which gives the price-yield sensitivity as

$$D_{\text{mod}} = \frac{1}{Y} \cdot \left(1 - \frac{1}{(1 + Y/\omega)^N}\right)$$

These formulas are useful for estimating the price-yield sensitivity of bonds. For example, if $N >> 1$ we can make the approximation $D_{mod} \approx 1/Y$. This approximation is exact for **perpetual** or **console bonds**, which are fixed income securities that pay a fixed coupon and have no redemption date. Because the maturity is infinite, the aforementioned formulas apply even if the console bond is not trading at par, by simply scaling the coupon. The modified duration of a console is exactly equal to $1/Y$. Moreover, it is easy to see that $Y = CN/B$.

Treasury bond prices are usually quoted in clean price or yield and bonds usually trade close to par (this is true for recently issued bonds). Historically, duration was introduced as a measure of the risk-exposure of a bond portfolio and hence as a hedging tool. The rationale for this is that if we assume that bond yields vary in the same direction and by the same amount, that is, if the yield curve shifts in parallel, we can measure the total exposure of a portfolio to a shift in the yield curve. In fact, a portfolio consisting of M bonds with n_1 dollars invested in bond, n_2 dollars invested in bond 2, etc., has, under the parallel shift assumption, a first-order variation with respect to yield of

$$\sum_j n_j \frac{dB_j}{B_j} = \left(\sum_j n_j D_{\text{mod } j} \right) dY$$

Thus, the sensitivity to a parallel shift in yields is equal to the dollar-weighted modified duration of the portfolio. A portfolio with vanishing dollar-weighted modified duration has no exposure to parallel shifts in the yield curve.

It has been recognized now for quite some time that duration-based hedging (under the explicit assumption of parallel shifts of the yield curve) is not precise enough to immunize a fixed income portfolio against interest rate risk. The reason is that yields of deferent maturities generally do not move together and by the same amount. Appropriate modelling of yield correlations is needed to produce efficient portfolio hedges and to correctly price fixed-income derivatives that are contingent on more than one yield. The modelling of yield correlations is an interesting subject.

15

Term-Structure Models

15.1 Martingale Models for the Short Rate

15.1.1 The *Q*-Dynamics

Let us again study an interest rate model where the *P*-dynamics of the short rate of interest are given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

As we saw in the previous section, the term structure (i.e. the family of bond price processes) will, together with all other derivatives, be completely determined by the general term-structure equation

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + \{\mu(t) - \lambda(t)\sigma(t)\}\dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 F^T}{\partial r^2} - r(t)F^T = 0 \\ F(r, T, T) = \Phi(r) \end{cases}$$

as soon as we have specified the following objects.

- The drift term μ.
- The diffusion term σ.
- The market price of risk λ.

Consider for a moment σ to be given a priori. Then it is clear from the term-structure equation that it is irrelevant exactly how we specify μ, and λ. The object, apart from σ, that really determines the term structure (and all other derivatives) is the term $\mu - \lambda\sigma$. Now, we recall

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that the term $\mu - \lambda\sigma$. is precisely the drift term of the short rate of interest under the martingale measure Q. This fact is so important that we stress it again.

The term structures, as well as the prices of all other interest rate derivatives, are completely determined by specifying the r-dynamics under the martingale measure Q.

Instead of specifying μ, and λ under the objective probability measure P we will henceforth specify the dynamics of the short-rate r directly under the martingale measure Q. This procedure is known as martingale modelling, and the typical assumption will thus be that r under Q has dynamics given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dV(t)$$

where μ, and σ are given functions. From now on the letter μ will thus always denote the drift term of the short rate of interest under the martingale measure. In the literature there are a large number of proposals on how to specify the Q-dynamics for r. We present a (far from complete) list of the most popular models. If a parameter is time dependent, this is written out explicitly. Otherwise all parameters are constant.

Vasicek

$$dr = (b - ar)\,dt + \sigma\,dV$$

Cox-Ingersoll-Ross (CIR)

$$dr = a\,(b - r)\,dt + \sigma\sqrt{r}dV$$

Dothan

$$dr = ardt + \sigma\,rdV$$

Black-Derman-Toy (BDT)

$$dr = a(t)rdt + \sigma(t)rdV$$

In some literature this SDE is written as:

$$d\ln(r) = \{\theta(t) + \rho(t)\ln(r)\}\,dt + \sigma(t)dV$$

where $\theta(t)$ will be shown to be the drift of the short-term rate and $\rho(t)$ the mean reversing term to an equilibrium short-term rate which depends on the interest rate local volatility as follows:

$$\rho(t) = \frac{d}{dt} \ln [\sigma(t)] = \frac{\dot{\sigma}(t)}{\sigma(t)}.$$

That is,

$$d \ln (r) = \left\{ \theta(t) + \frac{\dot{\sigma}(t)}{\sigma(t)} \ln (r) \right\} dt + \sigma(t) dV$$

Since the volatility is time dependent, there are two independent functions of time, $\theta(t)$ and $\sigma(t)$, chosen so that the model fits the term structure of spot interest rates and the structure of the spot rate volatilities.

Ho-Lee (HL)

$$dr = a(t)dt + \sigma(t)dV$$

Since dV is normally distributed Wiener process, this is a normal process for the short-term rate.

Hull-White (extended Vasicek) (HW)

$$dr = (b(t) - a(t)r)\, dt + \sigma(t)dV$$

In this model, where there is an extra term giving an additional degree of freedom. For that reason a trinomial tree can be used to model the stochastic process.

Hull-White (extended CIR)

$$dr = (b(t) - a(t)r)\, dt + \sigma(t)\sqrt{r} dV$$

The **Kalotay-Williams-Fabozzy** model (KWF), the short-rate dynamics is given by:

$$d \ln r = a(t)dt + \sigma(t)dV$$

This is a log normal process interest rate model, similar to the BDT.

The **Black-Karasinski** model (BK) the short-rate dynamics is given by:

$$d \ln r = (a - \theta \ln r)\, dt + \sigma dV$$

This is logarithmic analogue to the HW model. So for the same reason a tree model is used to model BK.

When we choose a short-rate model, we then have to consider the following questions:

1. What do the dynamics of the model imply for the short rate, r?
2. Is r positive at each time, t?
3. Are you dealing with a fat-tailed distribution?
4. Can the bond prices, $p(t, T)$ and the bond option prices be calculated explicitly?
5. Does the short rate tend towards a long-term mean?
6. How suited is the model for recombining lattices and Monte Carlo simulation?
7. Can historical estimation methods be used for parameter estimation?

15.1.2 Inverting the Yield Curve

Let us now address the question of how we will estimate the various model parameters in the previous martingale models. To take a specific case, assume that we have decided to use the Vasicek model. Then we have to get values for a, b and σ in some way, and a natural procedure would be to use SDE theory. This procedure, however, is unfortunately completely non-sensical and the reason is as follows.

We have chosen to model our r-process by giving the Q-dynamics, which means that a, b and σ are the parameters which hold under the martingale measure Q. When we observe in the real world we are not observing r under the martingale measure Q, but under the objective measure P. This means that if we apply standard statistical procedures to our observed data we will not get our Q-parameters. What we get instead is pure nonsense.

This looks extremely disturbing but the situation is not hopeless. It is in fact possible to show that the diffusion term is the same under P and under Q, so "in principle" it may be possible to estimate diffusion parameters using P-data. Since we are familiar with martingale theory, we will at this point recall that a Girsanov transformation will only affect the drift term of a diffusion process but not the diffusion term. When it comes to the estimation of parameters affecting the drift term of r we have to use completely different methods.

Therefore we ask us the following question

Who chooses the martingale measure?

The answer to this question is

The market!

At the same time two parties agree on a price and make a deal on which the (their) risk neutral probability is fixed. This is equivalent to say that when we know the risk neutral probability we also know the market price of the deal price.

Thus, in order to obtain information about the Q-drift parameters we have to collect price information from the market, and the typical approach is that of **inverting the yield curve** that works as follows

1. Choose a particular model involving one or several parameters. Let us denote the entire parameter vector by α. Thus we write the r-dynamics (under Q) as:

$$dr(t) = \mu(t, r(t); \alpha)dt + \sigma(t, r(t); \alpha)dV(t)$$

2. Solve, for every conceivable time of maturity T, the term-structure equation

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + \mu \dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2 \frac{\partial^2 F^T}{\partial r^2} - rF^T = 0 \\ F^T(r, T) = 1 \end{cases} .$$

   In this way we have computed the theoretical term structure as

$$p(t, T; \alpha) = F^T(r, t; \alpha)$$

   Note that the form of the term structure will depend upon our choice of parameter vector. We have not made this choice yet.

3. Collect price data from the bond market. In particular, we may today (i.e. at $t = 0$) observe $p(0, T)$ for all values of T. Denote this **empirical term structure** by $\{p^*(0, T); T \geq 0\}$.
4. Now choose the parameter vector α in such a way that the theoretical curve

$\{p(0, T; \alpha); T \geq 0\}$ fits the empirical curve $\{p^*(0, T); T \geq 0\}$ as well as possible (according to some objective function). This gives us our estimated parameter vector α^*

5. Insert α^* into μ and σ. Now we have pinned down exactly which martingale measure we are working with. Let us denote the result of inserting α^* into μ and σ by μ^* and σ^*, respectively.
6. We have now fixed our martingale measure Q, and we can go on to compute prices of interest rate derivatives, like, say, $X = \Lambda(r(T))$. The price process is then given by $\Pi(t; \Lambda) = G(t, r(t))$ where G solves the term-structure equation

$$\begin{cases} \dfrac{\partial G}{\partial t} + \mu^* \dfrac{\partial G}{\partial r} + \frac{1}{2}(\sigma^*)^2 \dfrac{\partial^2 G}{\partial r^2} - rG = 0 \\ G(r, T) = \Gamma(r) \end{cases}$$

If the program is to be carried out within reasonable time limits, it is of course of great importance that the PDEs involved are easy to solve. It turns out that some of the previous models are much easier to deal with analytically than the others, and this leads us to the subject of so-called **affine term structures** that we will discuss in detail.

15.1.3 Affine Term Structure

Definition 15.1.0.1. If the term structure $\{p(t, T); 0 \leq t \leq T, T > 0\}$ has the form

$$p(t, T) = F(r(t), t, T)$$

where

$$F(r, t, T) = e^{A(t,T) - B(t,T)r}$$

and A and B are deterministic functions, then the model is said to possess an **affine term structure** (ATS).

In some literature, the affine bond prices are written as:

$$p(t, T) = A(t, T)e^{-B(t,T)r}.$$

We can also use the following definition:

Definition 15.1.0.2. A model is said to have the ATS if the continuously compounded short-rate $R(t, T)$ is an affine structure of the short-rate $r(t)$

$$R(t, T) = \alpha(t, T) + \beta(t, T) \cdot r(t)$$

where α and β are deterministic function of time, were we set

$$\alpha(t, T) = \frac{A(t, T)}{T - t}$$
$$\beta(t, T) = \frac{B(t, T)}{T - t}$$

The previous functions A and B are functions of the two real variables t and T, but conceptually it is easier to think of A and B as being functions of t, while T serves as a parameter. It turns out that the existence of an ATS is extremely pleasing from an analytical and a computational point of view, so it is of considerable interest to understand when such a structure appears. In particular we would like to answer the following question:

For which choices of μ, and σ in the Q-dynamics for r do we get an ATS?

We will try to give at least a partial answer to this question, and we start by investigating some of the implications of an ATS. Assume then that we have the Q-dynamics

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dV(t)$$

and assume that this model actually possesses an ATS. In other words we assume that the bond prices have the form of $F(r, t, T)$ in the previous equation. Then we may easily compute the various partial derivatives of F, and since F must solve the term-structure equation, we thus obtain

$$\frac{\partial A(t, T)}{\partial t} - \left\{1 + \frac{\partial B(t, T)}{\partial t}\right\} r(t) - \mu(t, r(t))B(t, T) + \frac{1}{2}\sigma^2(t, r(t))B^2(t, T) = 0$$

The boundary value $F(r, T, T) = 1$ implies

$$A(T, T) = B(T, T) = 0$$

The aforementioned equation gives us the relations that must hold between A, B, μ. and σ in order for an ATS to exist. For a certain choice of μ and σ there may or may not exist functions A and B such that the equation is satisfied. Our immediate task is thus to give conditions on μ and σ which guarantee the existence of functions A and B.

We observe that if μ and σ^2 are both affine (i.e. linear plus a constant) functions of r, with possibly time-dependent coefficients, then the previous equation becomes a separable differential equation for the unknown functions A and B. Therefore, we assume that

$$\begin{cases} \mu(t,r) = a(t)\cdot r + b(t) \\ \sigma^2(t,r) = c(t)\cdot r + d(t) \end{cases}$$

Then, we get the following equation:

$$\frac{\partial A}{\partial t}(t,T) - \left\{1 + \frac{\partial B}{\partial t}(t,T)\right\} r(t) - \{a(t)r + b(t)\}\, B(t,T)$$

$$+ \frac{1}{2}\{c(t)r(t) + d(t)\}\, B^2(t,T) = 0$$

which can be separated into two equations

$$\begin{cases} \dfrac{\partial A(t,T)}{\partial t} - b(t)B(t,T) + \dfrac{1}{2}d(t)B^2(t,T) = 0 \\ \dfrac{\partial B(t,T)}{\partial t} + a(t)B(t,T) - \dfrac{1}{2}c(t)B^2(t,T) = -1 \\ A(T,T) = B(T,T) = 0 \end{cases}$$

The last equation is a Ricatti equation in B. If we solve this, we can insert this in the first equation in order determine A. When we are studying different models they will in general be on the form

$$dr(t) = \{a(t)r(t) + b(t)\}\, dt + \sqrt{c(t)r(t) + d(t)}dV(t)$$

Therefore we will use the functions (a, b, c and d) in the equations of A and B previously to find the bond prices

To see that our choice of μ and σ^2 is a good one, we take the derivative of the affine TSE

$$\frac{\partial A(t,T)}{\partial t} - \left\{1 + \frac{\partial B(t,T)}{\partial t}\right\} r(t) - \mu(t, r(t))B(t,T) + \frac{1}{2}\sigma^2(t, r(t))B^2(t,T) = 0$$

with respect to r. We then get

$$-\left\{1 + \frac{\partial B(t,T)}{\partial t}\right\} - \frac{\partial}{\partial r}\left(\mu(t, r(t))\right) B(t,T) + \frac{1}{2}\frac{\partial}{\partial r}\left(\sigma^2(t, r(t))\right) B^2(t,T) = 0$$

A second differentiation with respect to r gives:

$$-B(t,T)\frac{\partial^2}{\partial r^2}\left(\mu(t, r(t))\right) + \frac{1}{2}B^2(t,T)\frac{\partial^2}{\partial r^2}\left(\sigma^2(t, r(t))\right) = 0$$

We must then have

$$\frac{\partial^2}{\partial r^2}\left(\mu(t, r(t))\right) = 0$$

and

$$\frac{\partial^2}{\partial r^2}\left(\sigma^2(t, r(t))\right) = 0$$

which gives us

$$\begin{cases} \mu(t, r) = a(t) \cdot r + b(t) \\ \sigma^2(t, r) = c(t) \cdot r + d(t) \end{cases}$$

Lemma 15.1.2. *If μ and σ^2 are both affine (i.e. linear plus a constant) functions of r, then the term structure is affine.*

Theorem 15.1.3. *Suppose that the model is affine. We have, under Q:*

$$dp(t,T) = r(t)p(t,T)dt - \sigma(t, r(t))B(t,T)p(t,T)dV(t)$$

and

$$df(t,T) = \sigma^2(t, r(t))B(t,T)\frac{\partial B(t,T)}{\partial T}dt + \sigma(t, r(t))\frac{\partial B(t,T)}{\partial T}dV(t)$$

Proof: We use Itô on $p(t,T) = \exp\{A(t,T) - B(t,T)r(t)\}$:

$$
\begin{aligned}
dp(t,T) &= \frac{\partial p}{\partial A}\frac{\partial A}{\partial t}dt + \frac{\partial p}{\partial B}\frac{\partial B}{\partial t}dt + \frac{\partial p}{\partial r}dr + \frac{1}{2}\frac{\partial^2 p}{\partial r^2}(dr)^2 \\
&= p\frac{\partial A}{\partial t}dt - rp\frac{\partial B}{\partial t}dt + \frac{\partial p}{\partial r}(\mu dt + \sigma dV) + \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial r^2}dt \\
&= p\frac{\partial A}{\partial t}dt - rp\frac{\partial B}{\partial t}dt - Bp(\mu dt + \sigma dV) + \frac{1}{2}\sigma^2 B^2 p dt \\
&= p\left\{\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} - B\mu + \frac{1}{2}\sigma^2 B^2\right\}dt - \sigma BpdV \\
&= r(t)p(t,T)dt - \sigma(t,r)B(t,T)p(t,T)dV(t)
\end{aligned}
$$

where we used the equation

$$
\frac{\partial A(t,T)}{\partial t} - \left\{1 + \frac{\partial B(t,T)}{\partial t}\right\}r(t) - \mu(t,r(t))B(t,T) + \frac{1}{2}\sigma^2(t,r(t))B^2(t,T) = 0
$$

from the previous equation. Furthermore, from

$$
dp(t,T) = r(t)p(t,T)dt + \nu(t,T)p(t,T)dV(t)
$$

and

$$
df(t,T) = \nu(t,T)\nu_T(t,T)dt - \nu_T(t,T)dV(t)
$$

we see that

$$
\nu(t,T) = -\sigma(t,r)B(t,T)
$$

so, we must have

$$
df(t,T) = \sigma^2(t,r(t))B(t,T)\frac{\partial B(t,T)}{\partial T}dt + \sigma(t,r(t))\frac{\partial B(t,T)}{\partial T}dV(t)
$$

From the ATS we also get

$$
\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}\left\{\ln\left[p(t,T)\right]\right\} \\
&= -\frac{\partial}{\partial T}\{A(t,T) - r(t)B(t,T)\} = r(t)B_T(t,T) - A_T(t,T)
\end{aligned}
$$

Especially, we will use the following expression in calibration of the at time t = 0 observed forward prices:

$$f^*(0,T) = r(0)B_T(0,T) - A_T(0,T)$$

15.1.3.1 The Vasicek Model

The model proposed by Vasicek in 1977 is a yield-based one-factor equilibrium model given by the dynamic

$$dr = (b - ar)\,dt + \sigma\,dV$$

or sometimes

$$dr = \kappa\,(\theta - r)\,dt + \sigma\,dV$$

This model assumes that the short rate is normally distributed (such models are called Gaussian) and has a so-called "mean reverting process" (under Q). If we put $r = \theta = b/a$, the drift in interest rate will disappear. So this value represents the mean value of the short rate. So a is a measure of how fast the short rate will reach the long-term mean value. The model is popular in the academic community (mainly due to its analytic tractability). Because the model is not necessarily arbitrage-free with respect to the actual underlying securities in the marketplace, the model is not used much. With $a(t) = -a, b(t) = b$, $c(t) = 0$ and $d(t) = \sigma^2$ (as shown earlier) the equation of B is given by

$$\begin{cases} \dfrac{\partial B}{\partial t}(t,T) - aB(t,T) = -1 \\ B(T,T) = 0 \end{cases}$$

This can easily be solved:

$$\begin{aligned} \dot{B}(t,T) - a \cdot B(t,T) &= -1 \\ e^{-at}\dot{B}(t,T) - a \cdot e^{-at}B(t,T) &= -e^{-at} \\ \frac{d}{dt}\left\{e^{-at}B(t,T)\right\} &= -e^{-at} \\ e^{-at}\int_t^T dB(u,T) &= -\int_t^T e^{-au}du \\ B(t,T) &= \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\} \end{aligned}$$

Inserting this in the equation of A, we obtain

$$\begin{cases} \dfrac{\partial A(t,T)}{\partial t} - bB(t,T) + \frac{\sigma^2}{2}B^2(t,T) = 0 \\ A(T,T) = 0 \end{cases}$$

$$A(t,T) = -b\int_t^T B(s,T)ds + \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds$$

$$= -\frac{b}{a}\int_t^T \left\{1 - e^{-a(T-s)}\right\} ds + \frac{\sigma^2}{2a^2}\int_t^T \left\{1 - e^{-a(T-s)}\right\}^2 ds$$

$$\Rightarrow \qquad = -\frac{b}{a}\left[s - \frac{1}{a}e^{-a(T-s)}\right]_t^T + \frac{\sigma^2}{2a^2}\left[s - \frac{2}{a}e^{-a(T-s)} + \frac{1}{2a}e^{-2a(T-s)}\right]_t^T$$

$$= -\frac{b}{a}\left\{T - t - \frac{1}{a}\left(1 - e^{-a(T-t)}\right)\right\}$$

$$+ \frac{\sigma^2}{2a^2}\left\{(T-t) - \frac{2}{a}\left(1 - e^{-a(T-t)}\right) + \frac{1}{2a}\left(1 - e^{-2a(T-t)}\right)\right\}$$

We then have the solution to the term-structure equation. We only have to calibrate the model parameters so that the model will replicate the observed market prices of some instruments.

There are good probabilistic reasons why some of the models in our list are easier to handle than others. We see that the models of Vasicek, Ho-Lee and HW (extended Vasicek) all describe the short rate using a **linear** SDE. Such models are easy to solve and the r-processes can be shown to be normally distributed.

We can also solve the Vasicek model like

$$d(e^{at}r) = e^{at}dr + ae^{at}rdt = e^{at}(b - ar)\,dt + e^{at}\sigma\,dV + are^{at}dt = e^{at}bdt + e^{at}\sigma\,dV$$

giving

$$e^{at}r(t) = r(0) + b\int_0^t e^{au}du + \sigma\int_0^t e^{au}dV_u$$

which simplifies to

$$r(t) = r(0)e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) + \sigma \int_0^t e^{-a(t-u)} dV_u$$

The calculation should have started at any time, thus

$$r(t) = r(s)e^{-a(t-s)} + \frac{b}{a}\left(1 - e^{-a(t-s)}\right) + \sigma \int_s^t e^{-a(t-u)} dV_u$$

The mean value in this model is given by

$$E\left[r(t)\right] = r(0)e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right)$$

and the variance by

$$Var\left[r(t)\right] = \sigma^2 E\left[\left(\int_0^t e^{-a(t-u)} dV_u\right)^2\right] = \sigma^2 \int_0^t e^{-2a(t-u)} du = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

We can see this if we write the short rate as

$$r(t) = h(t, r) + \sigma \int_s^t g(u) dV_u$$

and calculate

$$\begin{aligned}
Var\left[r(t)\right] &= E\left[r(t)^2\right] - \left(E\left[r(t)\right]\right)^2 \\
&= E\left[\left[h(t, r)\right]^2 + 2\sigma h(t, r) \int_s^t g(u) dV_u + \left(\sigma \int_s^t g(u) dV_u\right)^2\right] \\
&\quad - h(t, r)^2 \\
&= \sigma^2 E\left[\left(\int_0^t g(u) dV_u\right)^2\right]
\end{aligned}$$

since the midterm in $E[\ldots]$ will vanish. We learn from this calculation that only the stochastic part contribute to the variance. This will be used next. The bond prices are then given by

$$p(t,T) = E^Q\left[\exp\left\{-\int_t^T r(u)du\right\} |\mathcal{F}_t\right]$$

It is little tricky to calculate the bond prices with this expression. To do so, we can use the following theorem:

Theorem 15.1.4. *For $X \sim N(m, \sigma^2)$ and $\gamma \in R$ we have*

$$E\left[e^{-\gamma X}\right] = \exp\left\{-\gamma m + \frac{1}{2}\gamma^2\sigma^2\right\}$$

For a proof, see Analytical Finance Vol. 1, for $\gamma = 1$ we have

$$E\left[e^{-X}\right] = \exp\left\{-m + \frac{1}{2}\sigma^2\right\} = \exp\left\{-E[X] + \frac{1}{2}Var[X]\right\}$$

If we let

$$X = \int_t^T r(u)du$$

We can write the bond prices $p(0,T)$ as

$$p(0,T) = \exp\left\{-E^Q\left[\int_0^T r(u)du\right] + \frac{1}{2}Var\left[\int_0^T r(u)du\right]\right\}$$

Taking part by part we obtain

$$\begin{aligned} E\left[\int_0^T r(u)du\right] &= \int_0^T \left\{r(0)e^{-au} + \frac{b}{a}\left(1 - e^{-au}\right)\right\} du \\ &= \frac{r(0)}{a}\left(1 - e^{-aT}\right) + \frac{b}{a}T + \frac{b}{a^2}\left(1 - e^{-aT}\right) \\ &= \frac{1}{a^2}\left(ar(0) - b\right)\left(1 - e^{-aT}\right) + \frac{b}{a}T \end{aligned}$$

Instead, integrating from t to T gives:

$$E\left[\int_t^T r(u)du\right] = \frac{1}{a^2}\left(b - ar(0)\right)e^{-a(T-t)} + \frac{b}{a}(T-t)$$

In order to calculate

$$Var\left[\int_0^T r(u)du\right]$$

we need the following two results from stochastic calculus:

Result 1: If $W(t)$ is a Brownian Motion and $g(t)$ a non-random function, then:

$$X(t) = \int_0^t g(u)dW(u)$$

is a Gaussian Process with $E[X(t)] = 0$ and $Var[X(t)] = \int_0^t g^2(u)du$.

Result 2: If $W(t)$ is a Brownian Motion and $g(t)$ and h(t) non-random functions defined as

$$\begin{cases} X(t) = \int_0^t g(u)dW(u) \\ Y(t) = \int_0^t h(u)X(u)du \end{cases}$$

Then $Y(t)$ is a Gaussian Process with $E[Y(t)] = 0$ and

$$Var[Y(t)] = \int_0^t g^2(u)\left(\int_u^t h(y)dy\right)^2 du$$

We will now use what we learned before; only the stochastic part contribute to the variance. Therefore we have

$$\begin{aligned} Var\left[\int_0^T r(u)du\right] &= \int_0^T \left[\int_u^T \sigma e^{-a(y-u)}dy\right]^2 du \\ &= \sigma^2 \int_0^T \left(e^{au}\right)^2 \left[\int_u^T e^{-ay}dy\right]^2 du \end{aligned}$$

$$= \frac{\sigma^2}{a^2} \int_0^T \left(e^{au}\right)^2 \left[e^{-au} - e^{-aT}\right]^2 du$$

$$= \left(\frac{\sigma}{a}\right)^2 \int_0^T \left(1 - e^{a(u-T)}\right)^2 du$$

$$= \left(\frac{\sigma}{a}\right)^2 \int_0^T \left(1 - 2e^{a(u-T)} + e^{2a(u-T)}\right) du$$

$$= \frac{\sigma^2}{a^3} \left\{aT - 2\left(1 - e^{-aT}\right) + \frac{1}{2}\left(1 - e^{-2aT}\right)\right\}$$

If we put all together we find

$$P(0,T) = \exp \left\{ \begin{array}{l} \frac{1}{a}\left[1 - e^{-aT}\right]\left(\frac{b}{a} - \frac{1}{2}\left(\frac{\sigma}{a}\right)^2 - r(0)\right) \\ -T\left[\frac{b}{a} - \frac{1}{2}\left(\frac{\sigma}{a}\right)^2\right] - \frac{\sigma^2}{4a^3}\left[1 - e^{-aT}\right]^2 \end{array} \right\}$$

Remark! This is the same result obtained earlier where we solved the equations for $A(t,T)$ and $B(t,T)$.

If we use the result when we derived the TSE we find:

$$\sigma_p = \frac{\sigma_r F_r(r(t),t,T)}{F(r(t),t,T)} \equiv \frac{\sigma_r}{p(t,T)} \frac{\partial p(t,T)}{\partial r} = -\sigma_r B(t,T)$$

The bond price volatility in the Vasicek model is then given by[1]:

$$\sigma_p = \frac{\sigma_r}{a}\left(1 - e^{-a(T-t)}\right)$$

This model allows that the short rates have a positive probability to become negative. A few years ago, such models was said to have a

[1] We have changed the sign so that the volatility becomes positive. The reason for being negative is that an increase in rate gives a decrease in price. The volatility on the other hand only reflects changes in price or rates.

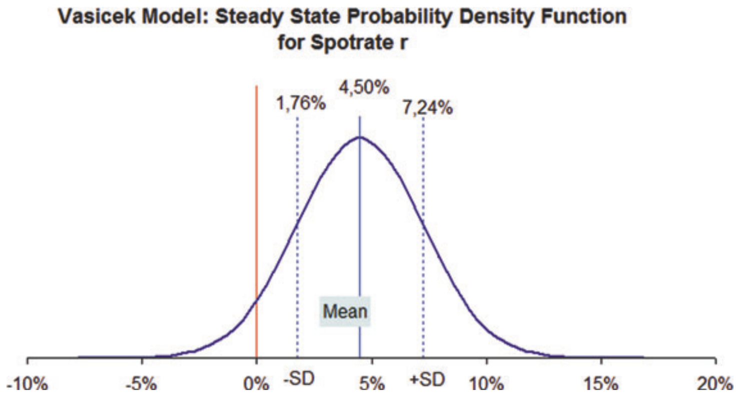

Fig. 15.1 The Vasicek probability density function

disadvantage because it allows for negative interest rates. But now, we all know that negative interest rates may accrue in reality. A simulation of the distribution of the short rates with $a = 0.15$, $b/a = 4.5$ % and $\sigma = 1.5$ % is shown in the Fig. 15.1.

The probability distribution of the spot rate is normally distributed with mean and variance given by

$$E\left[r(t)\right] = r(0)e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) = \frac{b}{a} + \left(r(0) - \frac{b}{a}\right)e^{-at}$$

$$Var\left[r(t)\right] = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

The probability density function is then given by

$$\phi(r) = \frac{1}{\sqrt{2\pi \cdot Var\left[r(t)\right]}} \exp\left\{-\frac{\left(r - E\left[r(t)\right]\right)^2}{2 \cdot Var\left[r(t)\right]}\right\} = N\left(\frac{r - E\left[r(t)\right]}{\sqrt{Var\left[r(t)\right]}}\right)$$

That is,

$$P_\infty = \sqrt{\frac{a}{\pi}\frac{1}{\sigma^2}}e^{-\frac{a(r-b/a)^2}{\sigma^2}}$$

If $r(0)$ is 2.0 % the simulated term structure of interest rates is shown in Fig. 15.2.

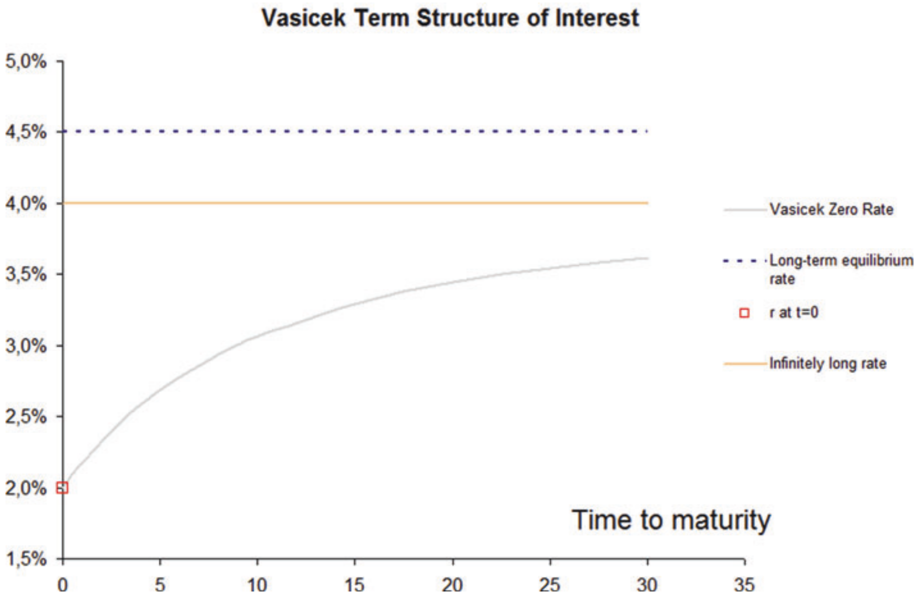

Fig. 15.2 The Vasicek term structure of interest rates

This gives a discount function as (Fig. 15.3)

We will in a subsequent section show the same simulation for the CIR model where the short rate always is positive.

Option Pricing

In many term-structure models, it is possible to find analytical solutions for European options on discount bonds. In a paper by Jamshidan (1989), a method for pricing options on coupon bonds is developed. These options are in fact options on a portfolio of discount bonds. Jamshidan shows how the valuation procedure can be changed so that the option can be calculated as a portfolio of options on discount bonds with appropriate strike prices.

The method works for one-parameter models, since all bond prices are decreasing functions of the interest rate used as the state variable.

Consider a European option on a coupon bond (or a general fixed cash flow pattern) with strike price X expiring at time τ. The value of the bond at any time t can be written

$$B(r,t) = \sum_{i=1}^{n} c_i \cdot p(r,t,T_i)$$

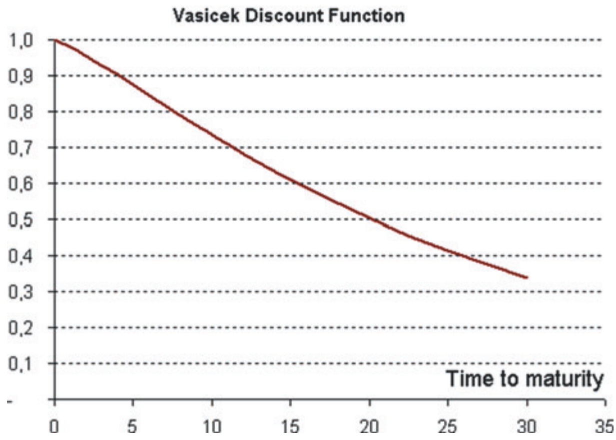

Fig. 15.3 The Vasicek discount function

where c_i is the coupons. The payoff from the option at maturity, τ is: $\max(0, B(r,\tau) - X)$. The value of r when the option is exactly at-the-money is called $r*$ and defined by: $B(r^*,\tau) = X$. The option will be exercised when $r(\tau) < r^*$. It can be shown that

$$\max\left\{0, \sum_{i=1}^{n} c_i \cdot p(r,\tau,T_i) - X\right\} = \sum_{i=1}^{n} c_i \cdot \max\left\{0, p(r,\tau,T_i) - p(r^*,\tau,T_i)\right\}$$

The second summation is the exact payoffs of a portfolio of options on discount bonds.

Jamshidan has also shown that options on zero-coupon bonds can be valued using Vasicek's model. A European call option is given by

$$\Pi = L \cdot p(0,S)N(h) - K \cdot p(0,T)N(h - \sigma_p)$$

where L is the face value of the bond, S the bond maturity, K the option strike and

$$\begin{cases} h = \dfrac{1}{\sigma_p} \ln\left(\dfrac{L \cdot p(0,S)}{K \cdot p(0,T)}\right) + \dfrac{\sigma_p}{2} \\ \sigma_p = \dfrac{\sigma}{a}\left(1 - e^{-a(S-T)}\right)\sqrt{\dfrac{1 - e^{-2aT}}{2a}} \end{cases}$$

Similarly, a European put option is given by

$$\Pi = K \cdot p(0,T)N(-h + \sigma_p) - L \cdot p(0,S)N(-h)$$

The volatility given will be interpreted as the volatility of proportional changes in the short rate, in order to obtain values in the same units as in the Black-Scholes model. If the volatility is given in yield it has to be converted to price volatility:

$$\sigma_p = \sigma_y \frac{y}{p} \cdot \frac{dp}{dy} = \sigma_y \cdot y \cdot ModDur$$

where T is the option maturity and τ the maturity of the bond. For options on cash flows with floating rates, an additional procedure must be used. It can be shown that the present value and interest rate sensitivity of a cash flow depending on the implied forward rate can be made identical to two fixed cash flows. For Swaptions, this procedure is used to first convert all floating cash flows and then apply the method described previously.

Example 15.1.5

Consider a European call option on a zero-coupon bond. Time to expiration is two years, the strike price is 92, the volatility is 3%, the mean-reverting level is 9%, and the mean reverting rate is 5%. The face value of the bond is 100 with time to maturity three years, and initial risk-free rate of 8%.

$$L = 100, K = 92, T = 2, S = 3, b = 0.0045, a = 0.05, r = 0.08, \sigma = 0.03.$$

$$c = 100 \cdot p(0,3) \cdot N(h) - 92 \cdot p(0,2) \cdot N(h - \sigma_p)$$

$$\sigma_p = \sigma \cdot B(T,S) \cdot \sqrt{\frac{1 - e^{-2aT}}{2a}}$$

$$B(t,T) = B(0,2) = \frac{1 - e^{-0.05\cdot(2-0)}}{0.05} = 1.9032$$

$$B(T,S) = B(2,3) = \frac{1 - e^{-0.05\cdot(3-2)}}{0.05} = 0.9754$$

$$B(t,S) = B(0,3) = \frac{1 - e^{-0.05\cdot(3-0)}}{0.05} = 2.7858$$

$$A(t,T) = A(0,2) = \frac{\{B(0,2) - 2 + 0\}\left\{0.0045 \cdot 0.05 - \frac{0.03^2}{2}\right\}}{0.05^2} - \frac{0.03^2}{4 \cdot 0.05} B^2(0,2)$$
$$= -0.00763$$

$$A(t,S) = A(0,3) = \frac{\{B(0,2) - 3 + 0\}\left\{0.0045 \cdot 0.05 - \frac{0.03^2}{2}\right\}}{0.05^2} - \frac{0.03^2}{4 \cdot 0.05} B^2(0,3)$$
$$= -0.01562$$

$$p(t,T) = p(0,2) = \exp\{A(0,2) - 0.08 \cdot B(0,2)\} = 0.8523$$

$$p(t,S) = p(0,3) = \exp\{A(0,3) - 0.08 \cdot B(0,3)\} = 0.7878$$

$$\sigma_p = 0.03 \cdot B(2,3)\sqrt{\frac{1 - e^{-2(2-0)\cdot 0.05}}{2 \cdot 0.05}} = 0.0394$$

$$h = \frac{1}{\sigma_p} \ln\left(\frac{100 \cdot p(0,3)}{92 \cdot p(0,2)}\right) + \frac{\sigma_p}{2} = 0.1394$$

The call value for one USD in face value is

$$\begin{aligned} c &= L \cdot p(0,S)N(h) - K \cdot p(0,T)N(h - \sigma_p) \\ &= 100 \cdot p(0,3) \cdot N(h) - 92 \cdot p(0,2) \cdot N(h - \sigma_p) \\ &= 100 \cdot 0.7878 \cdot N(0.1394) - 92 \cdot 0.8523 \cdot N(0.1394 - 0.0394) \\ &= 0.0143 \end{aligned}$$

With a face value of 100 the call value is 1.43 USD (100×0.0143).

Example 15.1.6

Consider a European call option on a coupon bond. Time to expiration is four years, the strike price 99.5, the volatility is 3.0%, the mean-reverting level is 1.0%, and the mean-reverting rate is 5.0%. The face value of the bond is 100, and it pays a semi-annual coupon of four. Time to maturity is seven years, and the risk-free rate is initially 9.0%.

First find the rate r that makes the value of the coupon bond equal to the strike price at the options expiry. Trial and error gives $r = 8.0050\%$. To find the value of the option, we have to determine the value of six different options:

1. A four-year option with strike price 3.8427 on a 4.5-year zero-coupon bond with a face value of four.

2. A four-year option with strike price 3.6910 on a five-year zero-coupon bond with a face value of four.
3. A four-year option with strike price 3.5452 on a 5.5-year zero-coupon bond with a face value of four.
4. A four-year option with strike price 3.4055 on a six-year zero-coupon bond with a face value of four.
5. A four-year option with strike price 3.2717 on a 6.5-year zero-coupon bond with a face value of four.
6. A four-year option with strike price 81.7440 on a seven-year zero-coupon bond with a face value of 104.

The values of the six options are, respectively, 0.0256, 0.0493, 0.0713, 0.0917, 0.1105 and 3.3219. This gives a total value of 3.6703.

15.1.3.2 The Ho-Lee Model

Ho and Lee (1986) published the first arbitrage-free yield-based model. It assumes a normally distributed short-term rate. This enables analytical solutions for European bond options. The short rate's drift depends on time, thus making the model arbitrage-free with respect to observed prices (the input to the model). The model does not incorporate mean reversion. In the Ho and Lee model, the short-rate dynamics are represented by

$$dr = \theta(t) \cdot dt + \sigma \cdot dV$$

In this model, where the risk neutral process $\theta(t)$ includes the market price of risk is of interest since it is easy to calibrate with real market data since the volatility is the same for Q as for P. But the model is not very realistic since the drift in the model does not follow market prices. The calibration to analytical solutions on bonds and bond option can be done without numerical calculations.

The model has an ATS

$$p(t, T) = F(r(t), t, T) = e^{A(t,T)-B(t,T)r}$$

where

$$\begin{cases} \dfrac{\partial B}{\partial t}(t, T) = -1 \\ B(T, T) = 0 \end{cases}$$

gives

$$B(t,T) = T - t$$

and

$$\begin{cases} \dfrac{\partial A(t,T)}{\partial t} = \theta(t)B(t,T) - \dfrac{\sigma^2}{2}B^2(t,T) \\ A(T,T) = 0 \end{cases}$$

Inserting $B(t,T) = T - t$ into this in the equation, we obtain

$$A(t,T) = \int_t^T \theta(s)(T-s)ds - \frac{\sigma^2}{2}\frac{(T-t)^3}{3}$$

We then only have to calibrate the model to the observed initial yield curve, that is, the observed bond prices at $t = 0$: $p^*(0,T)$ and to the historical volatility σ. From $p^*(0,T)$ we can also get the forward rates f^* $(0, T)$. The initial forward rates are given by

$$f^*(0,T) = B_T(0,T) \cdot r(0) - A_T(0,T) = r(0) + \int_t^T \theta(s)ds - \frac{\sigma^2}{2}T^2$$

Taking derivative with respect to T gives

$$f_T^*(0,T) = \theta(T) - \sigma^2 T$$

Theorem 15.1.7. *For each observed yield-curve $\{p*(0, T): T \geq 0\}$ there exists a unique function $\theta(t)$ that fit the theoretical bond prices at $t = 0$, where*

$$\theta(t) = f_T^*(0,t) + \sigma^2 t$$

Given $p*(0,T)$ and $\theta(t)$ we have decided which martingale measure we are working with. The next step is to calculate the theoretical term structure under this martingale measure. Therefore we will use $\theta(t)$ to calculate A and B to get

$$p(t, T) = F(r(t), t, T) = e^{A(t,T) - B(t,T)r}.$$

This is quite comprehensive calculations. A better method is to calculate the forward prices given by

$$df(t,T) = \sigma^2(t,r(t))B(t,T)\frac{\partial B(t,T)}{\partial T}dt + \sigma(t,r(t))\frac{\partial B(t,T)}{\partial T}dV(t)$$

With the simple expression for B we get

$$f(t,T)=f(0,T)+\sigma^2\int_0^t (T-s)ds+\sigma\int_0^t dV(t)=f(0,T)+\sigma^2 t(T-\frac{t}{2})+\sigma V(t)$$

To get the bond prices we use

$$\begin{aligned}
p(t,T) &= \exp\left\{-\int_t^T f(t,u)du\right\} \\
&= \exp\left\{-\int_t^T f(0,u)du+\sigma^2\int_t^T t(\frac{t}{2}-u)du-\sigma\int_t^T V(u)du\right\} \\
&= \exp\left\{-\int_t^T f(0,u)du-\frac{\sigma^2 Tt}{2}(T-t)-\sigma(T-t)V(t)\right\}
\end{aligned}$$

That is,

$$p(t,T)=\frac{p(0,T)}{p(0,t)}\exp\left\{-\frac{\sigma^2 Tt}{2}(T-t)-\sigma(T-t)V(t)\right\}$$

Before we use this in a real situation we would like to remove the Wiener process. This is done by using

$$f(t,T)=f(0,T)+\sigma^2 t(T-\frac{t}{2})+\sigma V(t)$$

for $T=t$. That is,

$$r(t)=f(t,t)=f(0,t)+\frac{\sigma^2 t^2}{2}+\sigma V(t)$$

We finally get, by eliminating $V(t)$:

$$p(t,T)=\frac{p^*(0,T)}{p^*(0,t)}\exp\left\{(T-t)f^*(0,t)-\frac{\sigma^2}{2}t\cdot(T-t)^2-(T-t)r(t)\right\}$$

As there is no dependence of the drift on the level of the short rate, the volatility structure for the bond prices is determined by the constant σ.

$$\sigma_p(t,T)=\sigma_r(T-t)$$

Calibration of Volatility Data

The simple structure of this model allows us to illustrate how models with closed form solutions can be calibrated to interest rate options data.

The HL model involves only one volatility parameter and, like Black-Scholes, it can be inferred from the market prices of actively traded interest rate options. Suppose, for example, that we have a set of m pure discount bond put options, the market price of which we denote by $market_i$ $(i = 1, \ldots m)$. One way to calibrate the model is to minimize the following function with respect to the parameter σ:

$$\min_{\sigma} \sqrt{\sum_{i=1}^{m} \left(\frac{\mathrm{model_i}(\sigma) - market_i}{market_i} \right)^2}$$

A problem with calibrating term-structure consistent models with market caps data is that the quotes obtainable from brokers are not cash prices, but instead are Black volatilities. The first step in the calibration procedure, therefore, is to obtain cash prices from the quoted volatilities via the pricing formula.

Option Pricing

To price a European call option with maturity T and strike price K on an S-bond, we get the arbitrage-free price as

$$\pi_0 [X] = E^Q \left[\max \{L \cdot p(S, T) - K, 0\} \cdot \exp \left\{ - \int_0^T r(s) ds \right\} \right]$$

It is possible to get an analytical result from this, but the calculations are quite complex. We will in a later section learn about forward measure, and then it will be easier to calculate such prices. The result for a European call option with strike price K and maturity T on a zero-coupon bond with a face value L and maturity S is given by

$$C(t, T, K, S) = L \cdot p(t, S) \cdot N(d) - K \cdot p(t, T) \cdot N(d - \sigma_p)$$

where

$$d = \frac{\ln\left\{\frac{L \cdot p(t,S)}{K \cdot p(t,T)}\right\} + \frac{\sigma_p^2}{2}}{\sigma_p}$$

$$\sigma_p = \sigma(S - T) \cdot \sqrt{T}$$

Binomial Tree

To build a binomial tree using the Ho-Lee model, we use the general dynamic

$$dr = \mu(t) \cdot dt + \sigma(t) \cdot dz(t)$$

where $dz(t) \sim N(0, 1)$ to write the discrete dynamics as

$$\Delta r(t) = \mu(t) \cdot \Delta t + \sigma(t) \cdot \Delta z(t)$$

for the time period $[t, t + \Delta t]$ and where $\Delta z(t)$ is a normally distributed random variable. $r(t)$ and $\sigma(t)$ are the short rate and the volatility of the short rate at time t for the interval from t to $t + \Delta t$. Without loss of generality, let $\Delta t = 1$ and let $t = 0$. We can write the evolution of the short rate as

$$\Delta r(t) = \mu(t) \cdot \Delta t + \sigma(t) \cdot \Delta z(t)$$

With constant time steps, we can expand the dynamics as:

$$\Delta r(0) \equiv r(1) - r(0) = \mu(0) + \sigma(0) \cdot \Delta z(0)$$

This yields, for example

$$\begin{aligned} r(1) &= r(0) + \mu(0) + \sigma(0) \cdot \Delta z(0), \\ r(2) &= r(1) + \mu(1) + \sigma(1) \cdot \Delta z(1) = r(0) + \mu(0) + \sigma(0) \cdot \Delta z(0) \\ &\quad + \mu(1) + \sigma(1) \cdot \Delta z(1) \\ &= r(0) + \{\mu(0) + \mu(1)\} + \{\sigma(0) \cdot \Delta z(0) + \sigma(1) \cdot \Delta z(1)\} \end{aligned}$$

and

$$\begin{aligned} r(3) = r(0) &+ \{\mu(0) + \mu(1) + \mu(2)\} \\ &+ \{\sigma(0) \cdot \Delta z(0) + \sigma(1) \cdot \Delta z(1) + \sigma(2) \cdot \Delta z(2)\} \end{aligned}$$

In general then

$$
\begin{aligned}
r(t) &= r(t-1) + \mu(t-1) + \sigma(t-1) \cdot \Delta z(1) \\
&= r(0) + \sum_{i=1}^{t-1} \mu(i-1) + \sum_{i=1}^{t-1} \sigma(i-1)\Delta z(i-1)
\end{aligned}
$$

This expression shows that the short rate is the sum of a set of non-stochastic drift terms and a set of stochastic terms; all of the latter are normally distributed. Consequently, all short interest rates are normally distributed (albeit with changing parametric values). For example,

$$
\begin{aligned}
&r(1) \sim N\big(r(0) + \mu(0), \sigma^2(r(1))\big), \\
&r(2) \sim N\big(r(0) + \mu(0) + \mu(1), \sigma^2(r(1) + r(2))\big), \\
&r(3) \sim N\big(r(0) + \mu(0) + \mu(1) + \mu(2), \sigma^2(r(1) + r(2) + r(3))\big),
\end{aligned}
$$

and general

$$
r(t) \sim N\left(r(0) + \sum_{i=1}^{t-1} \mu(i-1), \sigma^2\left(\sum_{i=1}^{t} r(i)\right)\right)
$$

The inputs for a Ho and Lee no-arbitrage interest rate model in discrete time are (1) a set of known (pure) discount bond prices, $\{p(0, 1), p(0, 2), p(0, 3), \ldots, p(0, n)\}$,2 and (2) the volatility (standard deviation) of future one-period short rates, $\{\sigma(0), \sigma(1), \ldots, \sigma(\text{n-1})\}$.

An evolution of the short rate that precludes arbitrage must satisfy the local expectation conditions that bonds of any maturity offer the same expected rate of return in a given period. This is equivalent to the expectation of the discounted value of each bond's terminal payment being equal to its given market value. Let the present values, at date 0, of a bond's terminal payments be given by

$$
b(0, n) = \exp\left(-\sum_{i=0}^{n-1} r(i)\right)
$$

Therefore, the no-arbitrage conditions will be stated as

$$
\begin{aligned}
&p(0, 1) = e^{-f(0,0)} \equiv E^Q\left[b(1)|\mathcal{F}_0\right] = E^Q\left[e^{-r(0)}|\mathcal{F}_0\right] = e^{-r(0)} \\
&p(0, 2) = e^{-\{f(0,0)+f(0,1)\}} \equiv E^Q\left[b(2)|\mathcal{F}_0\right] = E^Q\left[e^{-\{r(0)+r(1)\}}|\mathcal{F}_0\right] \\
&p(0, 3) = e^{-\{f(0,0)+f(0,1)+f(1,2)\}} \equiv E^Q\left[b(3)|\mathcal{F}_0\right] = E^Q\left[e^{-\{r(0)+r(1)+r(2)\}}|\mathcal{F}_0\right]
\end{aligned}
$$

and in general

$$p(0,n) = \exp\left(-\sum_{i=0}^{n-1} f(i-1,i)\right) \equiv E^Q\left[b(n)\right] = E^Q\left[\exp\left(-\sum_{i=0}^{n-1} r(j)\right)\right]$$

where $f(i-1,i)$ is the one period forward rate observed at time i. We know that if $X \sim \mathrm{N}(\mu, \sigma^2)$, then

$$E\left[e^{-X}\right] = e^{-\mu+\frac{1}{2}\sigma^2}$$

The zero-coupon bond price at $t = 2$ is then given by:

$$\begin{aligned} p(0,2) &= E^Q\left[e^{-(r(0)+r(1))}|\mathcal{F}_0\right] = e^{-r(0)}E^Q\left[e^{-r(1)}|\mathcal{F}_0\right] \\ &= e^{-r(0)}e^{-E^Q[r(1)]+\frac{1}{2}\sigma^2(r(1))} \end{aligned}$$

That is,

$$\ln p(0,2) = -r(0) - E^Q\left[r(1)\right] + \frac{1}{2}\sigma^2(r(1))$$

or

$$E^Q\left[r(1)\right] = -\ln p(0,2) - r(0) + \frac{1}{2}\sigma^2(r(1))$$

We know that

$$\ln P(0,2) = -f(0,0) - f(0,1) = -r(0) - f(0,1)$$

Therefore we can write

$$E^Q\left[r(1)\right] = -f(0,1) + \frac{1}{2}\sigma^2(r(1))$$

Thus, the expectation at date 0 of the short rate at date 1 is the forward rate plus a term determined by the variance, $\frac{1}{2}\sigma^2(r(1))$. Further, applying the expectations operator to $r(t)$, we get a second expression for the expectation of the short rate,

$$E^Q\left[r(1)\right] = r(0) + \mu(0)$$

Thus

$$\mu(0) = f(0,1) - r(0) + \frac{1}{2}\sigma^2(r(1))$$

This expression tells us that the drift term, $\mu(0)$ is given by the combination of two effects: (1) $f(0, 1) - r(0)$ is the difference between the forward rate and the short rate (i.e. the short rate drifts up or down towards the forward rate). (2) $\frac{1}{2}\sigma^2(r(1))$ is a positive drift adjustment term that is required to preclude arbitrage.

Let $\delta(t)$ denote the drift adjustment term for date t. Then, $\delta(0) = \frac{1}{2}\sigma^2(r(1))$. We can then work out the details in step 3

$$p(0,3) = E^Q\left[e^{-\{r(0)+r(1)+r(2)\}}|\mathcal{F}_0\right] = e^{-r(0)}E^Q\left[e^{-\{r(1)+r(2)\}}|\mathcal{F}_0\right]$$
$$= e^{-r(0)}e^{-E^Q[r(1)+r(2)]+\frac{1}{2}\sigma^2(r(1)+r(2))}$$

Further

$$\ln p(0,3) = -r(0) - E^Q\left[r(1)\right] - E^Q\left[r(2)\right] + \frac{1}{2}\sigma^2(r(1) + r(2))$$
$$= -r(0) - f(0,1) - \frac{1}{2}\sigma^2(r(1)) - E^Q\left[r(2)\right] + \frac{1}{2}\sigma^2(r(1) + r(2))$$

or

$$E^Q\left[r(2)\right] = -\ln p(0,3) - r(0) - f(0,1) + \frac{1}{2}\sigma^2(r(1) + r(2)) - \frac{1}{2}\sigma^2(r(1))$$

We know that $\ln p(0,3) = -f(0,0) - f(0,1) - f(1,2) = -r(0) - f(0,1) - f(1,2)$. Therefore, upon substitution,

$$E^Q\left[r(2)\right] = f(1,2) + \frac{1}{2}\sigma^2(r(1) + r(2)) - \frac{1}{2}\sigma^2(r(1))$$

Thus, the expectation at date 0 of the short rate at date 2 is the forward rate plus a term determined by the variance,

$$\frac{1}{2}\sigma^2(r(1) + r(2)) - \frac{1}{2}\sigma^2(r(1))$$

Further, applying the expectations operator, we get a second expression for the expectation of the short rate,

$$E^Q\left[r(2)\right] = r(0) + \mu(0) + \mu(1)$$

That is,

$$\mu(1) = f(1,2) - r(0) - \mu(0) + \frac{1}{2}\sigma^2(r(1) + r(2)) - \sigma^2(r(1))$$
$$= f(1,2) - f(0,1) + \frac{1}{2}\sigma^2(r(1) + r(2)) - \sigma^2(r(1))$$

This expression tells us that the drift term, $\mu(1)$, is given by the combination of two effects: (1) $f(2) - f(1)$ is the difference between the forward rate at date 2 and the forward rate at date 1, that is, the nearby forward short rate drifts up or down towards the distant forward rate. (2) $\delta(1) = \frac{1}{2}\sigma^2(r(1)+r(2)) - \sigma^2(r(1))$ is a positive drift adjustment term at time 1 that is required to preclude arbitrage.

Then, if we add $\delta(0)$ and $\delta(1)$ we get

$$\sum_{t=0}^{1} \delta(t) = \frac{1}{2}\sigma^2(r(1)+r(2)) - \frac{1}{2}\sigma^2(r(1))$$

Similarly, if we add $\mu(0)$ and $\mu(1)$ we get

$$\begin{aligned}\mu(0)+\mu(1) &= f(0,1) - r(0) + \frac{1}{2}\sigma^2(r(1)) + f(1,2) + \frac{1}{2}\sigma^2(r(1)+r(2)) \\ &\quad - \sigma^2(r(1)) \\ &= f(1,2) - r(0) + \frac{1}{2}\sigma^2(r(1)+r(2)) - \frac{1}{2}\sigma^2(r(1))\end{aligned}$$

which can be simplified to

$$\sum_{t=0}^{1} \mu(t) = f(1,2) - r(0) + \sum_{t=0}^{1} \delta(t)$$

If we continue with the same process as earlier to $t = 4$ we will find that:

$$\sum_{t=0}^{2} \mu(t) = f(2,3) - r(0) + \sum_{t=0}^{2} \delta(t)$$

etc. If we generalize this we will have the following result:

$$E^{Q}\left[r(t)|\mathcal{F}_0\right] = f(t-1,t) + \frac{1}{2}\sigma^2\left(\sum_{j=1}^{t} r(j)\right) - \frac{1}{2}\sigma^2\left(\sum_{j=1}^{t-1} r(j)\right) \quad \forall\, 1 < t \leq T-1$$

$$\mu(0) = f(0,1) - r(0) + \frac{1}{2}\sigma^2(r(1))$$

$$\mu(1) = f(1,2) - f(0,1) + \frac{1}{2}\sigma^2\left(\sum_{j=1}^{2} r(j)\right) - \sigma^2(r(1))$$

$$\mu(t-1) = f(t-1,t) - f(t-2,t-1) + \frac{1}{2}\sigma^2\left(\sum_{j=1}^{t} r(j)\right) - \sigma^2\left(\sum_{j=1}^{t-1} r(j)\right) + \frac{1}{2}\sum_{n=1}^{t-2}\sigma^2\left(\sum_{j=1}^{n} r(j)\right) \quad \forall\, t \geq 3$$

$$\sum_{n=0}^{t}\delta(n) = \frac{1}{2}\sigma^2\left(\sum_{j=1}^{t+1} r(j)\right) - \frac{1}{2}\sigma^2\left(\sum_{j=1}^{t} r(j)\right) \quad \forall\, t \geq 1$$

and

$$\sum_{n=0}^{t}\mu(n) = f(t,t+1) - r(0) + \sum_{n=0}^{t}\delta(n) \quad \forall\, t \geq 1$$

These equations give the necessary recursive relations to evolve the Ho-Lee no-arbitrage model of short interest rate. The inputs are the set of market prices of (pure) discount bonds and a structure of volatilities for the short rates.

The aforementioned discussion is general in the sense that it applies equally well to implementation based on the binomial models and Monte Carlo simulation. If we adopt the tree approach to depict the evolution, we would write the evolutionary equation as

$$r(t) = \begin{cases} r(t-\Delta t) + \mu(t-\Delta t)\Delta t + \sigma(t-\Delta t)\sqrt{\Delta t} \\ r(t-\Delta t) + \mu(t-\Delta t)\Delta t - \sigma(t-\Delta t)\sqrt{\Delta t} \end{cases}$$

where we use equal probabilities = ½. If $\Delta t = 1$ we then have

For $t = 1$

$$\begin{cases} r_0(1) = r(0) + \mu(0) - \sigma(0) \\ r_1(1) = r(0) + \mu(0) + \sigma(0) \end{cases}$$

For $t = 2$

$$\begin{cases} r_0(2) = r(0) + \{\mu(0) + \mu(1)\} - \{\sigma(0) + \sigma(1)\} \\ r_1(2) = r(0) + \{\mu(0) + \mu(1)\} - \{\sigma(0) - \sigma(1)\} \\ r_2(2) = r(0) + \{\mu(0) + \mu(1)\} + \{\sigma(0) - \sigma(1)\} \\ r_3(2) = r(0) + \{\mu(0) + \mu(1)\} + \{\sigma(0) + \sigma(1)\} \end{cases}$$

An alternative is to express the tree using the positive drift adjustment term. Then we have

For $t = 1$

$$\begin{cases} r_0(1) = f(0,1) + \delta(0) - \sigma(0) \\ r_1(1) = f(0,1) + \delta(0) + \sigma(0) \end{cases}$$

For $t = 2$

$$\begin{cases} r_0(2) = f(1,2) + \{\delta(0) + \delta(1)\} - \{\sigma(0) + \sigma(1)\} \\ r_1(2) = f(1,2) + \{\delta(0) + \delta(1)\} - \{\sigma(0) - \sigma(1)\} \\ r_2(2) = f(1,2) + \{\delta(0) + \delta(1)\} + \{\sigma(0) - \sigma(1)\} \\ r_3(2) = f(1,2) + \{\delta(0) + \delta(1)\} + \{\sigma(0) + \sigma(1)\} \end{cases}$$

Here $r_n(t)$ denotes the nth node at date t. If the evolution can be depicted as a lattice, then the nth node means $n$up-moves. On the other hand, if the evolution is depicted as a tree, then the nth node is an ordinal rank, starting with $n = 0$ at the bottom of the tree and ending with $n = t_2$ at the top of the tree at date t. Depending upon the context, one must infer whether the n th node shows n up-moves or shows the ordinal rank.

The binomial tree is built as in Fig. 15.5.

Using constant volatilities σ_c the binomial tree is simplified to the tree in Fig. 15.5

Under both approaches, however, we recognize that we need the variances of the sums of short rates. Remember

$$r(t) = r(0) + \sum_{i=1}^{t-1} \mu(i-1) + \sum_{i=1}^{t-1} \sigma(i-1)\Delta z(i-1)$$

For the ease of exposition, let the (time) indexes in the parentheses be designated as a subscript. Then,

$$\sigma^2(r_1) = \sigma^2(r_0 + \mu_0 + \sigma_0 \Delta z_0) = \sigma^2(\sigma_0 \Delta z_0) = \sigma_0^2$$

$$\begin{aligned} \sigma^2(r_1 + r_2) &= \sigma^2\left(r_0 + \mu_0 + \sigma_0 \Delta z_0 + r_0 + \mu_0 + \mu_1 + \sigma_0 \Delta z_0 + \sigma_1 \Delta z_1\right) \\ &= \sigma^2\left(2\sigma_0 \Delta z_0 + \sigma_1 \Delta z_1\right) \\ &= \sigma^2\left(2\sigma_0 \Delta z_0\right) + \sigma^2\left(\sigma_1 \Delta z_1\right) + 2\mathrm{Cov}\left(2\sigma_0 \Delta z_0, \sigma_1 \Delta z_1\right) \\ &= 4\sigma_0^2 + \sigma_1^2 \end{aligned}$$

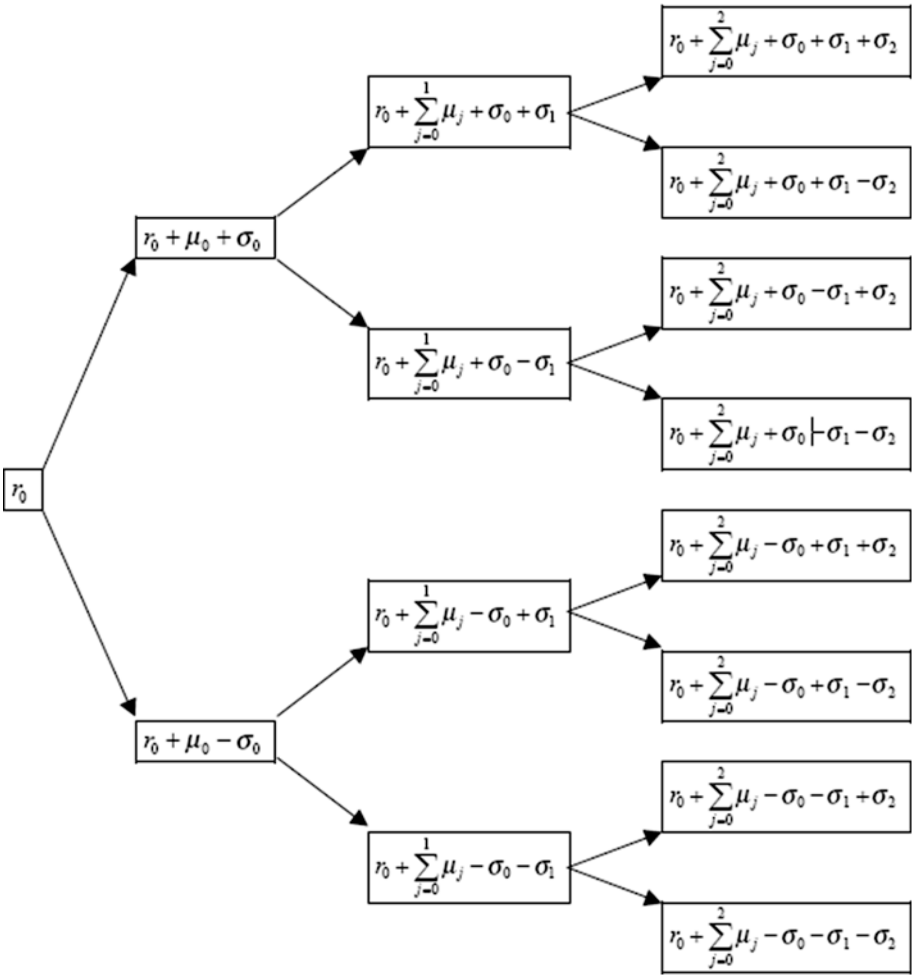

Fig. 15.4 The Ho-Lee binominal tree

$$\begin{aligned}
\sigma^2(r_1 + r_2 + r_3) &= \sigma^2\left(\sigma_0\Delta z_0 + \sigma_0\Delta z_0 + \sigma_1\Delta z_1 + \sigma_0\Delta z_0 + \sigma_1\Delta z_1 + \sigma_2\Delta z_2\right)\\
&= \sigma^2\left(3\sigma_0\Delta z_0 + 2\sigma_1\Delta z_1 + \sigma_2\Delta z_2\right)\\
&= \sigma^2\left(3\sigma_0\Delta z_0\right) + \sigma^2\left(2\sigma_1\Delta z_1\right) + \sigma^2\left(2\sigma_1\Delta z_1\right)\\
&\quad + 2\mathrm{Cov}\left(3\sigma_0\Delta z_0, 2\sigma_1\Delta z_1\right)\\
&\quad + 2\mathrm{Cov}\left(3\sigma_0\Delta z_0, 2\sigma_2\Delta z_2\right) + 2\mathrm{Cov}\left(2\sigma_1\Delta z_1, \sigma_1\Delta z_2\right)\\
&= 9\sigma_0^2 + 4\sigma_1^2 + \sigma_2^2
\end{aligned}$$

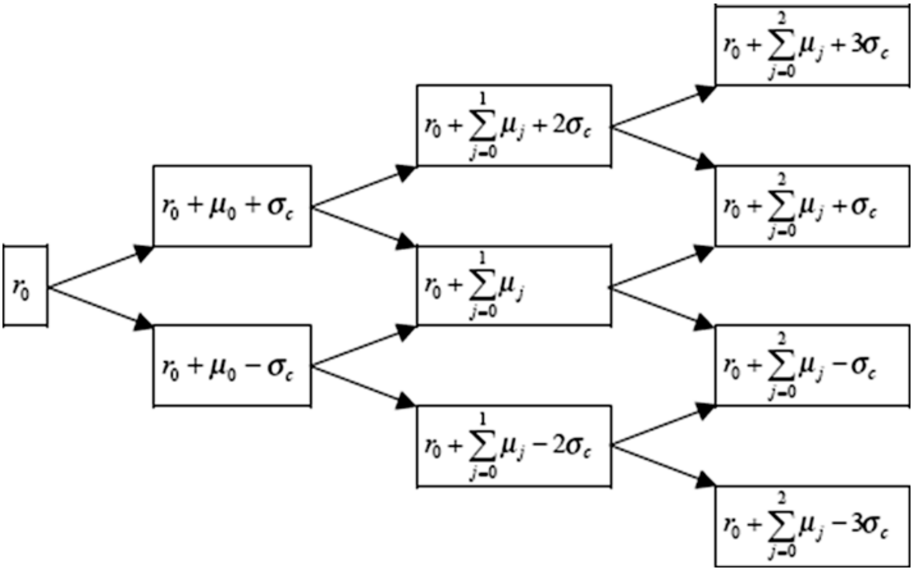

Fig. 15.5 The Ho-Lee binominal tree with constant volatility

Therefore, in general,

$$\sigma^2\left(\sum_{j=1}^{t} r(j)\right) = \sigma^2\left(\sum_{k=1}^{t}(t-k+1)\sigma_{k-1}\Delta z_{k-1}\right) = \sum_{k=1}^{t}(t-k+1)^2\sigma_{k-1}^2$$

For example,

$$\sigma^2(r_1+r_2+r_3+r_4) = 16\sigma_0^2 + 9\sigma_1^2 + 4\sigma_2^2 + \sigma_3^2$$

The implementation of can be made easier if we use matrix notation. Let $\mathbf{D}_t$ denote a diagonal t x t matrix whose elements are $d_{ii} = \sigma_{i-1}^2$. Let $\mathbf{w}_t$ denote a t-dimensional column vector whose elements are the integer values of the index t in reverse order. Then, the previous expression can be written as

$$\sigma^2\left(\sum_{j=1}^{t} r(j)\right) = \mathbf{w}_\mathbf{t}^\mathbf{T}\mathbf{D}_\mathbf{t}\mathbf{w}_\mathbf{t} \quad \forall t$$

where T denotes transposition.

For example, for $t = 4$

$$\sigma^2\left(\sum_{j=1}^{4} r(j)\right) = \mathbf{w_t^T D_t w_t} = \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \sigma_0^2 & 0 & 0 & 0 \\ 0 & \sigma_1^2 & 0 & 0 \\ 0 & 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 & \sigma_3^2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= 16\sigma_0^2 + 9\sigma_1^2 + 4\sigma_2^2 + \sigma_3^2$$

Implementation of the Model

The model can be implemented in some different ways, depending on the volatility structure.

First the volatility of the short rate is not constant (i.e. it differs as time changes). For example, the volatility of the short rate at any given time can be $\{\sigma(0), \sigma(1), \ldots, \sigma(T-1)\}$where T denotes the horizon of the analysis. When we compute the evolution of the short interest rate as a binomial model, this example produces a short-rate tree.

Secondly the short rate is constant at all times, this is the canonical Ho and Lee model. When we compute the evolution of the short interest rate as a binomial model, the canonical model produces a short-rate lattice. This model requires a numerical solution for the short interest rate one period ahead and forward induction (see previous BDT trees).

Note that the assumption of constant volatility is not necessary for producing a lattice. The volatility structure allows the volatility of the short rate to vary across short rates but to be constant for each short rate as time elapses. For example, the volatility of the short rate from date 3 to date 4 can differ from the volatility of the short rate from date 4 to date 5 but those two different volatilities do not change as time elapses. The effect of this non-constant volatility structure is quite different from that of the first model.

The third kind of model are quite dissimilar. The tree is built from the evolution of the short interest rate as well as the satisfaction of no-arbitrage conditions. These conditions are that the bond prices are recovered at date 0 and that volatility of interest rates obtains at every date. In addition, the equality of one-period rates of returns is thereby satisfying the interpretation of no-arbitrage as equality of local expectations. In other words, at any vertex (except those on the last date), we can calculate the expectation of the rate of return on a bond. This expectation of the rate of return should equal the short rate evolved at that vertex. If this equality is not obtained arbitrage profits are possible.

15.1.3.3 HW Models

If the parameters are dependent of time, the possibility to calibrate them is much better. The HW model (1990) is a generalization of the Vasicek model with time-dependent parameters

$$dr = (\theta(t) - a(t)r)\, dt + \sigma(t)dV(t)$$

where $\theta(t)$ is a deterministic function of time. Typically, the parameters a and σ is calibrated against the volatility and thereafter $\theta(t)$ is calibrated against the theoretical bond prices, $\{p(0,T): T \geq 0\}$ to the observed curve $\{p*(0,T): T \geq 0\}$. To see why both a and σ is calibrated against the volatility we recall the term-structure equation:

$$\begin{cases} \dfrac{\partial F^T}{\partial t} + \{\mu(t,r) - \lambda(t,r)\sigma(t)\} \dfrac{\partial F^T}{\partial r} + \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 F^T}{\partial r^2} - r(t)F^T = 0 \\ F(r,T,T) = 1 \end{cases}$$

The drift is given by: $\mu(t,r) - \lambda(t,r)\sigma(t)$. If we compare the drift terms we see that the parameters in the HW models include the market price of risk and the volatility.

$$\mu(t,r) - \lambda(t,r)\sigma(t) = \theta(t) - a(t)r(t).$$

A breakthrough with this model is that it is possible to use trinomial trees. Also HW (and the Ho-Lee model) allows negative interest rates. Note that the HW model with $a(t) = 0$ is equivalent to the Ho-Lee model.

The model has an ATS

$$p(t,T) = F(r(t),t,T) = e^{A(t,T) - B(t,T)r}$$

and can be simplified if we let a to be a constant. We then get the following equation for $B(t,T)$

$$\begin{cases} \dfrac{\partial B}{\partial t}(t,T) - aB(t,T) = -1 \\ B(T,T) = 0 \end{cases}$$

This can easily be solved, and the answer is (as we have seen earlier)

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

Inserting this in the equation of A, we obtain the equation

$$\begin{cases} \dfrac{\partial A(t,T)}{\partial t} - \theta(t)\cdot B(t,T) + \dfrac{\sigma^2}{2}B^2(t,T) = 0 \\ A(T,T) = 0 \end{cases}$$

If we integrate this, we get

$$\begin{aligned} A(t,T) &= \int_t^T \left\{\frac{\sigma^2}{2}B^2(s,T) - \theta(s)\cdot B(s,T)\right\}ds \\ &= \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds - \int_t^T \theta(s)\cdot B(s,T)ds \end{aligned}$$

We will now calibrate the model to the observed initial yield curve, using

$$f(t,T) = -\frac{\partial\left[\ln p(t,T)\right]}{\partial T}$$

and

$$p(t,T) = \exp\left\{A(t,T) - rB(t,T)\right\}.$$

Since

$$B_T(t,T) = \frac{\partial}{\partial T}\left(\frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}\right) = e^{-a(T-t)}$$

and

$$\frac{\partial}{\partial\tau}\int_t^T \theta(s)\cdot B(s,\tau)ds = \int_t^T \theta(s)\cdot\frac{\partial}{\partial\tau}B(s,\tau)ds = \int_t^T \theta(s)e^{-a(T-s)}ds$$

the initial forward rates are given by

$$\begin{aligned} f^*(0,T) &= B_T(0,T)\cdot r(0) - A_T(0,T) \\ &= r(0)e^{-aT} + \int_0^T \theta(s)e^{-a(T-s)}ds - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 \end{aligned}$$

We recognize the first two terms as the solution to an ordinary differential equation (ODE) of the first order. So, the simplest way to solve this is to do the following trick and write this as

$$f^*(t,T) = x(T) - g(T)$$

where

$$\begin{cases} \dot{x}(T) = -ax(T) + \theta(T) \\ x(0) = r(0)e^{-aT} \end{cases}$$

and[2]

$$\begin{aligned}
x(T) &= r(0)e^{-aT} + \int_0^T \theta(s)e^{-a(T-s)}ds \\
g(T) &= \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 = \frac{\sigma^2}{2}B^2(0,T) \\
g_T(T) &= \frac{\sigma^2}{2}\frac{\partial}{\partial T}B^2(0,T) = \sigma^2 B(0,T)\frac{d}{dT}B(0,T) = \sigma^2 B(0,T)e^{-aT}
\end{aligned}$$

We then get

$$\begin{aligned}
\theta(T) &= \dot{x}(T) + a \cdot x(T) = \left\{x(T) = f^*(0,T) + g(T)\right\} \\
&= f_T^*(0,T) + g_T(T) + a\left\{f^*(0,T) + g(T)\right\} \\
&= f_T^*(0,T) + a \cdot f^*(0,T) + \sigma^2 \cdot B(0,T) \cdot e^{-aT} + a \cdot \frac{\sigma^2}{2} \cdot B^2(0,T) \\
&= f_T^*(0,T) + a \cdot f^*(0,T) + \sigma^2 \cdot B(0,T)\left(e^{-aT} + \frac{1}{2}\left(1-e^{-aT}\right)\right) \\
&= f_T^*(0,T) + a \cdot f^*(0,T) + \frac{\sigma^2}{2a}\left(1-e^{-aT}\right)\left(1+e^{-aT}\right) \\
&= f_T^*(0,T) + a \cdot f^*(0,T) + \frac{\sigma^2}{2a}\left(1-e^{-2aT}\right)
\end{aligned}$$

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2 $$\dot{x} - ax = \theta \Rightarrow e^{-at}\dot{x} - e^{-at}ax = e^{-at}\theta \Rightarrow \frac{d}{dt}\left(e^{-at}x\right) = e^{-at}\theta \Rightarrow d\left(e^{-at}x\right) = e^{-at}\theta dt$$
$$\int_{x(0)}^{x(T)} \frac{d}{dx}e^{-at}x dx = \int_0^T e^{-at}\theta dt \Rightarrow x(T) = e^{-aT}x(0) + e^{-aT}\int_0^T e^{-as}\theta(s)ds$$

With this function $\theta(T)$ we have fixed the values of a and σ and decided the martingale measure to use. Remember that

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds - \int_t^T \theta(s)B(s,T)ds$$

We start to calculate the integral

$$\begin{aligned}\frac{\sigma^2}{2}\int_t^T B^2(s,T)ds &= \frac{\sigma^2}{2a^2}\int_t^T \left(e^{-a(T-s)}-1\right)^2 ds \\ &= \frac{\sigma^2}{2a^2}\left[\frac{1}{2a}\left(1-e^{-2a(T-t)}\right)+(T-t)-\frac{2}{a}\left(1-e^{-a(T-t)}\right)\right]\end{aligned}$$

Next we compute the integral

$$\begin{aligned}\int_t^T \theta(s)B(s,T)ds &= \frac{1}{a}\int_t^T \theta(s)\left(1-e^{-a(T-s)}\right)ds \\ &= -\frac{1}{a}\int_t^T \theta(s)e^{-a(T-s)}ds + \frac{1}{a}\int_t^T \theta(s)ds \\ &= -\frac{1}{a}\int_t^T e^{-a(T-s)}\left(f_s^*(0,s)+a\cdot f^*(0,s)\right)ds \\ &\quad + \frac{1}{a}\int_t^T \left(f_s^*(0,s)+a\cdot f^*(0,s)\right)ds \\ &\quad + \frac{\sigma^2}{2a}\int_t^T \left(1-e^{-a(T-s)}\right)\left(1-e^{-2as}\right)ds\end{aligned}$$

The last integral is

$$\begin{aligned}&\frac{\sigma^2}{2a}\int_t^T \left(1-e^{-a(T-s)}\right)\left(1-e^{-2as}\right)ds \\ &\quad = -\frac{\sigma^2}{2a^3}\left[1-e^{-a(T-t)}+\frac{1}{2}e^{-2aT}-e^{-a(T+t)}+\frac{1}{2}e^{-2at}\right]+\frac{\sigma^2}{2a^2}(T-t)\end{aligned}$$

We now have

$$\int_t^T \theta(s)B(s,T)ds = -\frac{1}{a}\int_t^T e^{-a(T-s)}f_s^*(0,s)ds - \int_t^T e^{-a(T-s)}f^*(0,s)ds$$
$$+\frac{1}{a}\left[f^*(0,s)\right]_t^T + \int_t^T f^*(0,s)ds$$
$$-\frac{\sigma^2}{2a^3}\left[1-e^{-a(T-t)}+\frac{1}{2}e^{-2aT}-e^{-a(T+t)}+\frac{1}{2}e^{-2at}\right]$$
$$+\frac{\sigma^2}{2a^2}(T-t)$$

Next, we compute

$$\frac{1}{a}\int_t^T e^{-a(T-s)}f_s^*(0,s)ds = \frac{1}{a}\left[e^{-a(T-s)}f^*(0,s)\right]_t^T - \int_t^T e^{-a(T-s)}f^*(0,s)ds$$

to get

$$\int_t^T \theta(s)B(s,T)ds$$
$$= -\frac{1}{a}\left[e^{-a(T-s)}f^*(0,s)\right]_t^T + \int_t^T e^{-a(T-s)}f^*(0,s)ds$$
$$-\int_t^T e^{-a(T-s)}f^*(0,s)ds + \frac{1}{a}\left(f^*(0,T)-f^*(0,t)\right) + \int_t^T f^*(0,s)ds$$
$$-\frac{\sigma^2}{2a^3}\left[1-e^{-a(T-t)}+\frac{1}{2}e^{-2aT}-e^{-a(T+t)}+\frac{1}{2}e^{-2at}\right]+\frac{\sigma^2}{2a^2}(T-t)$$

Now, simplification gives

$$\int_t^T \theta(s)B(s,T)ds = f^*(0,t)B(t,T) + \int_t^T f^*(0,s)ds$$
$$-\frac{\sigma^2}{2a^3}\left[1-e^{-a(T-t)}+\frac{1}{2}e^{-2aT}-e^{-a(T+t)}+\frac{1}{2}e^{-2at}\right]+\frac{\sigma^2}{2a^2}(T-t)$$

Combining the two integrals we get

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds - \int_t^T \theta(s)B(s,T)ds$$

$$= \frac{\sigma^2}{2a^2}\left[\frac{1}{2a}\left(1-e^{-2a(T-t)}\right) + (T-t) - \frac{2}{a}\left(1-e^{-a(T-t)}\right)\right]$$

$$+ f^*(0,t)B(t,T) - \int_t^T f^*(0,s)ds$$

$$- \frac{\sigma^2}{2a^3}\left[1-e^{-a(T-t)} + \frac{1}{2}e^{-2aT} - e^{-a(T+t)} + \frac{1}{2}e^{-2at}\right] + \frac{\sigma^2}{2a^2}(T-t)$$

$$= f^*(0,t)B(t,T) - \int_t^T f^*(0,s)ds + \frac{\sigma^2}{4a}B^2(t,T)\left(e^{-2at}-1\right)$$

If we use

$$\int_t^T f(0,s)ds = -\ln\left(\frac{p(0,T)}{p(0,t)}\right)$$

we finally have

$$A(t,T) = f^*(0,t)B(t,T) + \ln\left(\frac{p(0,T)}{p(0,t)}\right) + \frac{\sigma^2}{4a}B^2(t,T)\left(e^{-2at}-1\right)$$

Thus, the bond prices are given by

$$p(t,T) = \exp\left\{A(t,T) - rB(t,T)\right\}$$

$$= \exp\left\{f^*(0,t)B(t,T) + \ln\left(\frac{p(0,T)}{p(0,t)}\right) + \frac{\sigma^2}{4a}B^2(t,T)\left(e^{-2at}-1\right) - rB(t,T)\right\}$$

$$= \frac{p^*(0,T)}{p^*(0,t)}\exp\left\{\frac{1}{a}\left(1-e^{-a(T-t)}\right)\left(f^*(0,t)-r\right) - \frac{\sigma^2}{4a^3}\left(1-e^{-a(T-t)}\right)^2\left(1-e^{-2at}\right)\right\}$$

The price volatility is the same as in the Vasicek model:

$$\sigma_p = \frac{\sigma_r}{a}\left(1-e^{-a(T-t)}\right)$$

We now return to the HW stochastic process:

$$dr = (\theta(t) - a(t)r)\,dt + \sigma(t)dV(t)$$

We can also solve this like

$$\begin{aligned} d(e^{at}r) &= e^{at}dr + ae^{at}r(t)dt = e^{at}\left(\theta(t) - ar(t)\right)dt + e^{at}\sigma(t)dV + ar(t)e^{at}dt \\ &= e^{at}\theta(t)dt + e^{at}\sigma(t)dV \end{aligned}$$

Integration gives

$$e^{at}r(t) = e^{as}r(0) + \int_s^t \theta(u)e^{au}du + \int_s^t \sigma(u)e^{au}dV_u$$

which simplifies to

$$r(t) = r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma\int_s^t e^{-a(t-u)}dV_u$$

This gives the short rate at time t conditional on the information given at time s.

Since $E^Q\left[\sigma\int_t^T e^{-a(T-u)}dV_u \middle| \mathcal{F}_t\right] = 0$ we have

$$E^Q\left[r(T)\right] = r(t)e^{-a(T-t)} + \int_t^T e^{-a(T-u)}\theta(u)du$$

$$\begin{aligned} Var\left[r(T)\right] &= \sigma^2 E\left[\left(\int_t^T e^{-a(T-u)}dV_u\right)^2\right] \\ &= \sigma^2\int_t^T e^{-2a(T-u)}du = \frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right) \end{aligned}$$

Notice that

$$\int_t^T e^{-a(T-u)}\theta(u)du = \int_t^T \frac{\partial}{\partial T}B(u,T)\theta(u)du = \frac{\partial}{\partial T}\int_t^T B(u,T)\theta(u)du$$

where we used that $B(T,T) = 0$ in the last part. From the previous equation, we do know that

$$\int_t^T \theta(u)B(u,T)du = f^*(0,t)B(t,T) + \int_t^T f^*(0,u)du$$
$$-\frac{\sigma^2}{2a^3}\left[1 - e^{-a(T-t)} + \frac{1}{2}e^{-2aT} - e^{-a(T+t)} + \frac{1}{2}e^{-2at}\right] + \frac{\sigma^2}{2a^2}(T-t)$$

so

$$\int_t^T e^{-a(T-u)}\theta(u)du = f^*(0,t)\frac{\partial}{\partial T}B(t,T) + \frac{\partial}{\partial T}\int_t^T f^*(0,s)ds$$
$$-\frac{\sigma^2}{2a^2}\left[e^{-a(T-t)} + e^{-2aT} - e^{-a(T+t)} - 1\right]$$
$$= f^*(0,t)e^{-a(T-t)} + f^*(0,T)$$
$$-\frac{\sigma^2}{2a^2}\left[e^{-a(T-t)} + e^{-2aT} - e^{-a(T+t)} - 1\right]$$

This can be written as

$$\int_t^T e^{-a(T-u)}\theta(u)du = \gamma(T) - \gamma(t)e^{-a(T-t)}$$

where

$$\gamma(t) = f^*(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2$$

Thus

$$E^Q\left[r(T)\right] = r(t)e^{-a(T-t)} + \gamma(T) - \gamma(t)e^{-a(T-t)}$$

$$Var\left[r(T)\right] = \frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)$$

We also have that the price at time t of a bond paying 1 (cash unit) at time T (maturity) is given as the solution to the aforementioned term-structure equation with following Feynmann-Kac formula

$$p(t,T) = E^Q\left[\exp\left\{\int_t^T r(u)du\right\} \times 1\,|\mathcal{F}_t\right]$$
$$= \exp\left\{-E^Q\left[\int_t^T r(u)du\right] + \frac{1}{2}Var\left[\int_t^T r(u)du\right]\right\}$$

The mean (expectation value) and variance can be calculated analytically.

A general pricing formula for a payoff function $\varphi(X(T))$ is given as:

$$p(t,T) = E^Q\left[\exp\left\{\int_t^T r(u)du\right\} \times \varphi\left(X(T)\right)|\mathcal{F}_t\right]$$

Option Pricing

To price a European call option with maturity T and strike price K on an S-bond, we get the arbitrage-free price as

$$\pi_0\left[X\right] = E^Q\left[\max\left\{L \cdot p(S,T) - K, 0\right\} \cdot \exp\left\{-\int_0^T r(s)ds\right\}\right]$$

It is possible to get an analytical result from this, but again the calculations are quite complex. We therefore wait until we learned about forward measure. The result for a European call option is given by

$$C(t,T,K,S) = L \cdot p(t,S) \cdot N(d) - K \cdot p(t,T) \cdot N(d - \sigma_p)$$

and a European put option by

$$P(t,T,K,S) = K \cdot p(t,T) \cdot N(-d + \sigma_p) - p(t,S) \cdot N(-d)$$

where

$$d = \frac{\ln\left\{\frac{L \cdot p(t,S)}{K \cdot p(t,T)}\right\} + \frac{\sigma_p^2}{2}}{\sigma_p}$$
$$\sigma_p = \frac{1}{a}\left(1 - e^{-a(S-T)}\right)\sqrt{\frac{\sigma^2}{2a}\left(1 - e^{2a\cdot(T-t)}\right)}.$$

Calibration of Volatility Data

In contrast to the Ho-Lee model, the HW spot rate volatility equation involves two parameters, a and σ: σ determines the overall the volatility of the short rate and a the relative volatility of long and short rates. In order to calibrate the model to the market prices we can follow the procedures outlined earlier for the Ho-Lee model, but now best fit both a and σ simultaneously to market data. If we assume that we have the prices of m individual European put options on pure discount bonds we now minimize the following function

$$\min_{a,\sigma} \sqrt{\sum_{i=1}^{m} \left(\frac{\mathrm{model_i}(\mathrm{a}, \sigma) - market_i}{market_i} \right)^2}$$

where $model_i(a, \sigma)$ is the option value derived from the equation for put options previously with the parameter values a and σ.

Cap/Floor Evaluation

The price of a cap is the sum of the prices of all its caplets. Therefore, knowing how to give a price of any caplet with the HW model is sufficient to compute the price of any cap. The price of a caplet can be expressed as the prices of a put on a zero-coupon bond, whose price is analytically computable with this model. More explicitly, if the rate convention is linear, the price of a caplet with strike X and nominal N is linked with the price of an option on a zero-coupon bond whose strike is

$$X' = \frac{1}{1 + \tau \cdot X},$$

where τ is the day-count fraction, whose nominal equals $N' = N(1 + \tau X)$.

The volatilities of the zero-coupon bond at S with maturity at T is given in the following formula for constant volatilities and mean reversion case

$$\sigma_p = \frac{1}{\sqrt{S}} \frac{1}{a} \left(1 - e^{-a(T-S)}\right) \sqrt{\frac{\sigma^2}{2a} \left(1 - e^{2aS}\right)}.$$

When model volatilities as a piecewise constant function as in the case for bootstrapping, the variance of the bond is given in the following formula:

$$\sigma_p^2 = \frac{1}{S}\frac{\left(e^{-a(T-S)} - 1\right)}{a^2}e^{-2aS}I(S)$$

where

$$I(S) = \sum_{i=1}^{n-1} \sigma_i^2 \frac{e^{2at_i} - e^{2at_{i-1}}}{2a}$$

where $t_0 = 0$ and $t_{n-1} = S$

Swaption Evaluation

A European swaption with strike rate X, maturity T, nominal N and payments dates $T_1, \ldots, T_n$ can be viewed as an option on a bond paying $c_i = X\tau_i$ for $i = 1, \ldots, n$ and $c_i = 1 + X\tau_i$ with a strike price of N.

Applying Jamshidians decomposition, N is written as a sum of discounted flows c_i for a certain short-rate r^* at time T which is evaluated using Newton-Raphsons algorithm.

$$N = \sum_{i=1}^{n} c_i \Pi(T, T_i, r^*)$$

The payoff is then rewritten as

$$\left[N\sum_{i=1}^{n} c_i \Pi(T, T_i, r^*) - \Pi(T, T_i, r(T))\right]^+$$

and since $\Pi(T, S, r)$ is a monotonic function of r for all S, T, the positive part of the sum can be converted as the sum of positive parts

$$N\sum_{i=1}^{n}\left[c_i \Pi(T, T_i, r^*) - \Pi(T, T_i, r(T))\right]^+$$

Therefore, pricing a swaption becomes equivalent to pricing a portfolio of options on zero-coupon bonds. As there exist some analytical formulas for such options, a swaption is analytically valuable with this model. During the calibration, the target price to match is the one giving by pricing a European swaption with a Black swap yield model.

Numerical Solution of the PDE

The term-structure equation for the HW model is given by:

$$\begin{cases} \dfrac{\partial U(r,t)}{\partial t} + \dfrac{1}{2}\sigma^2 \frac{\partial^2 U(r,t)}{\partial r^2} + \{\theta(t) - a \cdot r\} \dfrac{\partial U(r,t)}{\partial r} - r \cdot U(r,t) = 0 \\ U(r,T) = 1 \end{cases}$$

where

$$\theta(t) = \frac{df(t,T)}{dt} + a \cdot f(t,T) + \frac{\left(1 - e^{-2at}\right)\sigma^2}{2a}$$

This can be solved with Crank-Nicholson where

$$\begin{aligned} u_{ij} &= u(r_i, t_j) \\ \frac{du(r_i,t_j)}{dt} &= \frac{u_{ij+1} - u_{ij}}{k} \\ \frac{\partial u(r_i,t_j)}{\partial r} &= \frac{u_{i+1j} - u_{i-1j}}{2h} \\ \frac{\partial^2 u(r_i,t_j)}{\partial r^2} &= \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2} \end{aligned}$$

We then have

$$\frac{du_{ij}}{dt} = r \cdot u_{ij} - \left\{\theta(t_j) - a \cdot r_i\right\} \frac{u_{i+1j} - u_{i-1j}}{2h} - \frac{1}{2}\sigma^2 \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2}$$

This can be rewritten as

$$\begin{aligned} \frac{du_{ij}}{dt} = &\left(-\frac{\sigma^2}{2h^2} + \frac{\left\{\theta(t_j) - a \cdot r_i\right\}}{2h}\right) \cdot u_{i-1j} \\ &+ \left(\frac{\sigma^2}{h^2} + r_i\right) \cdot u_{ij} - \left(\frac{\sigma^2}{2h^2} + \frac{\left\{\theta(t_j) - a \cdot r_i\right\}}{2h}\right) \cdot u_{i+1j} \end{aligned}$$

or

$$\frac{du_{ij}}{dt} = x(r_i,t_j) \cdot u_{i-1j} + y(r_i) \cdot u_{ij} - z(r_i,t_j) \cdot u_{i+1j}$$

For a solution we need the initial condition $u(r_i, T)$ and the boundary conditions $u(r_0, t_j)$ and $u(r_N, t_j)$, where $t_j = jk$, $j = 0, 1, 2, \ldots, T$ and $r_i = ih$, $i = 0, 1, 2, \ldots, N$.

On matrix form, we can express this as

$$\frac{d\mathbf{A}}{dt} = \mathbf{A}(t)\mathbf{u}(t)$$

where

$$A(t) = \begin{pmatrix} x_{B0}(r_0,t) & y_{B0}(r_0) & z_{B0}(r_0,t) & 0 & 0 & \dots & 0 \\ 0 & x(r_1,t) & y(r_1) & z(r_1,t) & 0 & \dots & \vdots \\ 0 & 0 & x & y & z & \vdots & \vdots \\ \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \dots & x_{BN}(r_N,t) & y_{BN}(r_N) & z_{BN}(r_N,t) \end{pmatrix}$$

Crank-Nicholson gives

$$\frac{\mathbf{u}_{j+1} - \mathbf{u}_j}{k} = \frac{\mathbf{A}(t_j)\mathbf{u}_j + \mathbf{A}(t_{j+1})\mathbf{u}_{j+1}}{2}$$

which is solved as

$$\mathbf{u}_j = \left(\mathbf{I} + \frac{k}{2}\mathbf{A}(t_j)\right)^{-1}\left(\mathbf{I} - \frac{k}{2}\mathbf{A}(t_{j+1})\right)\mathbf{u}_{j+1}$$

We use this equation of $N+1$ unknown backward in time. When we solve this we use the previously mentioned forward rate $f(t,T)$ given by

$$f(t,T) = R(t,T) + t \cdot \frac{dR(t,T)}{dt}$$

where $R(t,T)$ is the continuously compounded interest rate given by the yield curve. Taking the derivative of $f(t,T)$ with respect to the time t, we have

$$\frac{df(t,T)}{dt} = 2 \cdot \frac{dR(t,T)}{dt} + t \cdot \frac{d^2R(t,T)}{dt^2}$$

Since we are used this derivative previously, which includes a second derivative of R (the yield curve) with respect to time, we get problems with piecewise linear yield curves. The solution to this problem is described by Antoon Pelsser in the book *Efficient Methods for Valuing*

Interest Rate Derivatives (Springer, 2000). We start by defining

$$\alpha(t) = f(t,T) + \frac{\left(1 - e^{-at}\right)^2 \sigma^2}{2a^2}$$

and the transformation

$$y = r - \alpha(t) \Leftrightarrow r = y + \alpha(t)$$

We then get the following PDE:

$$\frac{\partial V(r,t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(r,t)}{\partial y^2} - a \cdot y \frac{\partial V(r,t)}{\partial y} - y \cdot V(t,y) = 0$$

where the option value U is calculated as:

$$U = V \cdot e^{-k}$$

where

$$\begin{aligned} k &= \tau \cdot R(\tau,T) - t \cdot R(t,T) \\ &\quad + \left[a \cdot (\tau - t) - 2\left(e^{-at} - e^{-a\tau}\right) + \frac{1}{2}\left(e^{-2at} - e^{-2a\tau}\right) \right] \frac{\sigma^2}{2a^3} \end{aligned}$$

and τ any non-negative time. If $\tau = 0$ we then have.

$$k = -t \cdot R(t,T) + \left[-at - 2\left(e^{-at} - 1\right) + \frac{1}{2}\left(e^{-2at} - 1\right) \right] \frac{\sigma^2}{2a^3}$$

There are a couple of major advantages with this transformation:

- The “implied transition probability distribution” (i.e. the process x) in the PDE grid is independent of the yield curve, making the bucket Greeks more stable.
- Only f is needed (i.e. no derivative of f). Therefore the yield curve only needs to be differentiated once, so we do not have any problem with piecewise linear yield curves.

Note that the short-rate r must be calculated for every state x for every “exercise event”, since it’s used to calculate the state-dependent yield curve.

Trinomial Trees

Instead of solving the PDE as earlier, we can also build a trinomial tree for the HW model. As usually, we start with the stochastic process

$$dr = [\theta\,(t) - ar]\,dt + \sigma\,dz$$

where r is the instantaneous short rate and $\theta(t)$ a function of time t. a and σ are supposed to be constants. This shows that, at any given time, r reverts towards $\theta(t)/a$. If we replace r with $\ln(r)$ the model becomes BK and with $a(t) = -\sigma'(t)/\sigma(t)$, and $\sigma'(t) = \partial\sigma/\partial t$, the model becomes the BDT model. As the BDT model the tree is built with forward inductions.

We assume that the Δt and the interest rate R, follows the same process as r, where

$$dR = [\theta\,(t) - ar]\,dt + \sigma\,dz$$

We can then construct a tree for R^* that is initially zero and follows the process

$$dR^* = -aR^*dt + \sigma\,dz$$

The reason is that The interest rate $r(t)$ can be decomposed into a sum of $R^*(t)$ and $\alpha(t)$: $r(t) = R^*(t) + \alpha(t)$ where

$$\begin{cases} dR^*(t) = -\alpha x(t)dt + \sigma(t)dW(t) \\ R^*(0) = 0 \end{cases}$$

$$\begin{cases} d\alpha(t) = (\theta(t) - a \cdot \alpha(t))\,dt \\ \alpha(0) = r(0) \end{cases}$$

Proof: Integration gives

$$R^*(t) = -a\int_0^t R^*(s)ds + \int_0^t \sigma(s)dW(s)$$

$$\alpha(t) - r(0) = \int_0^t (\theta(s) - a \cdot \alpha(s))\,ds$$

giving

$$R^*(t) + \alpha(t) = r(0) - a \int_0^t R^*(s)ds + \int_0^t \sigma(s)dW(s) + \int_0^t (\theta(s) - a \cdot \alpha(s))\, ds$$

$$= r(0) + \int_0^t \left(\theta(s) - a \cdot \left\{\alpha(s) + R^*(s)\right\}\right) ds + \int_0^t \sigma(s)dW(s)$$

$$= r(0) + \int_0^t (\theta(s) - a \cdot r(s))\, ds + \int_0^t \sigma(s)dW(s)$$

We also see that we now have

$$r(t) = r(0) + \int_0^t (\theta(s) - a \cdot r(s))\, ds + \int_0^t \sigma(s)dW(s) = \alpha(t) + \int_0^t \sigma(s)dW(s)$$

Integration of dr from s to t (using integration factor) gives for constant volatility:

$$r(t) = r(s) \cdot e^{-a(t-s)} + \int_s^t \theta(u) \cdot e^{-a(t-u)} du + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

$$= \alpha(t) + (r(s) - \alpha(s)) \cdot e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW(u)$$

where

$$\alpha(t) = f(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-at}\right)^2$$

Therefore, $r(t)$ with the information known at time s is normally distributed with mean and variance given by

$$E\left[r(t) \,|\, \mathcal{F}_s\right] = \alpha(t) + (r(s) - \alpha(s)) \cdot e^{-a(t-s)}$$

$$Var\left[r(t) \,|\, \mathcal{F}_s\right] = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right)$$

Since HW have an ATS with zero-coupon prices given by

$$p(t,T) = e^{A(t,T) - B(t,T)r}$$

where

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

and

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)ds - \int_t^T \theta(s)B(s,T)ds$$

giving

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\left\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)\left(1 - e^{-2at}\right) - B(t,T)r(t)\right\}$$

Here $p^*(0,t)$ and $p^*(0,T)$ are zero-coupon bonds with maturity at t and T respectively.

By using $r(t) = R^*(t) + \alpha(t)$ as shown earlier, we have

$$C(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \cdot \exp\left\{\frac{1}{2}\left[V(t,T) - V(0,T) + V(0,t)\right]\right\}$$

where

$$V(t,T) = \int_t^T \left[\sigma(s)B(s,T)\right]^2 ds$$

For constant volatility we can integrate this

$$V(t,T) = \sigma^2\int_t^T \left[B(s,T)\right]^2 ds = \frac{\sigma^2}{a^2}\int_t^T \left[1 - e^{-a(T-s)}\right]^2 ds$$

$$= \frac{\sigma^2}{a^2}\int_t^T \left[1 - 2\cdot e^{-a(T-s)} + e^{-2a(T-s)}\right] ds$$

$$= \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}\cdot e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]$$

The process is symmetrical around $R^* = 0$ and $R^*(t + \Delta t) - R^*(t)$ is normally distributed. If higher order terms than Δt are ignored we have

$$E\left[R^*(t + \Delta t) - R^*(t)\right] = -aR^*(t)\,\Delta t$$

and

$$Var\left[R^*(t+\Delta t) - R^*(t)\right] = \sigma^2 \Delta t$$

We define ΔR as the spacing between interest rates on the tree and set

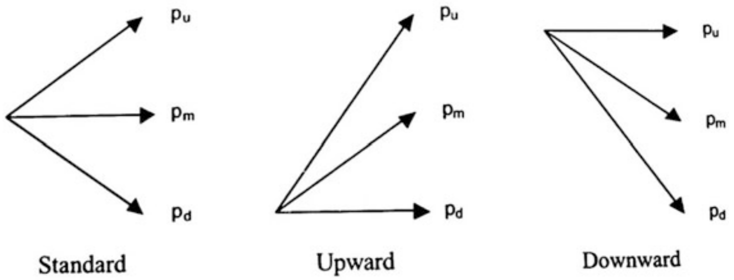
$$\Delta R = \sigma\sqrt{3\Delta t}$$

Remark! This is the same condition used to get good convergence when solving PDE's.

We are now going to build a tree as shown in Fig. 15.6.

We then face the three branching situations as

In order to calculate these three kinds of nodes, we need some formulas. First, we define (i, j) as the node where $t = i\Delta t$ and $R^* = j\Delta R$. HW showed that the probabilities are always positive if we set j_{max} equal



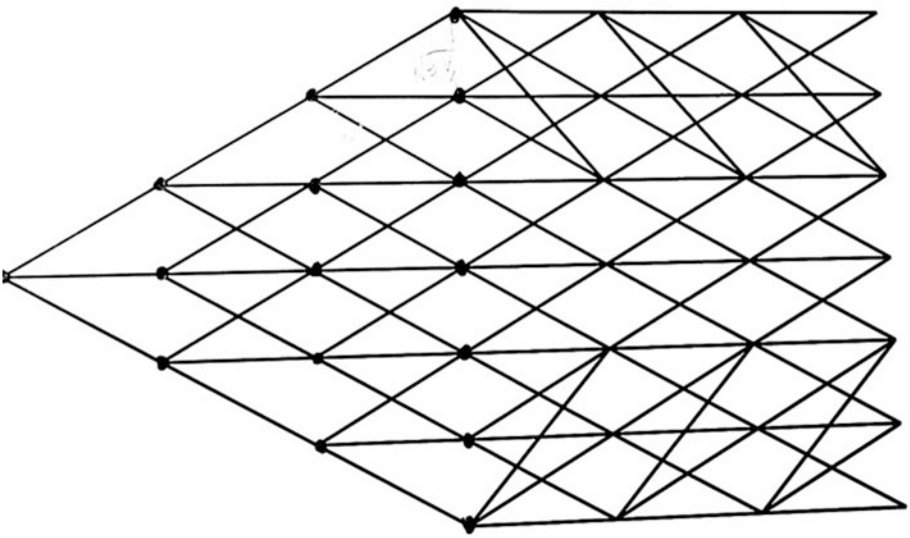


Fig. 15.6 The HW trinomial tree

to the smallest integer greater than $0.184/(a\Delta t)$. We then define the transition probabilities, p_u, p_m and p_d which are used to match the expected value and variance of $R^*(t + \Delta t) - R^*(t)$ over a time interval Δt. At the node (i, j) the standard branching must satisfy

$$\begin{aligned} p_u \Delta R - p_d \Delta R &= -aj\Delta R \Delta t \\ p_u \Delta R^2 + p_d \Delta R^2 &= \sigma^2 \Delta t + a^2 j^2 \Delta R^2 \Delta t^2 \\ p_u + p_m + p_d &= 1 \end{aligned}$$

Using $\Delta R = \sigma\sqrt{3\Delta t}$ to solve this equation system, we get the up-, middle- and down-branching probabilities:

$$\begin{aligned} P_u &= 1/6 + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2} \\ P_m &= 2/3 - a^2 j^2 \Delta t^2 \\ P_d &= 1/6 + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2} \end{aligned}$$

As we can see in the aforementioned tree, we cope with mean reversion by allowing the branching to be non-standard at the edge of the tree. At the top edge of the tree where the branching is non-standard the modified probabilities become

$$\begin{aligned} P_u &= 7/6 + \frac{a^2 j^2 \Delta t^2 - 3aj\Delta t}{2} \\ P_m &= -1/3 - a^2 j^2 \Delta t^2 + 2aj\Delta t \\ P_d &= 1/6 + \frac{a^2 j^2 \Delta t^2 - aj\Delta t}{2} \end{aligned}$$

and at the bottom edge of the tree where the branching is non-standard the modified probabilities become:

$$\begin{aligned} P_u &= 1/6 + \frac{a^2 j^2 \Delta t^2 + aj\Delta t}{2} \\ P_m &= -1/3 - a^2 j^2 \Delta t^2 - 2aj\Delta t \\ P_d &= 7/6 + \frac{a^2 j^2 \Delta t^2 + 3aj\Delta t}{2} \end{aligned}$$

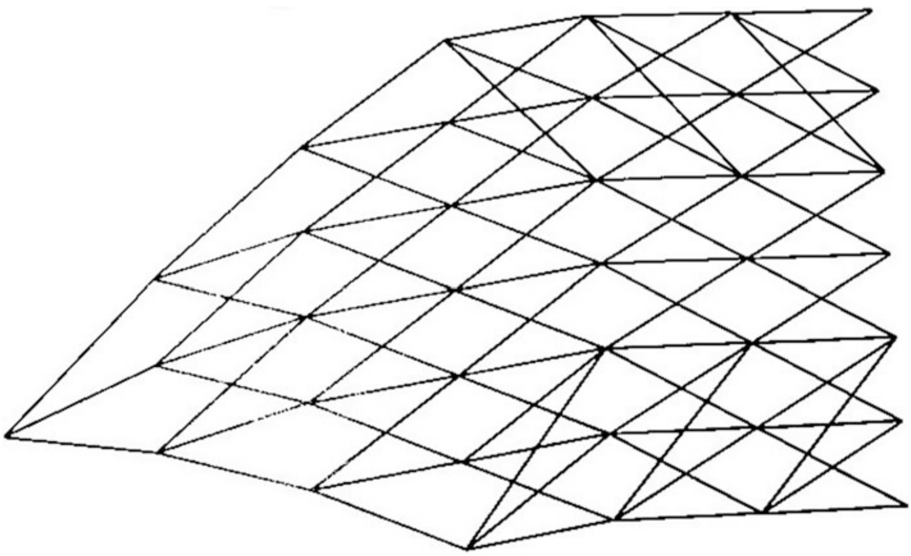


Fig. 15.7 The transformed HW tree

Fitting the Tree

We will now transform the tree of R^* into a free for R (Fig. 15.7)
We start by defining

$$\alpha(t) = R(t) - R^*(t)$$

giving

$$d\alpha = [\theta(t) - a\alpha(t)]\,dt$$

where

$$\alpha(t) = f(0, t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2$$

This equation can be used to create a tree for R from the corresponding tree ofR^* . The approach is to set the interest rate on the R-tree at time $i\Delta t$ to be equal the corresponding interest rates in the R^*-tree plus the value of α at time $i\Delta t$, to keep the probabilities the same. This is not exactly consistent with the initial term structure, so we have to calculate α iteratively to match the initial term structure.

To illustrate this approach, we define α_i as $R(i\Delta t) - R^*(i\Delta t)$, α_0 as the price of a zero-coupon bond maturing at time Δt (which is equal to the initial Δt-period interest rate) and $Q_{i,j}$ as the present value of a security with payoff one cash unit if the node (i, j) is reached and zero otherwise.

Suppose that $Q_{i,j}$ have been determined for $i \leq m (m \geq 0)$. We now have to determine α_m so that the tree correctly replicate the prices of zero-coupon bonds with maturity $(m+1)\Delta t$. The interest rate at node (m, j) is $\alpha_m + j\Delta R$, so that the price of a zero-coupon bond maturing at $(m+1)\Delta t$ is given by

$$p_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp\left[-\left(\alpha_m + j\Delta R\right)\Delta t\right]$$

Where n_m is the number of nodes on each side of the central node at time $m\Delta t$. The solution to this equation is given by

$$\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta R\Delta t} - \ln P_{m+1}}{\Delta t}$$

Once we know α_m, we can determine $Q_{i,j}$. For $i = m+1$ we have

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp\left[-\left(\alpha_m + k\Delta R\right)\Delta t\right]$$

where $q(k,j)$ is interpreted as the probability of moving from node (m,k) to $(m+1,j)$. The summation made over all values of k for this is non-zero.

The bond price at node (i,j), for each branch is calculated as:

$$v_{i,j} = \left(p_u v_{i+1,j+1} + p_m v_{i+1,j} + p_d v_{i+1,j-1}\right) \exp\left(-R_{i,j}\Delta t\right)$$

$$v_{i,j} = \left(p_u v_{i+1,j+2} + p_m v_{i+1,j+1} + p_d v_{i+1,}\right) \exp\left(-R_{i,j}\Delta t\right)$$

$$v_{i,j} = \left(p_u v_{i+1,j+1} + p_m v_{i+1,j-1} + p_d v_{i+1,j-2}\right) \exp\left(-R_{i,j}\Delta t\right)$$

Example 15.1.8

Consider the following tree in Fig. 15.8

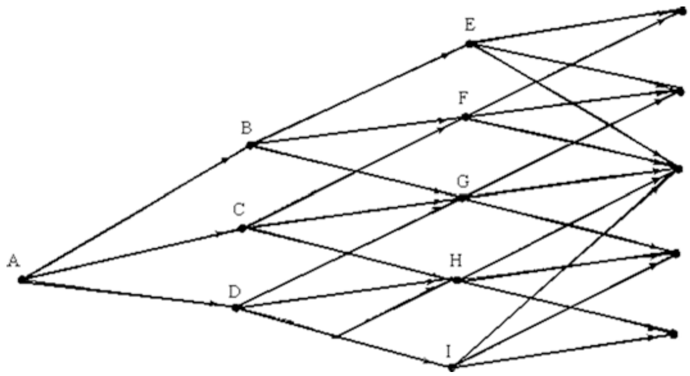
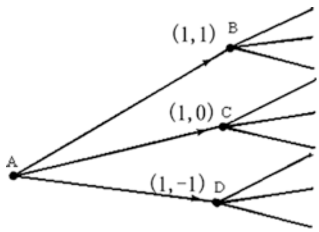


Fig. 15.8 A HW trinomial tree

Table with zero rate

Maturity	Rate %
0.5	3.43
1	3.824
1.5	4.183
2	4.512
2.5	4.812
3	5.086

Specify Δt as one year and the initial rate as 3.824%. Then $R_0 = 3.824\%$, $Q_{0,0} = 1$ and $\Delta R = \sigma\sqrt{3\Delta t} = 0.01 \cdot \sqrt{3} = 0.01732$. Calculate Q on node B, C and D



Node	A	B	C	D	E	F	G	H	I
R	3.82%								
P_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
P_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.0266	0.6666	0.6566	0.0266
P_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.0867	0.1667	0.1217	0.8867

$$
\begin{aligned}
Q_A &= Q_{0,0} = 1 \\
Q_B &= Q_{1,1} = p_{Au}e^{-R_A \Delta t} = 0.1667e^{-0.03824 \cdot 1} = 0.1604 \\
Q_C &= Q_{1,0} = p_{Am}e^{-R_A \Delta t} = 0.6666e^{-0.03824 \cdot 1} = 0.6417 \\
Q_D &= Q_{1,-1} = p_{Ad}e^{-R_A \Delta t} = 0.1667e^{-0.03824 \cdot 1} = 0.1604
\end{aligned}
$$

We calculate α_1 which is chosen to replicate the price of a zero-coupon bond maturing at time $2\Delta t$. Fitting Q to P

$$
\begin{aligned}
P_B &= P_{1,1} = Q_B e^{-(\alpha_1+\Delta R)} = 0.1604e^{-(\alpha_1+0.01732)} \\
P_C &= P_{1,0} = Q_B e^{-\alpha_1} = 0.6417e^{-\alpha_1} \\
P_D &= P_{1,-1} = Q_B e^{-(\alpha_1-\Delta R)} = 0.1604e^{-(\alpha_1-0.01732)}
\end{aligned}
$$

For the initial term structure, the bond price should be $e^{-0.0452 \cdot 2} = 0.9137$ giving

$$
\begin{aligned}
&P_B + P_C + P_D = 0.9137 \\
&0.1604e^{-(\alpha_1+0.01732)} + 0.6417e^{-\alpha_1} + 0.1604e^{-(\alpha_1-0.01732)} = 0.9137
\end{aligned}
$$

We then get

$$
\alpha_1 = \ln\left(\frac{0.1604e^{-0.01732} + 0.6417 + 0.1604e^{0.01732}}{0.9137}\right) = 0.05205
$$

And the rates in the nodes B, C and D

$$
\begin{aligned}
R_c &= \alpha_1 = 0.05205 \\
R_B &= \alpha_1 + 1 \cdot \Delta R = 0.05205 + 0.01732 = 0.06937 \\
R_D &= \alpha_1 + 1 \cdot \Delta R = 0.05205 - 0.01732 = 0.03473
\end{aligned}
$$

We can then fill in the rates in the previous table. The new table is

Node	A	B	C	D	E	F	G	H	I
R	3.82%	6.94%	5.03%	3.47%					
P_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
P_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.0266	0.6666	0.6566	0.0266
P_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.0867	0.1667	0.1217	0.8867

Next, we calculate the rates in the nodes E, F, G, H and I. F can only be reached from node B and C, then

$$
\begin{aligned}
Q_F = Q_{2,1} &= p_{Bm}e^{-R_B} Q_B + p_{Cu}e^{-R_C} Q_C \\
&= 0.6566e^{-0.06937} \cdot 0.1604 + 0.1667e^{-0.005205} \cdot 0.6407 \\
&= 0.1998
\end{aligned}
$$

G can only be reached from B, C and D:

$$
\begin{aligned}
Q_G = Q_{2,0} &= p_{Bd}e^{-R_B}Q_B + p_{Cm}e^{-R_C}Q_C + p_{Du}e^{-R_D}Q_D \\
&= 0.2217e^{-0.06937} \cdot 0.1604 + 0.6666e^{-0.05205} \cdot 0.6417 + 0.2217e^{-0.03473} \cdot 0.1604 \\
&= 0.4736
\end{aligned}
$$

other nodes can be calculated in a similar way. We finally get:

$$
\begin{aligned}
Q_E &= 0.0182 \\
Q_H &= 0.2033 \\
Q_I &= 0.0189
\end{aligned}
$$

Fitting P and calculate $\alpha_2 = 0.06522$

$$
\begin{aligned}
P_E &= Q_E e^{-(\alpha_2+2\Delta R)} \\
P_F &= Q_F e^{-(\alpha_2+\Delta R)} \\
P_G &= Q_G e^{-\alpha_2} \\
R_H &= Q_F e^{-(\alpha_2-\Delta R)} \\
P_I &= Q_I e^{-(\alpha_2-2\Delta R)} \\
P_E + P_F + P_G + R_H + P_I &= e^{-5.086\cdot 3}
\end{aligned}
$$

Solve these in order to get α_2, the rate in the nodes E and F will be

$$
\begin{aligned}
R_E &= \alpha_2 + 2\Delta R \\
R_F &= \alpha_2 + \Delta R \\
R_G &= \alpha_2 \\
R_I &= \alpha_2 - \Delta R \\
R_H &= \alpha_2 - 2\Delta R
\end{aligned}
$$

If we continue the calculation to find the prices and rates for each node we can calculate the bond price for each node for three kinds of branching:

$$
\begin{aligned}
v_{i,j} &= \left(p_u v_{i+1,j+1} + p_m v_{i+1,j} + p_d v_{i+1,j-1}\right) \exp\left(-R_{i,j}\Delta t\right) \\
v_{i,j} &= \left(p_u v_{i+1,j+2} + p_m v_{i+1,j+1} + p_d v_{i+1,}\right) \exp\left(-R_{i,j}\Delta t\right) \\
v_{i,j} &= \left(p_u v_{i+1,j+1} + p_m v_{i+1,j-1} + p_d v_{i+1,j-2}\right) \exp\left(-R_{i,j}\Delta t\right)
\end{aligned}
$$

Other Issues

There are a number of other practical issues to consider when implementing HW trees for valuing interest rate derivatives. In our description we assumed that the length of the time step is constant. In practice, it is sometimes desirable to change the length of the time step. Consider for example the situation where the model is used to value a European six-month option on a five-year bond. It might be appropriate to use a longer Δt between six months and five years than during the first six months. This is because the part of the tree between six months and five years is used only to value the underlying bond.

Barrier options present a further problem in the use of the tree because convergence tends to be slow when nodes do not lie exactly on barriers. In the case of an interest rate option the barrier is typically expressed in terms of a bond price or a particular rate. Analytic results can be used to express the barrier as a function of the Δt-period rate. Non-standard branching can then be used to ensure that nodes always lie on the barrier. Ritchken (JOD Winter 1995) describes such an approach, and shows that a substantial improvement in performance is possible with it.

A final problem in the use of interest rate trees is path dependence. This can sometimes be handled in the way described by Hull and White [1993]. The requirements for the HW method to work are:

1. The value of the derivative at each node must depend on just one function of the path for the short-rate r (e.g. the maximum, minimum or average value);
2. In order to update the path function as we move forward through the tree we need to know only the previous value of the function and the new value of r.

Hull and White show how their approach can be used for index amortizing Swaps and mortgage-backed securities. The relevant path function in each case is the remaining principal.

15.1.3.4 The Cox-Ingersoll-Ross Model (CIR)

As we know, the previously mentioned models also generate negative interest rates. A model to prevent this is the CIR model[3]:

$$dr(t) = (\theta - a \cdot r(t))dt + \sigma\sqrt{r(t)}dV(t)$$

---

[3] Be aware about the t negative interest rates that has become a fact in several countries in 2014 and 2015 (Sweden, Denmark, EURO etc).

Since the volatility is a function of the interest rate, this cannot be negative. If this happens the stochastic part will become weaker and the drift will take over and force the rate to increase. The term-structure equation for CIR is given by

$$\begin{cases} \frac{\partial p}{\partial t} + \{\theta - a \cdot r\} \frac{\partial p}{\partial r} + \frac{1}{2}\sigma^2 r \frac{\partial^2 p}{\partial r^2} - rp = 0 \\ p(r, T, T) = 1 \end{cases}$$

This equation has a closed form solution. We will look for a solution of the form

$$p(t,\ T) = F(r(t), t,\ T) = e^{A(t,\,T)-B(t,\,T)r}$$

where $A(T, T) = B(T, T) = 0$. Differentiating p with respect to r and t and differentiating p_r with respect to r we get the following:

$$\begin{cases} \frac{\partial p(t,T)}{\partial r} = -B(t,T)e^{A(t,T)-B(t,T)r} = -B(t,T)p(t,T) \\ \frac{\partial^2 p(t,T)}{\partial r^2} = B^2(t,T)e^{A(t,T)-B(t,T)r} = B^2(t,T)p(t,T) \\ \frac{\partial p(t,T)}{\partial t} = p(t,T)\left(\frac{\partial A(t,T)}{\partial t} - r\frac{\partial B(t,T)}{\partial t}\right) \end{cases}$$

Substituting these expressions in the previous term-structure equation, the PDE becomes

$$\begin{aligned} 0 &= p(A_t - rB_t) - \{\theta - ar\} Bp + \frac{1}{2}\sigma^2 rB^2 p - rp \\ &= rp\left\{-1 - B_t + aB + \frac{1}{2}\sigma^2 B^2\right\} + p\{A_t - \theta B\} \end{aligned}$$

A careful inspection of the previous equation reveals that the expression in the first bracket is the well-known Ricatti equation. We will now solve this equation. After solving this equation we need to set

$$A(t,T) = -\theta \int_t^T B(u,T)du$$

Since $A(T, T) = 0$ we have $A_t(t, T) = \theta B(t, T)$: Although the time starts at t and ends at T we will consider it to start at 0 for the purposes of derivation and end at t.

Hence, when we refer to t from this point onwards it is the amount of time that has elapsed. The Ricatti equation can be rewritten as

$$B_t = -1 + aB + \frac{1}{2}\sigma^2 B^2$$

We introduce another dependent variable u such that

$$B = -\frac{2u_t}{\sigma^2 u}$$

Differentiating the previous expression with respect to t we get

$$B_t = -\frac{2u_{tt}}{\sigma^2 u} + u_t \cdot \frac{2u_t}{\sigma^2 u^2} = -\frac{2u_{tt}}{\sigma^2 u} + 2\left(\frac{u_t}{\sigma u}\right)^2$$

We then get

$$-\frac{2u_{tt}}{\sigma^2 u} + 2\left(\frac{u_t}{\sigma u}\right)^2 = -1 - a\frac{2u_t}{\sigma^2 u} + \frac{1}{2}\sigma^2\left(\frac{2u_t}{\sigma^2 u}\right)^2$$
$$-\frac{2u_{tt}}{\sigma^2 u} = -1 - a\frac{2u_t}{\sigma^2 u}$$

Multiplying both sides of the aforementioned expression by $\sigma^2 u$ we get

$$2u_{tt} - 2au_t - u\sigma^2 = 0$$

This is a second-order, ODE with constant coefficients. We solve this with a characteristic polynomial

$$2\lambda^2 - 2a\lambda - \sigma^2 = 0$$

with roots

$$\lambda_{1,2} = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + 2\sigma^2}$$

The solution to the differential equation is therefore given by

$$\begin{aligned}
u(t) &= C_1 e^{\lambda_1(T-t)} + C_2 e^{\lambda_2(T-t)} = C_1 e^{\frac{1}{2}\left(a-\sqrt{a^2+2\sigma^2}\right)(T-t)} + C_2 e^{\frac{1}{2}\left(a+\sqrt{a^2+2\sigma^2}\right)(T-t)} \\
&= C_1 e^{\frac{1}{2}(a-\gamma)(T-t)} + C_2 e^{\frac{1}{2}(a+\gamma)(T-t)} \\
&= C_1 e^{a(T-t)/2} e^{-\gamma(T-t)/2} + C_2 e^{a(T-t)/2} e^{\gamma(T-t)/2} \\
\frac{du(t)}{dt} &= -\frac{1}{2}C_1(a-\gamma)\cdot e^{a(T-t)/2} e^{-\gamma(T-t)/2} - \frac{1}{2}C_2(a+\gamma)\cdot e^{a(T-t)/2} e^{\gamma(T-t)/2}
\end{aligned}$$

where we defined γ by

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

The boundary condition; $B(T,T) = 0$ gives $u_t(T,T) = 0$ and therefore

$$C_1 = -C_2 \frac{a+\gamma}{a-\gamma} = C_2 \frac{\gamma+a}{\gamma-a}$$

Inserting this and making some algebra we will finally get

$$\begin{aligned}
B &= -\frac{2u_t}{\sigma^2 u} = \frac{1}{\sigma^2} \frac{C_1(a-\gamma)\cdot e^{a(T-t)/2}e^{-\gamma(T-t)/2} + C_2(a+\gamma)\cdot e^{a(T-t)/2}e^{\gamma(T-t)/2}}{C_1 e^{a(T-t)/2}e^{-\gamma(T-t)/2} + C_2 e^{a(T-t)/2}e^{\gamma(T-t)/2}} \\
&= \frac{1}{\sigma^2} \frac{-C_1(\gamma-a)\cdot e^{-\gamma(T-t)/2} + C_2(\gamma+a)\cdot e^{\gamma(T-t)/2}}{C_1 e^{-\gamma(T-t)/2} + C_2 e^{\gamma(T-t)/2}} \\
&= \frac{1}{\sigma^2} \frac{-\frac{\gamma+a}{\gamma-a}(\gamma-a)\cdot e^{-\gamma(T-t)/2} + (\gamma+a)\cdot e^{\gamma(T-t)/2}}{\frac{\gamma+a}{\gamma-a}e^{-\gamma(T-t)/2} + e^{\gamma(T-t)/2}} \\
&= \frac{1}{\sigma^2} \frac{e^{\gamma(T-t)/2} - e^{-\gamma(T-t)/2}}{(\gamma+a)e^{\gamma(T-t)/2} + (\gamma-a)e^{-\gamma(T-t)/2}}
\end{aligned}$$

Then we can calculate A:

$$A(t,T) = -\theta \int_t^T B(u,T)du$$

We get

$$A(t,T) = \frac{2\theta}{\sigma^2} \ln \left[\frac{\gamma e^{\gamma(T-t)/2}}{(\gamma+a)\,e^{\gamma(T-t)/2} + (\gamma-a)\,e^{-\gamma(T-t)/2}} \right]$$

In other words, we have explicitly calculated the bond prices.

A simulation of the short rate with the same parameters ($a = 0.15$ and $\sigma = 4.5$ %, i.e. $\theta/a = 4,5$ %) as in the previous Vasicek model is shown in Fig. 15.9. As we can see, the probability of negative interest rates is zero.

If $r(0)$ is 2.0 % the simulated term structure of interest rates is shown in Fig. 15.10.

This give a discount function as in Fig. 15.11.

A square root process, like CIR can only take on non-negative values. To see this, note that if the value should become zero, then the drift is positive and the volatility zero, and therefore the value of the process

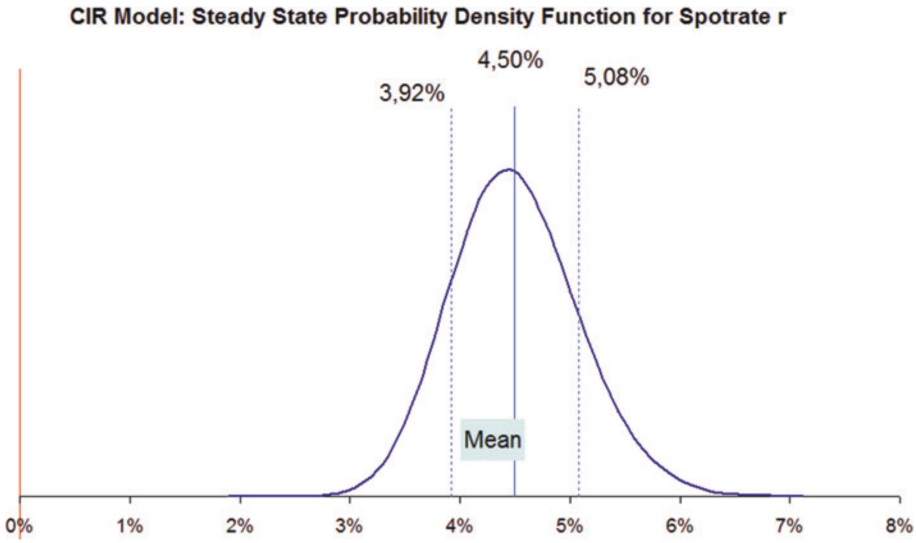


Fig. 15.9 The rate distribution in the CIR model

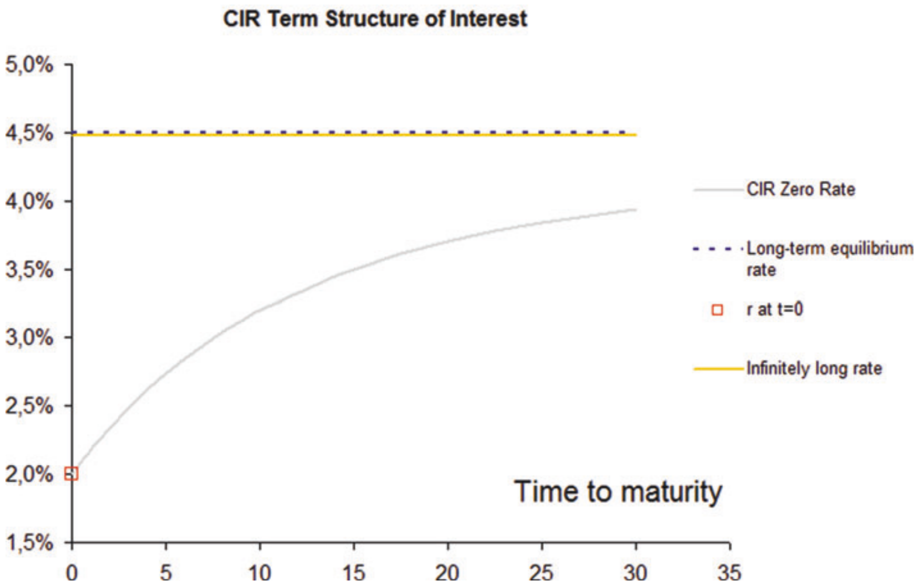


Fig. 15.10 The zero-rates in the CIR model

will with certainty become positive immediately after (zero is a so-called reflecting barrier). It can be shown that if $2\theta \geq \sigma^2$, the positive drift at low values of the process is so big relative to the volatility that the process cannot even reach zero, but stays strictly positive.

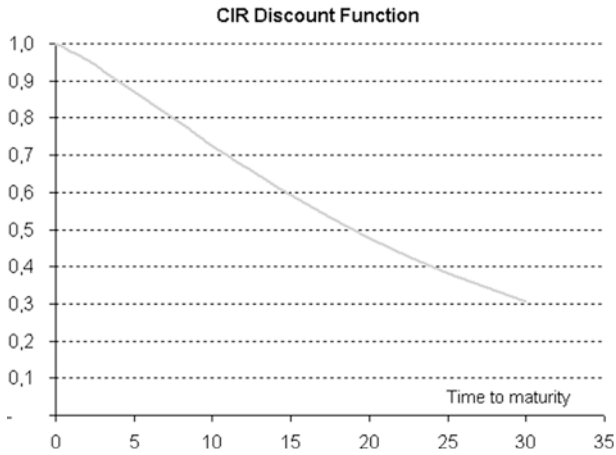


Fig. 15.11 The discount function in the CIR model

Paths for the square root process can be simulated by successively calculating

$$r_{n+1} = r_n + (\theta - a \cdot r_n)\,\Delta t + \sigma \cdot \sqrt{r_n} \cdot \varepsilon \cdot \sqrt{\Delta t}$$

Variations in the different parameters will have similar effects as for the Ornstein-Uhlenbeck process. Since a square root process cannot become negative, the future values of the process cannot be normally distributed. In order to find the actual distribution, let us try the same trick. Look at $y = e^{at}r$. By Itô's lemma,

$$dy = ae^{at}rdt + e^{at}\,(\theta - ar)\,dt + e^{at}\sigma\sqrt{r}dV = \theta e^{at}dt + e^{at}\sigma\sqrt{r}dV$$

so that

$$y(T) = y(t) + \int_t^T \theta e^{au}du + \int_t^T \sigma e^{au}\sqrt{r(u)}dV(u)$$

Computing the ordinary integral and substituting the definition of y, we get

$$r(T) = r(t)e^{-a(T-t)} + \frac{\theta}{a}\left(1 - e^{-a(T-t)}\right) + \sigma\int_t^T e^{-a(T-u)}\sqrt{r(u)}dV(u)$$

Since r enters the stochastic integral we cannot immediately determine the distribution of $r(T)$ given $r(t)$ from this equation. We can, however, use it to obtain the mean and variance of $r(T)$. Due to the fact that the stochastic integral has mean zero we easily get

$$E\left[r(T)\right] = r(t)e^{-a(T-t)} + \frac{\theta}{a}\left(1 - e^{-a(T-t)}\right) = \frac{\theta}{a} + \left(r(t) - \frac{\theta}{a}\right)e^{-a(T-t)}$$

We then have to calculate the variance:

$$\begin{aligned}
Var\left[r(T)\right] &= Var\left[\sigma\int_t^T e^{-a(T-u)}\sqrt{r(u)}dV(u)\right] = \sigma^2\int_t^T e^{-2a(T-u)}E\left[r(u)\right]du \\
&= \sigma^2\int_t^T e^{-2a(T-u)}\left(\frac{\theta}{a} + \left(r(t) - \frac{\theta}{a}\right)e^{-a(T-u)}\right)du \\
&= \frac{\sigma^2\theta}{a}\int_t^T e^{-2a(T-u)}du + \sigma^2\left(r(t) - \frac{\theta}{a}\right)e^{-2aT+at}\int_t^T e^{au}du \\
&= \frac{\sigma^2\theta}{2a^2}\left(1 - e^{-2a(T-t)}\right) + \frac{\sigma^2}{a}\left(r(t) - \frac{\theta}{a}\right)\left(e^{-a(T-t)} - e^{-2a(T-t)}\right) \\
&= \frac{\sigma^2\theta}{2a^2}\left(1 - e^{-a(T-t)}\right)^2 + \frac{\sigma^2 r(t)}{a}\left(e^{-a(T-t)} - e^{-2a(T-t)}\right)
\end{aligned}$$

For $T \to \infty$, the mean approaches θ/a and the variance approaches $\frac{\sigma^2\theta}{2a^2}$. For $a \to \infty$, the variance approaches 0. For $a \to 0$, the mean approaches the current value $r(t)$, and the variance approaches $\sigma 2r(t)(T-t)$. It can be shown that, given the value $r(t)$, the value $r(T)$ with $T > t$ is given by the non-central $\chi 2$-distribution. A non-central $\chi 2$-distribution is characterized by a number x of degrees of freedom and a non-centrality parameter y and is denoted by $\chi 2(x, y)$. More precisely, the distribution of $r(T)$ given $r(t)$ is identical to the distribution of the random variable $Y/c(T-t)$ where c is the deterministic function

$$c(\tau) = \frac{4a}{\sigma^2\left(1 - e^{-a\tau}\right)}$$

and Y is a $\chi 2(x, y(T-t))$-distribution random variable with

$$x = \frac{4a}{\sigma^2}, \qquad y(T-t) = r(t)c(T-t)e^{-b(T-t)}$$

The density function for a $\chi 2(x, y)$-distributed random variable is

$$f_{\chi^2(x,y)}(z) = \sum_{i=0}^{\infty} \frac{e^{-y/2}(y/2)^i}{i!} \frac{(1/2)^{i+x/2}}{\Gamma(i + x/2)} z^{i-1+x/2} e^{-z/2}$$

Here Γ denotes the so-called gamma-function defined as

$$\Gamma(m) = \int_0^{\infty} x^{m-1} e^{-x} dx$$

The mean and variance of a $\chi 2(x, y)$-distributed random variable are $x+y$ and $2(x+2y)$, respectively. This opens another way of deriving the mean and variance of $r(T)$ given $r(t)$. We leave it for the reader to verify that this procedure will yield the same results as given earlier. A frequently applied dynamic model of the term structure of interest rates is based on the assumption that the short-term interest rate follows a square root process. Since interest rates are positive and empirically seem to have a variance rate which is positively correlated to the interest rate level, the square root process gives realistic description of interest rates. On the other hand, models based on square root processes are complicated to analyse.

Option Pricing

We have seen that the CIR stochastic process can be written as

$$dr(t) = (\theta - a \cdot r)dt + \sigma \sqrt{r} dV$$

or equivalently as

$$dr(t) = a(\mu - r)dt + \sigma \sqrt{r} dV$$

To price a European call option with maturity T and strike price K on an S-bond, we get the arbitrage-free price as

$$C(t, T, K, S) = L \cdot p(t, S) \cdot \chi_n^2(x_1, \nu_1, \nu_2) - K \cdot p(t, T) \cdot \chi_n^2(x_2, \nu_1, \nu_3)$$

where

$$x_1 = 2 \cdot \frac{A(S, T) - \ln(K)}{B(S, T)} \cdot \left\{ \frac{2 \cdot \gamma}{\sigma^2 \left(e^{\gamma(T-t)} - 1\right)} + \frac{a + \gamma}{\sigma^2} + B(S, T) \right\}$$

$$x_2 = x_1 - 2 \cdot (A(S, T) - \ln(K))$$

and

$$\nu_1 = 4 \cdot \theta/\sigma^2$$

$$\nu_2 = \frac{\frac{8\cdot\gamma^2\cdot r\cdot e^{\gamma(T-t)}}{\sigma^4\left(e^{\gamma(T-t)}-1\right)^2}}{\frac{2\cdot\gamma}{\sigma^2\left(e^{\gamma(T-t)}-1\right)} + \frac{a+\gamma}{\sigma^2} + B(S,T)}$$

$$\nu_3 = \frac{\frac{8\cdot\gamma^2\cdot r\cdot e^{\gamma(T-t)}}{\sigma^4\left(e^{\gamma(T-t)}-1\right)^2}}{\frac{2\cdot\gamma}{\sigma^2\left(e^{\gamma(T-t)}-1\right)} + \frac{a+\gamma}{\sigma^2}}$$

The put price for this (and most of the other models) can be obtained from put-call parity using the value $p(t,S)$ as the value of the underlying and $p(t,T)$ as the discount function on the exercise price.

CIR also gives the price of a future contract expiring at T where the underlying pure discount bond expires at S with $S > T$ as

$$F(t,S,T) = \frac{\rho}{B(S,T)-\rho} \cdot \exp\left\{A(S,T) - r \cdot \frac{\rho \cdot B(S,T)e^{a(T-t)}}{\rho + B(S,T)}\right\}$$

$$\rho = \frac{2a}{\sigma^2\left(1-e^{a(T-t)}\right)}$$

15.1.4 Yield-Curve Fitting: For and Against

15.1.4.1 For

The building blocks of the bond pricing equation are delta hedging and no-arbitrage. If we are to use a one-factor model correctly then we must abide by the delta hedging assumptions. We must buy and sell instruments to remain delta neutral. The buying and selling of instruments must be done at the market prices. We cannot buy and sell at a theoretical price. But we are not modelling the bond prices directly, we model the spot rate and bond prices are then derivatives of the spot rate. This means that there is a real likelihood that our output bond prices will differ markedly from the market prices. This is useless if we are to hedge with these bonds. The model thus collapses and cannot be used for pricing other instruments, unless we can find a way to generate the correct prices for our hedging instruments from the model: this is yield-curve fitting.

15.1.4.2 Against

If the market prices of simple bonds were correctly given by a model, such as Ho and Lee or Hull and White, fitted at time $t*$ then, when we come back a week later, $t*$ + one week, say, to refit the function $\theta(t^*)$, we would find that this function *had not changed* in the meantime. This *never* happens in practice. We find that the function $\theta(t^*)$ has changed out of all recognition. What does this mean? Clearly the model is wrong.

By simply looking for a Taylor series solution of the bond-pricing equation for short times to expiry, we can relate the value of the risk-adjusted drift rate at the short end to the slope and curvature of the market yield curve. This is done as follows. Look for a solution in the TSE form

$$F(r, t, T) \sim 1 + a(r)(T-t) + b(r)(T-t)^2 + c(r)(T-t)^3 + \ldots.$$

Substitute this into the TSE equation

$$\begin{aligned}
&-a - 2b(T-t) - 3c(T-t)^2 \\
&+ \frac{1}{2}\left(\sigma^2 - 2(T-t)\sigma\frac{\partial\sigma}{\partial t}\right)\left((T-t)\frac{d^2a}{dr^2} + (T-t)^2\frac{d^2b}{dr^2}\right) \\
&+ \left(\mu - (T-t)\frac{\partial\mu}{\partial t}\right)(T-t)\left(\frac{da}{dr} + (T-t)^2\frac{db}{dr}\right) \\
&- r\left(1 + a(T-t) + c(T-t)^2\right) + \ldots = 0
\end{aligned}$$

Note how the drift and volatility terms are expanded around $t = T$; in the aforementioned expression these are evaluated at r and T. By equating powers of $(T-t)$ we find that

$$\begin{cases}
a(r) = -r \\
b(r) = \dfrac{1}{2}\left(r^2 - \mu^2\right) \\
c(r) = \dfrac{1}{12}\sigma^2\frac{\partial^2}{\partial r^2}\left(r^2 - r\mu\right) - \dfrac{1}{6}\mu\frac{\partial}{\partial r}\left(r^2 - r\mu\right) - \dfrac{1}{3}\dfrac{d\mu}{dt} + \dfrac{r^2}{6}(r-\mu)
\end{cases}$$

In all of these μ and σ are evaluated at r and T.

From the Taylor series expression for F we find that the *ytm* for short times to maturity is given by

$$-\frac{\ln(F(r,t,T))}{T-t} \sim -a + \left(\frac{1}{2}a^2 - b\right)(T-t) + \left(ab - c - \frac{1}{3}a^3\right)(T-t)^2 + \ldots$$

The yield curve takes the value $-a(r) = r$ at maturity, obviously. The slope of the yield curve is

$$\tfrac{1}{2}a^2 - -b = \tfrac{1}{2}\mu,$$

That is, one half of the risk-neutral drift. The curvature of the yield curve at the short end is proportional to

$$ab - c - 1/3a^3.$$

which contains a term that is the derivative of the risk-neutral drifts with respect to time via c. Let's stress the key points of this analysis. The slope of the yield curve at the short end depends on the risk-neutral drift, and vice versa. The curvature of the yield curve at the short end depends on the time derivative of the risk-neutral drift, and vice versa. If we choose time-dependent parameters within the risk-adjusted drift rate such that the market prices are fitted at time t^* then we have

$$F\left(r^*, t^*, T\right) = F_M\left(t^*, T\right)$$

which is one equation for the time-dependent parameters.

Thus, for Ho and Lee, for example, the value of the function $\theta(t)$ at the short $t = t^*$, depends on the slope of the market yield curve. Moreover, the slope of $\theta^*(t)$ depends on the curvature of the yield curve at the short end. Results such as these are typical for all fitted models. These, seemingly harmless, results are actually quite profound.

It is common for the slope of the yield curve to be quite large and positive, the difference between very short and not quite so short rates is large. But then for longer maturities typically the yield curve flattens out. This means that the yield curve has a large negative curvature. If one performs the fitting procedure as outlined here for the Ho and Lee or extended Vasicek models, one typically finds the following

- The value of $\theta^*(t)$ at $t = t^*$ is very large. This is because the yield-curve slope at the short end is often large;
- The slope of $\theta^*(t)$ at $t = t^*$ is large and negative. This is because the curvature of the yield curve is often large and negative.

If we plot $\theta^*(t)$ versus t today and we come back in a few months to look at how our fitted parameter is doing, we would not find that we have moved forward on the original curve. The recalibrated function looks nothing like the function a few months earlier. In fact, we don't even have to wait for a few months for the deviation to be significant; it becomes apparent in weeks or even days.

We can conclude from this that yield-curve fitting is an inconsistent and dangerous business. The results presented here are by no means restricted to the models we have named; no one-factor model will capture the suspected behaviour.

15.1.5 The BDT Model

The Black-Derman-Toy (1990) one-factor model is one of the most used yield-based models to price bonds and interest-rate options. In 1991 Black and Karasinski generalized this model. The model is arbitrage-free and thus consistent with the observed term structure of interest rates. The short-rate volatility is potentially time dependent, and the continuous process of the short-term interest rate is given by

$$d\ln(r(t)) = \left\{\theta(t) + \frac{\dot{\sigma}(t)}{\sigma(t)}\ln(r(t))\right\}dt + \sigma(t)dV$$

where the factor in front of $\ln(r)$ is the speed of mean reversion ("gravity"), and $\theta(t)$ divided by the speed of mean reversion is a time-dependent mean-reversion level.

The model assumes that the market in which the model is established is perfect

- Changes in all bond yields are perfectly correlated.
- Expected returns on all securities over one period are equal.

- Short rates at any time are lognormal distributed thus are positive for all times.

The model can be derived from a continuous short-rate process:

$$r(t) = u(t)e^{\sigma(t)z(t)}$$

If we take the logarithm, we get

$$\ln r(t) = \ln u(t) + \sigma(t)z(t)$$

and differentiate

$$d\ln r(t) = \frac{d\ln u(t)}{dt}dt + \sigma(t)\cdot dz(t) + \frac{d\sigma(t)}{dt}z(t).$$

Since $z(t)$ can be expressed as

$$z(t) = \frac{1}{\sigma(t)}\left(\ln r(t) + \ln u(t)\right)$$

we have

$$d\ln r(t) = \left(\frac{d\ln u(t)}{dt} + \frac{1}{\sigma(t)}\frac{d\sigma(t)}{dt}\left[\ln r(t) - \ln u(t)\right]\right)dt + \sigma(t)\cdot dz(t).$$

With some simplifications we have

$$d\ln(r(t)) = \left\{\theta(t) + \frac{\dot{\sigma}(t)}{\sigma(t)}\ln(r(t))\right\}dt + \sigma(t)dV$$

For constant volatility, the BDT model does not display any mean reversion. In this case the process reduces to a lognormal version of the Ho-Lee model.

The short rate evolves by diffusion with a drift that follows the logarithm of the median. If the volatility is decaying, the reversion speed will be positive and the logarithm of the short rate will reverse to $\ln[u(t)]$.

This means that the short rates will not assume implausibly large values over long time horizons, which is along the lines of market observations. However, care must be taken to ensure that the model is viable.

The BDT model incorporates as we see, two independent functions of time, $\theta(t)$ and $\sigma(t)$, chosen so that the model fits the term structure of spot interest rates and the term structure of spot rate volatilities. The changes in the short rate are lognormal distributed, with the resulting advantage that interest rates cannot become negative. Once $\theta(t)$ and $\sigma(t)$ are chosen, the future short-rate volatility, by definition, is entirely determined. An unfortunate consequence of the model is that for certain specifications of the volatility function σ (t) the short rate can be mean-fleeing rather than mean-reverting. The model has the advantage that the volatility unit is a percentage, conforming to the market convention. Unfortunately, due to its lognormality, neither analytic solutions for the prices of bonds or the prices of bond options are available, and numerical procedures are required to derive the short-rate tree that correctly returns the market term structures. Remark that this model does not have an ATS since the volatility term is proportional to the level of the short rate. Many practitioners choose to fit the rate structure only, holding the future short-rate volatility constant. The convergent limit therefore reduces to the following

$$d\ln(r) = \theta(t)dt + \sigma(t)dV$$

This process can be seen as a lognormal version of Ho-Lee.

The following example shows how to calibrate the BDT binomial tree to the current term structure of zero-coupon yields and zero-coupon volatilities.

15.1.5.1 A Simple Binomial Tree-Model

The best way to illustrate a simple tree-model is via an example.

We start by looking for the value of an American call option on a five-year zero-coupon bond with time to expiration of four years and a strike price of 85.50. The term structure of zero-coupon rates and volatilities is shown in Table 15.1. From the rates and volatilities, we will calibrate the BDT interest rate tree. To price the option by using backward induction, we build a tree for the bond prices, as shown in. (Fig. 15.12)

Table 15.1 Market data

Input to BDT Model		
Years of maturity	**Zero-coupon rates (%)**	**Zero-coupon volatilities**
1	9	24
2	9.5	22
3	10	20
4	10.5	18
5	11	16

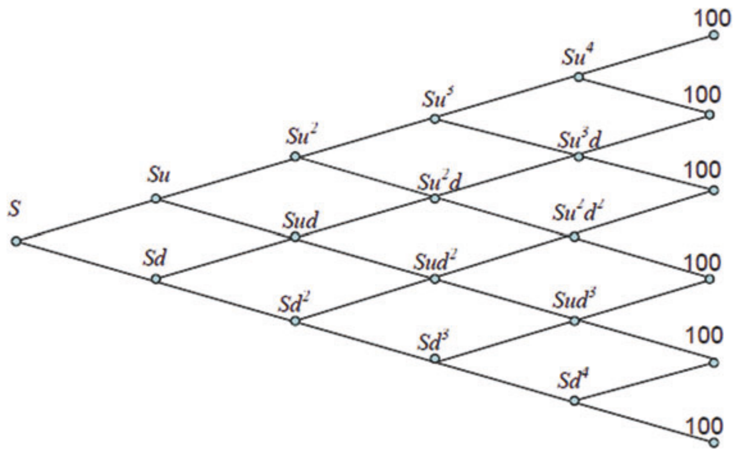


Fig. 15.12 The bond prices I BDT

To build the price tree, we have to build the rate tree as in Fig. 15.13.

We start by finding the prices of the zero-coupon bonds with maturity from one year to five years in the future. The face values are 100 (% of the nominal amount), the zero-coupon rates are given in the previous table.

$$\frac{100}{1+0.09} = 91.74$$

$$\frac{100}{(1+0.095)^2} = 83.40$$

$$\frac{100}{(1+0.10)^3} = 75.13$$

$$\frac{100}{(1+0.105)^4} = 67.07$$

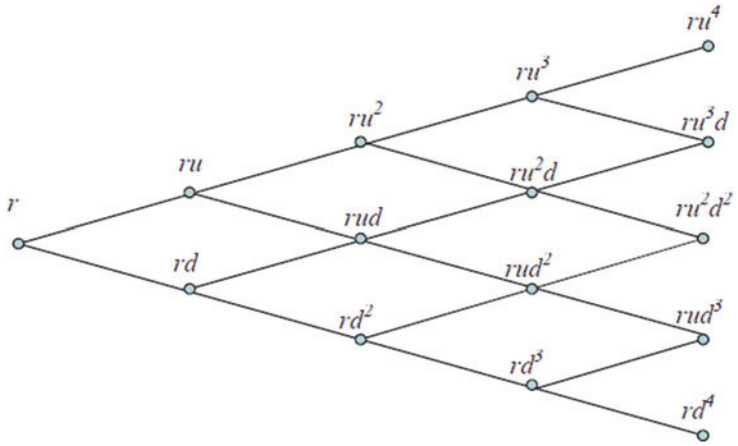


Fig. 15.13 The interest rate tree in BDT

and

$$\frac{100}{(1+0.11)^5} = 59.35$$

Then we add this to our data. In a different situation, we might know from the beginning, these prices from data or by bootstrapping of coupon-bonds.

This gives us the one-period price tree as in Fig. 15.14.
The next step is to build a two-period price tree. From Table 15.2, it is clear that the price today of a two-year zero-coupon bond with maturity of two years from today must be 83.40. To find the second-year bond prices at year one, we need to know the short rates at step one as in Fig. 15.15.

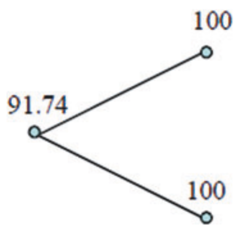


Fig. 15.14 A one-period tree

Table 15.2 Data by bootstrapping

Years of maturity	Zero-coupon rates (%)	Zero-coupon volatilities	Zero-bond prices
1	9	24	91.74
2	9.5	22	83.4
3	10	20	75.13
4	10.5	18	67.07
5	11	16	59.35

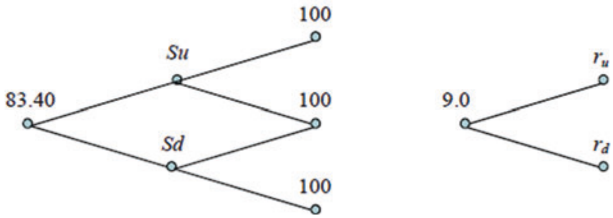


Fig. 15.15 How to find the rates in period one

Appealing to risk-neutral valuation, the following relationship must hold

$$\frac{0.5 \cdot \frac{100}{1+r_d} + 0.5 \cdot \frac{100}{1+r_u}}{1 + 0.09} = 83.40$$

In a standard binomial tree, we have

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}}, \qquad d = e^{-\sigma\sqrt{\Delta t}} \\ \frac{u}{d} &= e^{2\sigma\sqrt{\Delta t}} \quad \Rightarrow \quad \ln\left(\frac{u}{d}\right) = 2\sigma\sqrt{\Delta t} \\ \sigma &= \frac{1}{2\sqrt{\Delta t}} \ln\left(\frac{u}{d}\right) \end{aligned}$$

and

$$\begin{cases} S_u = \dfrac{100}{1 + r_u} \\ S_d = \dfrac{100}{1 + r_d} \end{cases}$$

where Δt in the previous situation is one year. Similarly, in the BDT tree, the rates are assumed to be lognormally distributed. This implies that

$$\sigma_n = \frac{1}{2\sqrt{\Delta t}} \ln\left(\frac{r_u}{r_d}\right) = 0.5 \cdot \ln\left(\frac{r_u}{r_d}\right) = 0.22$$

As we did in the OAS model earlier, we define the volatility factor Z_n by

$$Z_n = e^{2\sigma_n\sqrt{\Delta t}}$$

We are left with two equations in two unknowns, r_u and r_d. We know that $r_u = r_d Z_n = r_d e^{0.44}$, which leads to the following quadratic equation

$$\frac{0.5 \cdot \frac{100}{1+r_d} + 0.5 \cdot \frac{100}{1+r_d \cdot e^{0.44}}}{1 + 0.09} = 83.40$$

This gives a second order polynomial equation that can be solved and we get the following rates at step one

$$r_d = 7.87\%, \qquad\qquad r_u = 12.22\%.$$

Using these solutions, it is now possible to calculate the bond prices that correspond to these rates. The two-step tree of prices then becomes as in Fig. 15.16.

The next step is to fill in the two-period rate tree, see Fig. 15.17.

Last time, there were two unknown rates, and two sources of information:

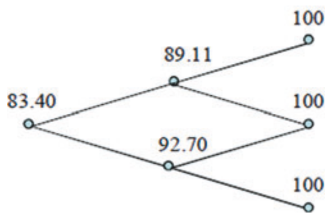


Fig. 15.16 The price-tree in period two

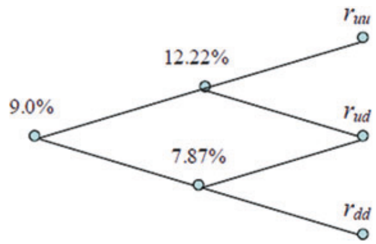


Fig. 15.17 The interest rate in period two

1. Zero-coupon rates.
2. The volatility of the zero-coupon rates.

This time, we have three unknown rates, but still only two sources of information. To get around this problem, remember that the BDT model is built on the following assumptions:

- Rates are lognormal distributed.
- The volatility is only dependent on time, not on the level of the short rates. There is thus only one level of volatility at the same time step in the rate tree.

Hence, the "steps" between the rates is given by

$$Z_3 = e^{2\sigma_3\sqrt{\Delta t}} = e^{2\cdot 0.20} = e^{0.40}$$

that is,

$$\frac{r_{uu}}{r_{ud}} = \frac{r_{ud}}{r_{dd}} = Z_3$$

So we are left with only two unknowns. Based on the risk-neutral valuation principle, the following relationships must hold

$$S_{uu} = \frac{100}{1 + r_{uu}} = \frac{100}{1 + r_{dd} \cdot Z_3^2}, \qquad S_{ud} = \frac{100}{1 + r_{ud}} = \frac{100}{1 + r_{dd} \cdot Z_3},$$

$$S_{dd} = \frac{100}{1 + r_{dd}} \qquad S_u = \frac{0.5 \cdot S_{uu} + 0.5 \cdot S_{ud}}{1 + 0.1222},$$

$$S_d = \frac{0.5 \cdot S_{ud} + 0.5 \cdot S_{dd}}{1 + 0.0787}, \qquad 75.13 = \frac{0.5 \cdot S_u + 0.5 \cdot S_d}{1 + 0.09}$$

If the bond only has two years left to maturity, the bond yield or rate of return must satisfy

$$75.13 = \frac{0.5 \cdot \frac{0.5 \cdot \frac{100}{1+r_{dd} \cdot Z_3^2} + 0.5 \cdot \frac{100}{1+r_{dd} \cdot Z_3}}{1+0.1222} + 0.5 \cdot \frac{0.5 \cdot \frac{100}{1+r_{dd} \cdot Z_3} + 0.5 \cdot \frac{100}{1+r_{dd}}}{1+0.0787}}{1+0.09}$$

By solving this equation, we get r_{dd}. By multiplying with Z_2 we then also get r_{ud} and r_{uu}:

$$r_{dd} = 7.47\%, \qquad r_{ud} = 10.76\%, \qquad r_{uu} = 15.50\%.$$

As the bond yields must be approximately lognormal distributed, it also follows that

$$0.5 \cdot \ln\left(\frac{y_u}{y_d}\right) = 0.20$$

$$\frac{y_u}{y_d} = e^{0.40}$$

Then, y_u can be expressed as

$$y_u = \frac{y_u}{y_d} y_d = e^{0.40}\left(\sqrt{\frac{100}{S_d}} - 1\right)$$

and S_u can be expressed in terms of S_d as

$$S_u = \frac{100}{\left[1 + e^{0.40}\left(\sqrt{\frac{100}{S_d}} - 1\right)\right]^2}$$

This equation must be solved by a numerical model as Newton-Raphson. The solution is

$$\begin{cases} S_u = 78.81 \\ S_d = 84.98 \end{cases}$$

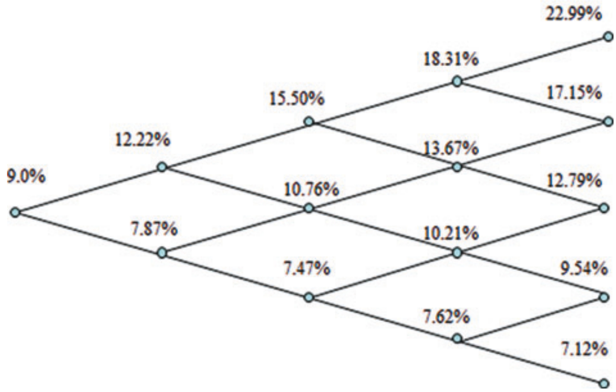


Fig. 15.18 The four-year short-rate tree

giving

$$\begin{cases} r_{dd} = 7.47\% \\ r_{ud} = 10.76\% \\ r_{dd} = 15.50\% \end{cases}$$

This gives the missing information in the two-period rate tree. The consecutive time steps can be computed by forward induction, as introduced by Jamshidan (1991), or more easily with the Bjerksund and Stensland (1996) analytical approximation of the short-rate interest-rate tree. Finally, we get the four-year short-rate tree as in Fig. 15.18.

From the short-rate tree, we can calculate the short-rate volatilities by using the relationship

$$\sigma_n = \frac{1}{2\sqrt{\Delta t}} \ln\left(\frac{r_u}{r_d}\right)$$

$$\begin{cases} \sigma_0 = 24.00\% \\ \sigma_1 = 22.00\% \\ \sigma_2 = 18.24\% \\ \sigma_3 = 14.61\% \\ \sigma_4 = 14.66\% \end{cases}$$

The four-year rate tree supplies input to the solution to the five-year price tree, Fig. 15.19.

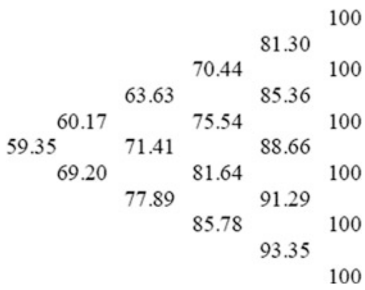


Fig. 15.19 The price tree at five year

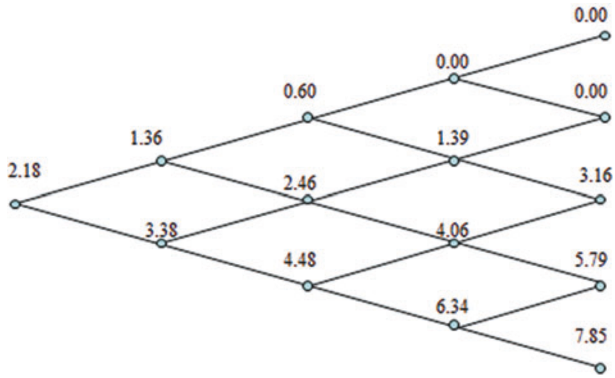


Fig. 15.20 The price tree of an American option

The value of the American call option with strike 85.50 and time to expiration of four years can now easily be found by standard backward induction. It follows as in (Fig. 15.20)

$$C_{j,i} = \max\left\{S_{j,i} - 85.50, \frac{\left(0.5 \cdot C_{j+1,i+1} + 0.5 \cdot C_{j+1,i}\right)}{i + r_{j,i}}\right\}$$

The price of the American call option on the five-year bond is thus 2.18.

15.1.5.2 Binomial Interest Trees with Forward Inductions

To solve the tree mentioned previously, more effectively we will introduce forward inductions. But first we will recall the backward induction. When we refer to nodes in a binomial model we will use the following index notation as shown in Fig. 15.21.

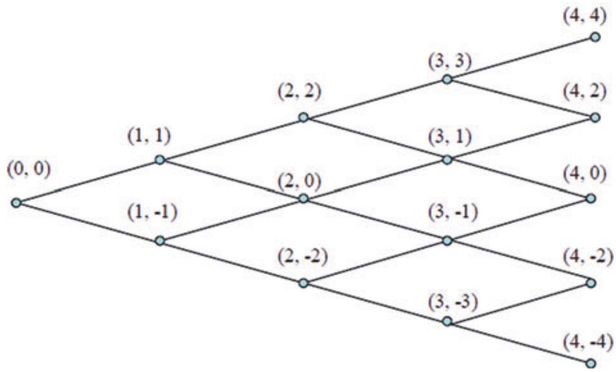


Fig. 15.21 The index notation of the nodes in the BDT model

Let the unit time be divided into M periods of length $\Delta t = 1/M$ each. At each period n, corresponding to time $t = n/M = n\Delta t$, there are $n + 1$ states. These states range from $i = -n, -n+2, \ldots, n-2, n$. At the present period $n = 0$, there is a single state $i = 0$. see the Fig. 15.21. Let $r(n, i)$ denote the annualized one-period rate at period n and state i. Denote the discount factor at period n and state i by

$$p(n, i) = \frac{1}{[1 + r(n, i)]^{\Delta t}}$$

The aforementioned tree is a discrete time representation of the stochastic process for the short rate, where we have used previous equation to express the discount. The probability for an up or down move in the three is chosen to be $1/2$.

The attraction of the binomial lattice model lies in the fact that, once the one period discount factors $p(n, i)$ are determined, securities are evaluated easily by **backward induction**. For example, let $C(n, i)$ denote the price of a security at period n and state i. This price is obtained from its prices at the up and down nodes in the next period by the backward equation. (Fig. 15.22)

$$C(n, i) = 1/2\, p(n, i)\, [C(n + 1, i + 1) + C(n + 1, i - 1)]$$

This iteration is continued backward all the way to period $n = 0$. The price of the security today is given by $C(0, 0)$.

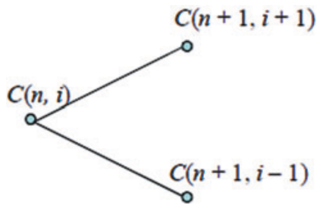


Fig. 15.22 The relation of the node index

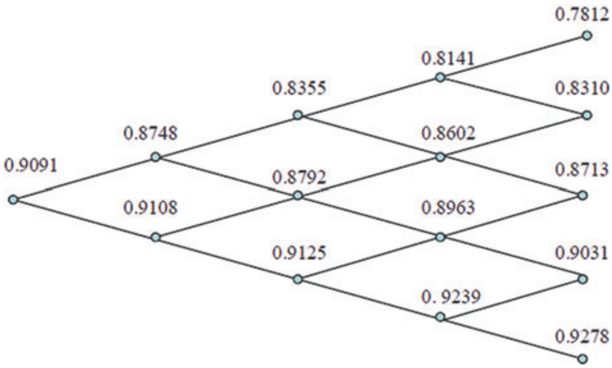


Fig. 15.23 The solution of zero-coupon prices (discount factors)

Suppose we were given the market data in Table 15.3 where the zero-coupon prices are calculated from the *ytm*:

Table 15.3 Market data

Maturity: *m* (years)	Zero-Coupon Price: $P(0, m)$ (Paying \$1 in *m* years)	$YTM(m)$ (%)	Yield Volatility $\sigma_{term}(m)$ (%)
1	0.9091	10.0	20
2	0.8116	11.0	19
3	0.7118	12.0	18
4	0.6243	12.5	17
5	0.5428	13.0	16

We want to find the discount factors that assure matching between the model's term structure and the market term structure. First we present the solution, see Fig. 15.23.

The discount factors in the tree were found by using forward induction, a method first introduced by Jamshidan. **Forward induction** is an efficient tool in the generation of yield-curve binomial trees. It is an application of the binomial formulation of the Fokker-Planck forward equation. The forward induction method will be described for the general class of Brownian-Path Independent interest models that includes the BDT model.

Brownian Path-Independent Interest Models

An interest rate model is referred to as Brownian Path Independent (BPI) if there is a function $r(z(t),\ t)$ such that $r(t) = r(z(t),\ t)$, where $z(t)$ is the Brownian motion. The instantaneous interest rate and hence the entire yield curve depends, at any time t, on the level $z(t)$ but not on the prior history $z(s),\ s < t$ of the Brownian motion. Two BPI families are of major interest:

Normal BPI	$r(t) = U_N(t) + \sigma_N(t)z(t)$
Lognormal BPI	$r(t) = U_L(t)e^{\sigma_L(t)z(t)}$

Where $\sigma_N(t)$ and $\sigma_L(t)$ represent, respectively the absolute and the percentage volatility of the short-rate $r(t)$. $U_N(t)$ is the mean and $U_L(t)$ the median of $r(t)$.

The advantages with the lognormal BPI are as we have mentioned the positive interest rates and natural unit of volatility in percentage form, consistent with the way volatility is quoted in the market place. However, unlike the normal BPI, the lognormal BPI does not provide a closed form solution. It is possible to fit the yield curve by trial and error but this is inefficient. In fact, the total computational time needed to calculate all the discount factors of a tree with N periods is proportional to N^3. Since N should be at least 100, too many iterations are needed. Forward induction efficiently solves the yield-curve fitting problems, but before describing the procedure a discrete version of the BPI models is necessary.

In most literature, the BDT model is presented with all time periods equal to one. We will here present the most general model where the time-steps and the cash flows can vary over time. Then we can handle all kind of time structures, amortizing instruments etc.

Consider again the previously mentioned tree. Define the variable X_k as

$$X_k = \sum_{j=1}^{k} y_j$$

where $y_j = 1$ if an up move occurs at period k and $y_j = -1$ if a down move occurs at period k.

The variable X_k gives the state of the short rate at period k. At any period k, the X_k has a binomial distribution with mean zero and variance k. Now, let us investigate the mean and variance of $X_k\sqrt{\Delta t}$

$$\begin{cases} E\left[X_k\sqrt{\Delta t}\right] = \sqrt{\Delta t}E\left[X_k\right] = 0 \\ Var\left[X_k\sqrt{\Delta t}\right] = \Delta t Var\left[X_k\right] = k\Delta t = 1 \end{cases}$$

It follows that $X_k\sqrt{\Delta t}$ has the same mean and variance as the Brownian motion $z(t)$. Since the normal distribution is a limit of binomial distributions, and the binomial process X_k has independent increments, the binomial process $X_k\sqrt{\Delta t}$ converges to the Brownian motion $z(t)$ as Δt approaches zero.

The state of the short rate was denoted by i. Replacing X_k by i will lead to having $z(t)$ approximated as $i\sqrt{\Delta t}$. Now, replacing t by m (with $m = t/\Delta t$) gives the discrete version of the normal and lognormal BPI families.

Normal BPI	$r(m,i) = U_N(m) + \sigma_N(m)i\sqrt{\Delta t}$
Lognormal BPI	$r(m,i) = U_L(m)e^{\sigma_L(m)i\sqrt{\Delta t}}$

Where $i = -m,\ -m+2, \ldots, m-2,\ m$.

Green Functions and the Forward Induction Method

For a node (m,j) a pure-state security $s(m,j)$ is a short-rate security expiring at $t = m$ which pays \$1 at (m,j) and 0 elsewhere. The Green

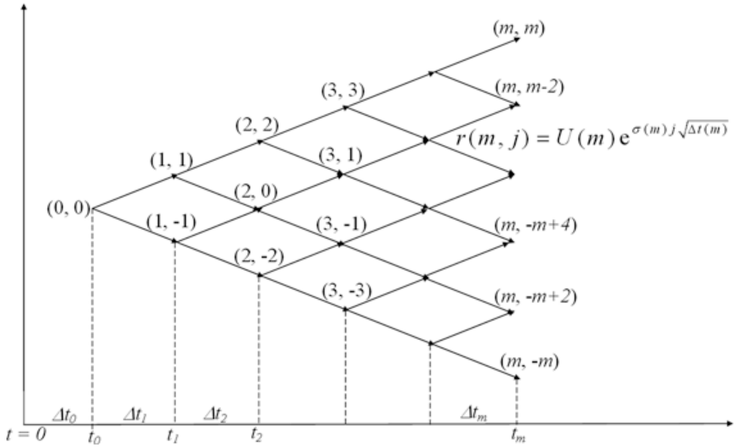


Fig. 15.24 How to build the BDT tree

function $G(m, 0)$ is the time-zero value of $s(m, 0)$. For example, $G(2,\ 0)$ is the value at $t = 0$ of $s(2,\ 0)$.

We build the tree using the nodes as in Fig. 15.24.

Example 15.1.1

A European short-rate security paying \$1 at (2, 0) and 0 elsewhere is given by

$$G(2,0) = \frac{1}{2}p(0,0)\left[\frac{1}{2}p(1,1) + \frac{1}{2}p(1,-1)\right]$$

where $p(m,j)$ is the one-period discount factor

$$p(m,j) = \exp\{-r(m,j)\Delta t\}$$

continuously compounded or

$$p(m,j) = \frac{1}{(1 + r(m,j))^{\Delta t}}$$

compounded with use of the simple rate. The nodes are given as in Fig. 15.25. If we put in the numbers from the following example, we see

$$G(2,0) = \frac{1}{2}\cdot 0.909091\cdot\left[\frac{1}{2}\cdot 0.8747 + \frac{1}{2}\cdot 0.9108\right] = 0.4058$$

$$G(2,2) = \frac{1}{2}\cdot 0.909091\cdot\frac{1}{2}\cdot 0.8747 = 0.1988$$

$$G(2,-2) = \frac{1}{2}\cdot 0.909091\cdot\frac{1}{2}\cdot 0.9108 = 0.2070$$

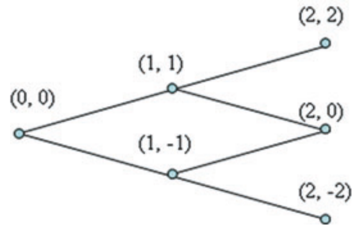


Fig. 15.25 The node indices

Where, for example,

$$G(2,2) = \frac{1}{2} \cdot G(1,1) \cdot p(1,1) = \frac{1}{2}\left\{\frac{1}{2}G(0,0) \cdot p(0,0)\right\} p(1,1)$$
$$= \frac{1}{2}\left\{\frac{1}{2} \cdot 1 \cdot 0.909091\right\} p(1,1)$$

and

$$G(2,0) = \frac{1}{2} \cdot (G(1,-1) \cdot p(1,-1) + G(1,1) \cdot p(1,1))$$
$$= \frac{1}{2} \cdot \left(\frac{1}{2}G(0,0) \cdot p(0,0) \cdot p(1,-1) + \frac{1}{2}G(0,0) \cdot p(0,0) \cdot p(1,1)\right)$$
$$= \frac{1}{2} \cdot \left(\frac{1}{2} \cdot 1 \cdot 0.90901 \cdot p(1,-1) + \frac{1}{2} \cdot 1 \cdot 0.90901 \cdot p(1,1)\right)$$

When we number the nodes in the tree as shown earlier, we have the value of the discount bond at (0, 0) as

$$p(0,0) = \frac{1}{[1 + r(0,0)]^{t_0}}$$

So we define the discount factors as

$$P(m) = \frac{1}{[1 + r(m)]^{t_m}}$$

These are known since we know the spot rate $r(m) \equiv r(t_m)$. The simply compounded forward rate is defined as

$$f(t,T,T+\Delta t) = \frac{1}{\Delta t}\left(\frac{p(t,T)}{p(t,T+\Delta t)} - 1\right)$$

Later, we will use simple rate compounding only, since this is the most common in the fixed income theory.

We can compute $G(m,j)$ with forward induction

$$G(m+1,j) = \begin{cases} \frac{1}{2}\left[p(m,j-1)G(m,j-1) + p(m,j+1)G(m,j+1)\right] & |j| \leq m-1 \\ \frac{1}{2}p(m,j-1)G(m,j-1) & j = m+1 \\ \frac{1}{2}p(m,j+1)G(m,j+1) & j = -m-1 \end{cases}$$

The initial condition is given by $G(0,0) = 1$.

Note that an arbitrary short-rate security expiring at $t = m$, with a payoff $p(m,j)$ can be considered as a combination of pure-state securities.

In discrete-time finance, the Green function is known as **Arrow-Debreu prices** where it represents prices of primitive securities. Let $G(n,i,m,j)$ denote the price at period n and state i of a security that has a cash flow of unity at period m ($m \geq n$) and state j. Note that $G(m,j,m,j) = 1$ and that $G(m,i,m,j) = 0$ for $i \neq j$. In most cases we ignore the first two arguments and say that $G(m,j) = 1$ if we reach the node (m,j) and zero else.

By intuitive reasoning, the previous forward induction function states how we discount a cash flow of unity for receiving it one period later. This is simply the dual of the backward binomial equation. Note that when $j = \pm m$, there is only one node (at the bottom or at the top), which gives a modified expression for these two cases.

The term structure $p(0, m)$, which represent the price today of a bond that pays unity at period m, can be obtained for all values of m, by the **maximum smoothness criterion** (see next section).

Arrow-Debreu prices are the building blocks of all securities. The price of a zero-coupon bond that matures at period $m+1$ can be expressed in terms of the Arrow-Debreu prices and the discount factors in period m.

$$p(0,m+1) = \sum_j G(0,0,m,j)p(m,j)$$

In most cases we simply write this equation as

$$p(m+1) = \sum_j G(m,j)p(m,j)$$

The term structure can be fitted to any BPI models using **forward induction**. First, assume that $\sigma(m) = \sigma$ is a given constant. The problem is then to solve $U(m)$ to match the given discount function $p(m, j)$ where

$$p(m,j) = \frac{1}{(1 + r(m,j))^{\Delta t(m)}} = \frac{1}{\left(1 + U(m)\exp\left(\sigma(m) j \sqrt{\Delta t(m)}\right)\right)^{\Delta t(m)}}$$

Constant Short-Rate Volatility

First, we let the short-rate volatility be a constant, $\sigma(m) = \sigma$. Let $m > 0$ and assume that $U(m-1)$, $G(m-1, j)$, $r(m-1, j)$ and $p(m-1, j)$ have been found. The values at the initial time $m = 0$ are $U(0) = r(0,0)$, $G(0,0) = 1$ and $p(0,0) = 1/[1 + r(0,0)]^{\Delta t(0)}$.

Step 1: Generate the Green functions:

$$G(m+1,j) = \begin{cases} \frac{1}{2}\left[p(m,j-1)G(m,j-1) + p(m,j+1)G(m,j+1)\right] & |j| \le m-1 \\ \frac{1}{2} p(m,j-1)G(m,j-1) & j = m+1 \\ \frac{1}{2} p(m,j+1)G(m,j+1) & j = -m-1 \end{cases}$$

or

$$\begin{cases} \frac{1}{2}\left[\frac{G(m,j-1)}{\left[1 + U(m)\exp\left(\sigma(m)(j-1)\sqrt{\Delta t(m)}\right)\right]^{\Delta t(m)}} \right. \\ \qquad \left. + \frac{G(m,j+1)}{\left[1 + U(m)\exp\left(\sigma(m)(j+1)\sqrt{\Delta t(m)}\right)\right]^{\Delta t(m)}}\right] \\ \frac{1}{2}\frac{G(m,j-1)}{\left[1 + U(m)\exp\left(\sigma(m)(j-1)\sqrt{\Delta t(m)}\right)\right]^{\Delta t(m)}} \\ \frac{1}{2}\frac{G(m,j+1)}{\left[1 + U(m)\exp\left(\sigma(m)(j+1)\sqrt{\Delta t(m)}\right)\right]^{\Delta t(m)}} \end{cases}$$

or

$$\begin{cases} \frac{1}{2}\left[\frac{G(m,j-1)}{\left[1+r(m,j-1)\right]^{\Delta t(m)}}+\frac{G(m,j+1)}{\left[1+r(m,j+1)\right]^{\Delta t(m)}}\right] \\ \frac{1}{2}\frac{G(m,j-1)}{\left[1+r(m,j-1)\right]^{\Delta t(m)}} \\ \frac{1}{2}\frac{G(m,j+1)}{\left[1+r(m,j+1)\right]^{\Delta t(m)}} \end{cases}$$

Step 2: Use $p(0, m+1)$ to solve $U(m)$ via:

$$p(0,m+1) = \sum_j G(m,j)\frac{1}{\left[1+U(m)\exp\left(\sigma j\sqrt{\Delta t(m)}\right)\right]^{\Delta t(m)}}$$

Step 3: From $U(m)$ calculate the short rate, and update the discount factors, for all nodes at time m:

$$r(m,j) = U(m)e^{\sigma j\sqrt{\Delta t(m)}}$$
$$p(m,j) = \frac{1}{\left[1+r(m,j)\right]^{\Delta t(m)}}$$

For the solution in step 2, we use a Newton-Raphson method

$$x_m^{n+1} = x_m^n - \frac{f(x_m^n)}{f'(x_m^n)}$$

We can easily do this since we have the derivatives. The procedure converges rapidly with three to four iterations.

Let $U(m) = x_m$ be the unknown. Then

$$f(x_m) = \sum_j G(m,j)\frac{1}{\left(1+x_m\cdot e^{\sigma j\sqrt{\Delta t(m)}}\right)^{\Delta t(m)}} - p(m+1) = 0$$

The derivatives are given by

$$f'(x_m) = -\sum_j G(m,j)\frac{e^{\sigma j\sqrt{\Delta t(m)}}}{\left[1+x_m\cdot e^{\sigma j\sqrt{\Delta t(m)}}\right]^{\Delta t(m)+1}}$$

Example 15.1.2

From the following given data, where we have calculated p from r:

$$p(i) = \frac{1}{(1 + r(i))^{t_i}}$$

$$r(0,\ 0) \equiv r(1) = 0.100 \Rightarrow p(0,\ 0) \equiv p(1) = 0.909091$$
$$r(0,\ 1) \equiv r(2) = 0.110 \Rightarrow p(0,\ 1) \equiv p(2) = 0.811622$$
$$r(0,\ 2) \equiv r(3) = 0.120 \Rightarrow p(0,\ 2) \equiv p(3) = 0.711780$$
$$r(0,\ 3) \equiv r(4) = 0.125 \Rightarrow p(0,\ 3) \equiv p(4) = 0.624295$$
$$r(0,\ 4) \equiv r(5) = 0.130 \Rightarrow p(0,\ 4) \equiv p(5) = 0.542760$$

and $\sigma = 19\%$, we calculate (with $U(0) = r(0,\ 0),\ G(0,\ 0) = 1.0$)

Step 1: Calculate the Green functions at time 1:

$$G(1,\ -1) = 0.5 \cdot G(0,\ 0) \cdot p(0,\ 0) = 0.5 \cdot 1.0 \cdot 0.9091 = 0.4545$$
$$G(1,\ 1) = 0.5 \cdot G(0,\ 0) \cdot p(0,\ 0) = 0.5 \cdot 1.0 \cdot 0.9091 = 0.4545$$

Step 2: Use $p(0,\ 1),\ G(1,\ -1)$ and $G(1,\ 1)$ to solve $U(1)$ via.

$$\frac{G(1,-1)}{[1 + r(1,-1)]^{\Delta t(1)}} + \frac{G(1,1)}{[1 + r(1,1)]^{\Delta t(1)}} = p(0,1)\text{ă}$$

giving

$$0.4545 \left\{ \frac{1}{1 + U(1) \cdot e^{0.19}} + \frac{1}{1 + U(1) \cdot e^{-0.19}} \right\} = 0.811622$$

The result is $U(1) = 0.1184$.

Step 3: From $U(1)$ calculate the short rate, and the discount factors at time 1:

$$r(1,\ -1) = 0.1184.e^{-0.19} = 9.7915\ \% \Rightarrow p(1,\ -1) = 0.9108$$
$$r(1,\ 1) = 0.1184.e^{0.19} = 14.3180\ \% \Rightarrow p(1,\ 1) = 0.8747$$

Step 1: Calculate the Green functions at time 2:

$$G(2,\ -2) = 0.5 \cdot G(1,\ -1) \cdot p(1,-1) = 0.5 \cdot 0.4545 \cdot 0.9108 = 0.2070$$
$$G(2,\ 2) = 0.5 \cdot G(1,\ 1) \cdot p(1,\ 1) = 0.5 \cdot 0.4545 \cdot 0.8747 = 0.1988$$
$$\begin{aligned} G(2,\ 0) &= 0.5 \cdot (G(1,\ -1) \cdot p(1,\ -1) + G(1,\ 1) \cdot p(1,\ 1)) \\ &= 0.5 \cdot (0.4545 \cdot 0.9108 + 0.4545 \cdot 0.8747)) = 0.4058 \end{aligned}$$

Step 2: Use $p(0,\ 2),\ G(2,\ -2),\ G(2,\ 0)$ and $G(2,\ 2)$ to solve $U(2) = 0.1374$.

Step 3: From $U(2)$ calculate the short rate, and the discount factors at time 2:

$$\begin{array}{ll} p(2,\ -2) = 0.9125 & r(2,\ -2) = 9.5862\,\% \\ p(2,\ 0) = 0.8792 & r(2,\ 0) = 13.7401\,\% \\ p(2,\ 2) = 0.8355 & r(2,\ 2) = 19.6941\,\% \end{array}$$

Step 1: Calculate the Green functions at time 3:

$$G(3,\ -3) = 0.5 \cdot G(2,\ -2) \cdot p(2,\ -2) = 0.5 \cdot 0.2070 \cdot 0.9125 = 0.0944$$
$$G(3,\ 3) = 0.5 \cdot G(2,\ 2) \cdot p(2,\ 2) = 0.5 \cdot 0.1988 \cdot 0.8355 = 0.0830$$
$$\begin{aligned} G(3,\ -1) &= 0.5 \cdot (G(2,\ -2) \cdot p(2,\ -2) + G(2,\ 0) \cdot p(2,\ 0)) \\ &= 0.5 \cdot (0.2070 \cdot 0.9125 + 0.4058 \cdot 0.8792) = 0.2728 \end{aligned}$$
$$\begin{aligned} G(3,\ 1) &= 0.5 \cdot (G(2,\ 0) \cdot p(2,\ 0) + G(2,\ 2) \cdot p(2,\ 2)) \\ &= 0.5(0.4058 \cdot 0.8792 + 0.1988 \cdot 0.8355) = 0.2614 \end{aligned}$$

Step 2: Use $p(0,\ 3),\ G(3,\ j)$ to solve $U(3) = 0.137156$.

Step 3: From $U(3)$ calculate the short rate, and the discount factors at time 3:

$$\begin{array}{ll} p(3,\ -3) = 0.9239 & r(3,\ -3) = 8.2361\ \% \\ p(3,\ -1) = 0.8963 & r(3,\ -1) = 11.5713\ \% \\ p(3,\ 1) = 0.8602 & r(3,\ 1) = 16.2571\ \% \\ p(3,\ 3) = 0.8141 & r(3,\ 3) = 22.8404\ \% \end{array}$$

etc.

In C/C++, a tree calibrated to yield can be written as

```
/* ------------------------------------------------------------------
    Build a Black-Derman-Toy tree calibrated to the interest rate
    Input:  yc   - a vector with short interest rates at the nodes
                   to build the tree
            vol  - a vector with short IR volatilities at the nodes
                   to build the tree
            N    - the number of nodes in the vectors
            dt   - the time intervals between the nodes

    Output: d    - the discount tree
            r    - the tree of forward rates
                   i = 0 -> i < N
                     j = -i -> j <= i(j+=2
                       r[i][N+j], d[i][N+j]
------------------------------------------------------------------*/


double buildBDT(double *yc, double *vol, int N, double *dt,
                double **d, double **r)
{
   double **Q; // State securities = 1 if (i,j) is reached, 0 if not
   double  *U; // Median of the (lognormal) distribution for r at
               // time t
   double *P;  // Bond prices

   const  double epsilon = 0.000001; // Error in N-R iterations
   const  int    maxit   = 20;       // Max number of iterations
   int    iter;
   double error, sum1, sum2, guess, Guess, Time, Disk, Gamma, Disk2;
```

```
U = (double  *)calloc(N + 2,   sizeof(double));
P = (double  *)calloc(N + 2,   sizeof(double));
Q = (double **)calloc(2*N + 1, sizeof(double));

for (int i = 0; i < 2*N + 1; i++)
  Q[i] = (double *)calloc(2*N + 1, sizeof(double));

Time = 0.0;

// Discount factors per time-node
for (int j = 0; j < N; j++) {
  Time += dt[j];
  P[j+1] = 1.0/pow(1.0 + yc[j], Time);
}

// initlize first node
Time    = 0.0;
Q[0][N] = 1.0;
U[0]    = yc[0];
r[0][N] = U[0];
d[0][N] = 1.0/pow(1.0 + r[0][N], dt[0]);

// evolve the tree for the short rate
for(int i = 1; i < N; i++){
  //update pure security prices at time i
  Q[i][N-i] = 0.5*Q[i-1][N-i+1]*d[i-1][N-i+1];
  Q[i][N+i] = 0.5*Q[i-1][N+i-1]*d[i-1][N+i-1];

  for (int j = -i+2; j <= i-2; j += 2)
    Q[i][N+j] = 0.5*Q[i-1][N+j-1]*d[i-1][N+j-1]
              + 0.5*Q[i-1][N+j+1]*d[i-1][N+j+1];

  // Use Ralph-Newton method to solve the mean rate, U[i]
  guess=r[0][N];
  iter=0;

  do {
    sum1 = 0; sum2 = 0;

    for (int j = -i; j <= i; j += 2) {
      Gamma = exp(vol[i]*j*sqrt(dt[i]));    //Volatility factor
      Disk  = pow(1.0+guess*Gamma,dt[i]);   //Discount   factor
      Disk2 = pow(Disk,dt[i]+1.0);
      sum1 += Q[i][N+j]/Disk;          //Should sum up to P[i+1]
      sum2 += Q[i][N+j]*Gamma*dt[i]/Disk2;  //Derivative
    }
    Guess = guess + (sum1 - P[i+1])/sum2;
    error = fabs(Guess - guess);
    guess = Guess;
    iter++;

    if (iter > maxit) {
      printf("===========WARNING!NOSOLUTION==============\n");
      break;
    }
  }
```

```
      while(error > epsilon);

      U[i] = guess;
      // set r[.] and d[.]
       for (int j = -i; j <= i; j+ =2 ) {
        r[i][N+j] = U[i]*exp(vol[i]*j*sqrt(dt[i]));
        d[i][N+j] = 1/pow(1 + r[i][N+j], dt[i]);
      }
    }
    return0.0;
}
```

Fitting Interest Rate Yield and Volatility Data

Next consider the case matching both the yield and volatility curves in BPI models of the same general form. This requires solving jointly for $U(m)$ and $\sigma(m)$. Let $P_U(1,m)$ and $P_D(1,m)$ represent the prices at period $n = 1$ states of m-maturity zero-coupon bonds, see Fig. 15.26.

Here, to get the present value of the cash flow at t_0, we use $r(0,0)$ to discount to time $t = 0$. We also define the steps in time: Δt_1 is the step from t_0 to t_1, so Δt_1 is the step from $t = 0$ to t_1. In the figure we have used the short-rate formula for the BDT model. Note that an up move of the short rate differs from a down move by a factor $\exp(2\sigma(m)\sqrt{\Delta t(m)})$.

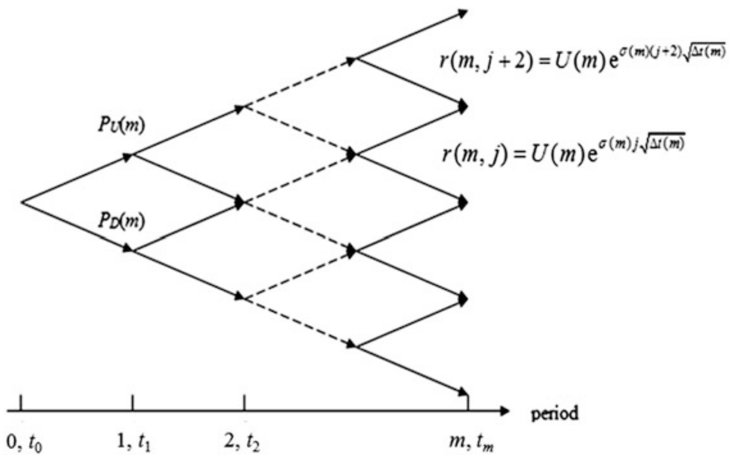


Fig. 15.26 The BDT tree

Therefore we have

$$\frac{y_U(m)}{y_D(m)} = \mathrm{e}^{2\sigma_{term}(m)\sqrt{\Delta t(m)}}$$

P_u and P_d are given by

$$P_U(m) = \frac{1}{\left[1 + y_U(m)\right]^{t_m - \Delta t_m}}$$

and

$$P_D(m) = \frac{1}{\left[1 + y_D(m)\right]^{t_m - \Delta t_m}}$$

A second equation is found by discounting back to the origin

$$P(m) = \frac{1}{2}\left[P_U(m) + P_D(m)\right] p(0,0)$$

If we introduce

$$T_m = t_m - \Delta t_m$$

and

$$\Gamma(m) = \mathrm{e}^{-2\sigma_{term}(m)\sqrt{\Delta t(m)}}$$

we have

$$\frac{1}{P_D(m)^{T_m}} - 1 = \left(\frac{1}{P_U(m)^{T_m}} - 1\right) \cdot \Gamma(m)$$

giving

$$P_D(m) = \left(1 - \Gamma(m) + \Gamma(m) P_U(m)^{-1/T_m}\right)^{-T_m}$$

We then have two equations to solve simultaneously:

$$\begin{cases} P_D(m) = \left(1 - \Gamma(m) + \Gamma(m) P_U(m)^{-1/T_m}\right)^{-T_m} \\ P_U(m) + \left(1 - \Gamma(m) + \Gamma(m) P_U(m)^{-1/T_m}\right)^{-T_m} = 2\dfrac{P(m)}{p(0,0)} \end{cases}$$

We first solve $P_U(m)$ via Newton-Raphson and then calculate $P_D(m)$.

Forward induction is then used to determine the time-dependent functions that ensure consistency with the initial yield-curve data. However, state prices are now determined from the nodes U and D requiring the following notation:

$G_U(m, j)$: the value, as seen from node U, of a security that pays off \$1 if node (m, j) is reached and zero otherwise.

$G_D(m, j)$: the value, as seen from node D, of a security that pays off \$1 if node (m, j) is reached and zero otherwise.

By definition $G_U(1, 1) = 1$ and $G_D(1, -1) = 1$. The tree is constructed from time Δt onward using a procedure similar to the previous section. Now we have two equations

$$\begin{cases} P_U(m+1) = \sum G_U(m,j)p(m,j) \\ P_D(m+1) = \sum G_D(m,j)p(m,j) \end{cases}$$

where $j = -m, -m+2, \ldots, m-2, m$ and

$$p(m,j) = \frac{1}{\left[1 + U(m)e^{\sigma(m)j\sqrt{\Delta t(m)}}\right]^{\Delta t(m)}}$$

The term structure of zero-coupon bonds $P(m)$ and the yield volatility term-structure $\sigma_{term}(m)$ are known. $P_U(m)$ and $P_D(m)$ can be found for all periods m by using the equation for the term structure of volatility in conjunction with the Arrow-Debreu prices in a Newton-Raphson iteration where we have two unknown $U(m)$ and $\sigma(m)$ to solve simultaneously. This can easily be done since, as we will see, the Jacobian is known.

The full set of steps to build the tree is therefore the following. Assume $m > 0$ and that $U(m-1)$, $\sigma(m-1)$, $G_U(m-1, j)$, $G_D(m-1, j)$ and $r(m-1, j)$ are known for all j at time step $m-1$. The values at initial time are $U(0) = r(0, 0)$, $\sigma(0) = \sigma_{term}(1)$, $G_U(1, 1) = 1$, $G_D(1, -1) = 1$ and $r(m-1, j)$ giving $p(0,0) = 1/(1 + r(0,0))^{\Delta t(0)}$.

Step 1: Derive $P_U(m)$ and $P_D(m)$ for $m = 2$ to N.

$$P_D(m) = \left(1 - \exp\left(-2\sqrt{\Delta t(m)}\sigma_{term}(m)\right)\right. \\ \left. + \exp\left(-2\sqrt{\Delta t(m)}\sigma_{term}(m)\right) P_U(m)^{-1/(t_m - \Delta t_m)}\right)^{-(t_m - \Delta t_m)}$$

$P_U(m)$ is found as the solution to

$$P_U(m) + \begin{pmatrix} 1 - \exp\left(-2\sqrt{\Delta t(m)}\sigma_{term}(m)\right) + \\ \exp\left(-2\sqrt{\Delta t(m)}\sigma_{term}(m)\right) P_U(m)^{-1/(t_m - \Delta t_m)} \end{pmatrix}^{-(t_m - \Delta t_m)} \\ - 2\frac{P(m)}{p(0,0)} = 0 \quad m \geq 2$$

The derivative used in Newton-Raphson is given

$$1 + \exp\left(-2\sqrt{\Delta t(m)}\sigma_{term}(m)\right) \cdot P_U(m)^{-1/(t_m-\Delta t_m)-1} \cdot$$
$$\left(1 - \exp\left(-2\sqrt{\Delta t(m)}\sigma_{term}(m)\right)\right.$$
$$\left. + \exp\left(-2\sqrt{\Delta t(m)}\sigma_{term}(m)\right) P_U(m)^{-1/(t_m-\Delta t_m)}\right)^{-(t_m-\Delta t_m)-1}$$

Step 2: Generate $G_U(m, j),\ G_D(m, j)$

$$\begin{cases} G_U(m, j) \\ \quad = \dfrac{1}{2}\left[p(m-1, j-1)G_U(m-1, j-1) + p(m-1, j+1)G_U(m-1, j+1)\right] \\ G_D(m, j) \\ \quad = \dfrac{1}{2}\left[p(m-1, j-1)G_D(m-1, j-1) + p(m-1, j+1)G_D(m-1, j+1)\right] \end{cases}$$

Step 3: Using the derived discount functions $P_U(m{+}1)$ and $P_D(m{+}1)$, solve $U(m)$ and $\sigma(m)$ from

$$\begin{cases} P_U(m+1) = \sum G_U(m, j)p(m, j) \\ P_D(m+1) = \sum G_D(m, j)p(m, j) \end{cases}$$

where $j = -m, -m+2, \ldots, m-2, m$ and

$$p(m, j) = \frac{1}{\left(1 + U(m)\mathrm{e}^{\sigma(m)j\sqrt{\Delta t(m)}}\right)^{\Delta t(m)}}$$

Step 4: From the calculated values of $U(m)$ and $\sigma(m)$ calculate the short rate, and one-period discount factors for all nodes j at time m

$$r(m, j) = U(m)\mathrm{e}^{\sigma(m)j\sqrt{\Delta t(m)}}$$
$$p(m, j) = \frac{1}{\left[1 + r(m, j)\right]^{\Delta t(m)}}$$

The method of forward induction requires only n arithmetic operations to construct the discount factors at period n. The iterations are made only on the nodes at the given period, not the whole way back to the root of the tree as in "trial and error". In a tree with N periods, the computational time is proportional to the number of nodes N^2, which is of the same order as when closed formed solutions for the discount factors are available.

The Newton-Raphson Method in 2 Dimensions

The Newton-Raphson method is a well-known numerical method to solve roots to equations from numerical analysis. The iteration scheme can be derived from a Maclaurin expansion of a given function $f(x)$

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

To the lowest order we have $f(x+h) = f(x) + h \cdot f'(x)$ where x + h is the root we are trying to calculate. Let the root be x_{n+1}. We can write an iteration scheme as

$$0 = f(x_{n+1}) = f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

This gives the well-known formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If we have to solve two equations simultaneously:

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$

We can write Newton-Raphson with vector notation:

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \frac{\mathbf{f}(\mathbf{x}_n)}{\mathbf{f'}(\mathbf{x}_n)}$$

where $\mathbf{x}_n = (x_n, y_n)$ and

$$\mathbf{f}(\mathbf{x}_n) = \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}$$

The derivative is called the Jacobian and is defined as

$$\mathbf{J}(\mathbf{x}_n) = \mathbf{f'}(\mathbf{x}_n) = \begin{pmatrix} \dfrac{\partial f(x_n, y_n)}{\partial x_n} & \dfrac{\partial f(x_n, y_n)}{\partial y_n} \\ \dfrac{\partial g(x_n, y_n)}{\partial x_n} & \dfrac{\partial g(x_n, y_n)}{\partial y_n} \end{pmatrix}$$

Since we cannot divide by the Jacobian, we have to multiply with the inverse Jacobian. Therefore, we remember how to invert a 2×2 matrix from the linear algebra

$$\mathbf{J} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \mathbf{J}^{-1} = \frac{1}{det(\mathbf{J})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},\ det(\mathbf{J}) = ad - bc$$

We then have the following system of equation to solve:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \frac{1}{D} \begin{pmatrix} \dfrac{\partial g(x_n, y_n)}{\partial y_n} & -\dfrac{\partial f(x_n, y_n}{\partial y_n} \\ -\dfrac{\partial g(x_n, y_n)}{\partial x_n} & \dfrac{\partial f(x_n, y_n)}{\partial x_n} \end{pmatrix} \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}$$

where

$$D = \begin{vmatrix} \dfrac{\partial f(x_n, y_n)}{\partial x_n} & \dfrac{\partial f(x_n, y_n)}{\partial y_n} \\ \dfrac{\partial g(x_n, y_n)}{\partial x_n} & \dfrac{\partial g(x_n, y_n)}{\partial y_n} \end{vmatrix} = \frac{\partial f(x_n, y_n)}{\partial x_n} \frac{\partial g(x_n, y_n)}{\partial y_n} - \frac{\partial f(x_n, y_n)}{\partial y_n} \frac{\partial g(x_n, y_n)}{\partial x_n}$$

In the previous BTD model we have to solve$U(m)$ and $\sigma(m)$ from

$$\begin{cases} P_U(m+1) = \sum G_U(m, j)\, p(m, j) \\ P_D(m+1) = \sum G_D(m, j)\, p(m, j) \end{cases}$$

where $j = -m, -m+2, \ldots, m-2, m$ and

$$p(m, j) = \frac{1}{\left(1 + U(m) e^{\sigma(m) j \sqrt{\Delta t(m)}}\right)^{\Delta t(m)}}$$

That is, we have two equations

$$\begin{cases} f(U, \sigma) = \sum G_U p(U, \sigma) - P_U = 0 \\ g(U, \sigma) = \sum G_D p(U, \sigma) - P_D = 0 \end{cases}$$

with

$$\begin{cases} f'_U(U,\sigma) = \sum G_U \dfrac{\partial p_U(U,\sigma)}{\partial U} \\ f'_\sigma(U,\sigma) = \sum G_U \dfrac{\partial p_U(U,\sigma)}{\partial \sigma} \\ g'_U(U,\sigma) = \sum G_D \dfrac{\partial p_D(U,\sigma)}{\partial U} \\ g'_\sigma(U,\sigma) = \sum G_D \dfrac{\partial p_D(U,\sigma)}{\partial \sigma} \end{cases}$$

$$f(x_m) = \sum_j G(m,j) \frac{1}{\left(1 + x_m \cdot \mathrm{e}^{\sigma j \sqrt{\Delta t(m)}}\right)^{\Delta t(m)}} - P(m+1) = 0$$

The derivatives are given by

$$\frac{\partial p(U,\sigma}{\partial U} = -\frac{\Delta t(m) \cdot e^{\sigma(m) j \sqrt{\Delta t(m)}}}{\left[1 + U(m) \cdot e^{\sigma(m) j \sqrt{\Delta t(m)}}\right]^{\Delta t(m)+1}}$$

and

$$\frac{\partial p(U,\sigma}{\partial \sigma} = -\frac{U(m) \cdot j \cdot \Delta t(m) \cdot \sqrt{\Delta t(m)} \cdot e^{\sigma(m) j \sqrt{\Delta t(m)}}}{\left[1 + U(m) \cdot e^{\sigma(m) j \sqrt{\Delta t(m)}}\right]^{\Delta t+1}}$$

Similar expressions yield for the derivatives of $P_D(U, \sigma)$. Therefore, we have the Jacobian and can use Newton-Raphson to build the tree.

Example 15.1.3

From the following data, where we now added volatility data:

$p(0, 0) = 0.909091$	$r(0, 0) = 0.100$	$\sigma(1) = 0.150$
$p(0, 1) = 0.811622$	$r(0, 1) = 0.110$	$\sigma(2) = 0.140$
$p(0, 2) = 0.711780$	$r(0, 2) = 0.120$	$\sigma(3) = 0.130$
$p(0, 3) = 0.624295$	$r(0, 3) = 0.125$	$\sigma(4) = 0.120$
$p(0, 4) = 0.542760$	$r(0, 4) = 0.130$	$\sigma(5) = 0.110$

and calculate ($\Delta t = 1$):

Step 1: Derive $P_U(m)$ and $P_D(m)$ for $m = 2$ to N.

Our target here is $P(2) = p(0,\ 1) = 0.811622$

$$\exp\left(-2\sqrt{\Delta t(2)}\sigma(2)\right) = 0.7557837$$

$$\exp\left(-2\sqrt{\Delta t(3)}\sigma(3)\right) = 0.7710515$$

$$\exp\left(-2\sqrt{\Delta t(4)}\sigma(4)\right) = 0.7866279$$

$$\exp\left(-2\sqrt{\Delta t(5)}\sigma(5)\right) = 0.8025188$$

And we calculate:

$$P_U(2) = 0.879073$$
$$P_D(2) = 0.906496$$
$$P_U(3) = 0.759125$$
$$P_D(3) = 0.806791$$
$$P_U(4) = 0.657237$$
$$P_D(4) = 0.716213$$
$$P_U(5) = 0.564908$$
$$P_D(5) = 0.629164$$

Since we have G_U(1, 1) = 1, and G_D(1, -1) = 1 we can calculate U(1) and σ(1) using Newton-Raphson. The values are:

$$U(1) = 0.119119 \text{ and } \sigma(1) = 0.143949$$

We also calculate r(1, –1), r(1, 1), p(1, –1) and p(1, 1) from

$$r(m,\ j) = U(m)e^{\sigma(m)j\sqrt{\Delta t(m)}}$$

and $p(m,\ j) = \frac{1}{[1+r(m,j)]^{\Delta t(m)}}$

The values are:

$$r(1,-1) = 0.103149$$
$$p(1,-1) = 0.906496$$
$$r(1,1) = 0.137561$$
$$p(1,1) = 0.879073$$

Step 2: Generate the Green functions G_U(2, 2), G_U(2, 0), G_D(2, 0) and G_D(2, -2)

$$\begin{cases} G_U(m,\ j) = \frac{1}{2}[p(m-1,\ j-1)G_U(m-1,\ j-1) + p(m-1,\ j+1)G_U(m-1,\ j+1)] \\ G_D(m,\ j) = \frac{1}{2}[p(m-1,\ j-1)G_D(m-1,\ j-1) + p(m-1,\ j+1)G_D(m-1,\ j+1)] \end{cases}$$

$$\begin{cases} G_U(2,\ 2) = \dfrac{1}{2}[p(1,1)G_U(1,1)] = \dfrac{1}{2}[0.879073 \cdot 1] & = 0.439537 \\ G_U(2,\ 0) = \dfrac{1}{2}[p(1,1)G_U(1,1)] = \dfrac{1}{2}[0.879073 \cdot 1] & = 0.439537 \\ G_D(2,\ 0) = \dfrac{1}{2}[p(1,-1)G_U(1,-1)] = \dfrac{1}{2}[0.906496 \cdot 1] & = 0.453248 \\ G_D(2,\ -2) = \dfrac{1}{2}[p(1,-1)G_D(1,-1)] = \dfrac{1}{2}[0.906496 \cdot 1] & = 0.453248 \end{cases}$$

where we omit indices "out of range".

Step 3: Using the derived discount functions $P_U(3)$ and $P_D(3)$ and the previous Green functions to solve $U(2)$ and $\sigma(2)$

$$U(2) = 0.138944, \sigma(2) = 0.123072$$

Step 4: From the calculated values of $U(2)$ and $\sigma(2)$ calculate the short rate, and one-period discount factors for all nodes j at time m:

$$r(2,-2) = 0.108628$$

$$p(2,-2) = 0.902016$$

$$r(2,0) = 0.138944$$

$$p(2,0) = 0.878006$$

$$r(2,2) = 0.177721$$

$$p(2,2) = 0.849097$$

Step 2: Generate the Green functions $G_U(3,3), G_U(3,1), G_U(3,-1), G_D(3,1), G_D(3,-1)$ and $G_D(3,-3)$ as shown earlier. The result is

$$G_U(3,-1) = 0.192958$$

$$G_U(3,1) = 0.379563$$

$$G_U(3,3) = 0.186605$$

$$G_D(3,1) = 0.198977$$

$$G_D(3,-1) = 0.403396$$

$$G_D(3,-3) = 0.204418$$

Step 3: Using the derived discount functions $P_U(4)$ and $P_D(4)$ and the previous Green functions to solve $U(3)$ and $\sigma(3)$:

$$U(3) = 0.139140, \sigma(3) = 0.102192$$

Step 4: From the calculated values of $U(2)$ and $\sigma(2)$ calculate the short rate, and one-period discount factors for all nodes j at time m:

$$
\begin{aligned}
r(3,-3) &= 0.102402\\
p(3,-3) &= 0.907110\\
r(3,-1) &= 0.125624\\
p(3,-1) &= 0.888396\\
r(3,1) &= 0.154111\\
p(3,1) &= 0.866468\\
r(3,3) &= 0.189059\\
p(3,3) &= 0.841001 \text{ etc.}
\end{aligned}
$$

The final interest rate tree is shown next.

				20.937
			18.906	
		17.772		17.705
	13.756		15.411	
10.000		13.894		14.972
	10.315		12.562	
		10.863		12.661
			10.240	
				10.706
Volatility				
15.000	14.395	12.307	10.219	8.384

In C/C++, a tree calibrated to yield and volatility can be written as:

```
/* ------------------------------------------------------------------
    Build a Black-Derman-Toy tree calibrated to the interest rate
    and volatility
    Input:  yc  - a vector with short interest rates at the nodes
                  to build the tree
            vc  - a vector with short IR volatilities at the nodes
                  to build the tree
            N   - the number of nodes in the vectors
            dt  - the time intervals between the nodes

    Output: d   - the discount tree
            r   - the tree of forward rates
                  i = 0 -> i < N
                    j = -i -> j ≤ i (j+= 2
```

```
                      r[i][N+j], d[i][N+j]
            vol - avector with forward volatilities
--------------------------------------------------------------- */
double buildBDT2(double *yc, double *vc, int N, double *dt,
                 double **d, double **r,
                 double *vol)
{
  double **Qu, **Qd; // State securities = 1 if (i,j) is reached, 0
                     // if not
  double *U;         // Median of the (lognormal) distribution for
                     // r at time t
  double *P, *Pu, *Pd;     // Bond prices

  const double epsilon = 0.000001; // Error in N-R iterations
  const int    maxit   = 20;       // Max number of iterations
  int    iter;
  double error, error1, f_Pd, f_Pu, df_PuU, df_PdU, df_PuS, df_PdS;
  double Sigss, sigss, Disk, Disk2, Time = 0.0;
  double D, T, guess, Guess, f_Der, Gamma, p00;

  U  = (double *)calloc(N + 2,    sizeof(double));
  P  = (double *)calloc(N + 2,    sizeof(double));
  Pu = (double *)calloc(N + 2,    sizeof(double));
  Pd = (double *)calloc(N + 2,    sizeof(double));
  Qu = (double **)calloc(2*N + 1, sizeof(double));
  Qd = (double **)calloc(2*N + 1, sizeof(double));

  for (int i = 0; i < 2*N + 1; i++) {
    Qu[i] = (double *)calloc(2*N + 1, sizeof(double));
    Qd[i] = (double *)calloc(2*N + 1, sizeof(double));
  }

  Time = 0.0;

  for (int j = 0; j < N; j++) {
    Time += dt[j];
    P[j+1] = 1.0/pow(1.0 + yc[j], Time);
  }

  //initialize nodes
  U[0]       = yc[0];
  r[0][N]    = yc[0];
  d[0][N]    = 1.0/pow(1.0 + r[0][N], dt[0]);
  vol[0]     = vc[0];
  Qu[1][N+1] = 1;      // N is used as "mid point" i.e., Qu[1][1]
                       // => Qu[1][N+1]
  Qd[1][N-1] = 1;      // and Qd[1][-11] => Qd[1][N-1] etc...
  Time = dt[0];        // This is the first (time) node in the tree.
  // compute Pu[.] and Pd[.]
  for (int i = 1; i < N+1; i++) {
    // solve the following for Pu[i]
    T     = Time;          // Previous time
    Time += dt[i];         //Now
    error = 0; iter = 0;   //Used to exit this loop
    guess = U[0];          //Initial guess of Pu[i]
    Gamma = exp(-2.0*vc[i-1]*sqrt(dt[i]));
    p00   = 1.0/pow(1.0 + r[0][N], dt[0]);
```

```
  do {
    f_Pu = guess + pow(1.0 - Gamma
          + Gamma*pow(guess, -1.0/T), -T) -2.0*P[i]/p00;
    f_Der = 1.0 + Gamma*pow(guess, -1.0/T - 1.0)*
            pow(1.0 - Gamma + Gamma*pow(guess, -1.0/T), -(1.0 + T));
    Guess = guess - f_Pu/f_Der;
    error = fabs(Guess - guess);
    guess = Guess;
    iter++;
    if (iter > maxit) break;
  }
  while (error > epsilon);

  Pu[i] = guess;
  Pd[i] = 2.0*P[i]/p00 - Pu[i];
}

// Evolve tree for the short rate
for (int i = 1; i < N; i++) {
  // Update pure security prices at time step i
  Qu[1][N+1] = 1;      // N is usedas "mid point" i.e., Qu[1][1]
                       //=> Qu[1][N+1]
  Qd[1][N-1] = 1;      //and Qd[1][-1] => Qd[1][N-1] etc...

  if (i > 1) {
    for (int j = -i+2; j ≤ i; j += 2) {
      Qu[i][N+j] = 0.5*Qu[i-1][N+j-1]*d[i-1][N+j-1]
                 + 0.5*Qu[i-1][N+j+1]*d[i-1][N+j+1];
    }

    for (int j = i-2; j ≥ -i; j -= 2) {
      Qd[i][N+j] = 0.5*Qd[i-1][N+j-1]*d[i-1][N+j-1]
                 + 0.5*Qd[i-1][N+j+1]*d[i-1][N+j+1];
    }
  }
  // Solve simultaneously U[i] and sig[i] using a 2-dim. Newton
  // initial guess
  guess = U[i-1];
  sigss = vc[i-1];
  iter  = 0;
  do {
    f_Pu = 0; f_Pd = 0; df_PuU = 0;
    df_PdU = 0; df_PuS = 0; df_PdS = 0;

    for (int j=-i+2; j ≤ i; j+=2) {
      Gamma = exp(sigss*j*sqrt(dt[i]));
     Disk     = pow(1.0 + guess*Gamma, dt[i]);
     Disk2   = pow(1.0 + guess*Gamma, dt[i] + 1.0);
      f_Pu   += Qu[i][N+j]/Disk;
     df_PuU -= Qu[i][N+j]*Gamma*dt[i]/Disk2;
     df_PuS -= Qu[i][N+j]*guess*j*dt[i]*sqrt(dt[i])*Gamma/Disk2;
    }
    f_Pu -= Pu[i+1];

    for (int j=-i; j ≤ i-2; j+=2) {
```

```
        Gamma = exp(sigss*j*sqrt(dt[i]));
       Disk     = pow(1.0 + guess*Gamma, dt[i]);
       Disk2   = pow(1.0 + guess*Gamma, dt[i] + 1.0);
       f_Pd   += Qd[i][N+j]/Disk;
       df_PdU -= Qd[i][N+j]*Gamma*dt[i]/Disk2;
       df_PdS -= Qd[i][N+j]*guess*j*dt[i]*sqrt(dt[i])*Gamma/Disk2;
      }
      f_Pd -= Pd[i+1];

      D       = df_PuU*df_PdS - df_PdU*df_PuS;
      Guess   = guess - (df_PdS*f_Pu - df_PuS*f_Pd)/D;
      error   = fabs(Guess - guess);
      guess   = Guess;

      Sigss   = sigss - (-df_PdU*f_Pu + df_PuU*f_Pd)/D;
      error1 = fabs(Sigss - sigss);
      sigss   = Sigss;
      iter++;
      if (iter > maxit) }
        printf("=========== WARNING! NO SOLUTION ============\n");
       break;
      }
    } while ((error > epsilon) && (error1 > epsilon));
    U[i]   = guess;
    vol[i] = sigss;

    // set r[.] and d[.]
    for (int j = -i; j ≤ i; j += 2) {
      r[i][N+j] = U[i]*exp(vol[i]*j*sqrt(dt[i]));
      d[i][N+j] = 1.0/pow(1.0 + r[i][N+j], dt[i]);
    }
  }
  return 0.0;
}
```

15.1.6 The Black–Karasinski Model

The BK model is developed in a perfect market environment. It assumes that the forward short rate developed by the model follows a lognormal distribution where the instantaneous spot rate evolves under the risk neutral measure, Q, according to the SDE

$$d\ln(r(t)) = [\theta(t) - a(t)\cdot\ln(r(t))]\,dt + \sigma(t)\,dV(t)$$

where $\theta(t)$, $a(t)$ and $\sigma(t)$ are deterministic functions of time and $r(0) = r_0$.

We let the $a(t) = a$ and $\sigma(t) = \sigma$ thus leaving $\theta(t)$ being the only parameter that changes in time. This allows us to use $\theta(t)$ to fit our model perfectly to the current term structure. The other two parameters, a and σ, will be used to calibrate the model to vanilla instruments, whose prices are observed in the market. The coefficient a can be interpreted as the rate at which the model reverts to a long-term mean.

From Itô lemma, we obtain

$$d(r(t)) = r(t)\left[\theta(t) + \frac{\sigma^2}{2} - a \cdot \ln(r(t))\right] dt + \sigma r(t)\, dV(t)$$

with the solution ($s \leq t$)

$$r(t) = \exp\left\{\ln(r(t)) \cdot e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma \int_s^t e^{-a(t-u)}dV(u)\right\}$$

The first moment of $r(t)$, with respect to the filtration $\mathcal{F}_s$, is given by,

$$E\left[r(t)|\mathcal{F}_s\right] = \exp\left\{\ln(r(t)) \cdot e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \frac{\sigma^2}{4a}\left[1 - e^{-2a(t-s)}\right]\right\}$$

Let

$$\alpha(t) = \ln(r_0) \cdot e^{-at} + \int_0^t e^{-a(t-u)}\theta(u)du$$

The long-term mean of the short rate cannot be calculated analytically. A numerical procedure such as the trinomial lattice can be implemented to derive a short-rate tree that matches the initial term structure.

Given the short-rate dynamics of the model, we can write the short rate as a function of time,

$$r(t) = e^{\alpha(t)+x(t)}$$

where the stochastic differential of the x process is given by,

$$dx(t) = -a \cdot x(t)dt + \sigma dV(t)$$

where $x(0) = 0$. We can integrate the SDE in order to obtain a formula for the process x,

$$x(t) = x(s) \cdot e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dV(u) \text{ for each } s < t.$$

We begin by implementing the trinomial tree for x by discretising the time horizon, where $\Delta t_i = t_{i+1} - t_i$ for each i. The process for $x$will evolve according to the trinomial tree where $x_{i;j} = j\Delta t_i$ is the value of the process at time t_i for the j-th node. From this node, the process can take on one of three values, $x_{i+1;k+1}$, $x_{i+1;k}$ or $x_{i+1;k-1}$, where $x_{i+1;k}$ is the central node. The level of k is set such that $x_{i+1;k}$ is as close as possible to $M_{i;j}$. The maximum and minimum number of nodes at each time step, t_i, is denoted by $\bar{j}$ and $\underline{j}$ respectively. $\bar{j}$ and $\underline{j}$ at time t_{i+1} can be determined by finding the values of k for the nodes $x_{i,\bar{j}}$ and $x_{i,\underline{j}}$ respectively.

The values of the nodes at each time must be calculated in an iterative manner, starting at the current time and working as far out into future as desired.

Once the $x$process for the whole tree is generated, the tree needs to be displaced in order for the model to match the current term structure. Since the BK model assumes that the instantaneous forward short rate is lognormal distributed, the model is not analytical tractable. This implies that one cannot solve αanalytically. In order to overcome this, a numerical procedure must be used to generate the value of _ at each time step. We denote $Q_{i;j}$ as the present value of an instrument paying 1 if node $(i,\ j)$ is reached and 0 otherwise. $Q_{i;j}$ can be thought of as the discrete analogue of the Arrow-Debreu security prices. We begin by calculating the current value of α, α_0. This is given by

$$\alpha_0 = \ln \left\{ -\ln \left(\frac{p(0,\ t_1)}{t_1} \right) \right\}$$

where $p(0,\ t_1)$ is the market discount factor for the maturity t_1. Once α_0 is computed, $Q_{1,j}$ can be calculated for all $j$by,

$$Q_{1,j} = \sum_h Q_{0,h} q(h,\ j) \cdot \exp\{-\Delta t_0 \cdot \exp(\alpha_0 + h\Delta x_0)\}$$

where $q(h,j)$ is the probability of moving from node (i, h) to node $(i + 1, j)$. Aftercalculating $Q_{1,j}$ for all j = $\bar{j}_1, \ldots, \underline{j}_1$ we can calculate α_1 by matching it to the marketdiscount factor for the maturity t_2. This can be calculated by numerically solvingthe equation,

$$\phi(\alpha_1) = p(0, t_2) - \sum_{\bar{j}_1}^{\underline{j}_1} Q_{1,j} \cdot \exp\{-\exp(\alpha_1 + j\Delta x_1)\,\Delta t_1\} = 0$$

This can be solved using the Newton-Raphson. Thus, in general we have,

$$Q_{1+1,j} = \sum_h Q_{i,h} q(h, j) \cdot \exp\{-\Delta t_i \cdot \exp(\alpha_i + h\Delta x_i)\}$$

where α_i is calculated numerically using

$$\phi(\alpha_i) = p(0, t_{i+1}) - \sum_{\bar{j}_i}^{\underline{j}_i} Q_{i,j} \cdot \exp\{-\exp(\alpha_i + j\Delta x_i)\,\Delta t_i\} = 0$$

The first and the second derivative of $\phi(\alpha_i)$ is given by:

$$\phi'(\alpha_i) = \sum_{\bar{j}_i}^{\underline{j}_i} Q_{i,j} \cdot \exp\{-\exp(\alpha_i + j\Delta x_i)\,\Delta t_i\}\ \exp(\alpha_i + j\Delta x_i)\,\Delta t_i$$

and

$$\phi''(\alpha_i) = \phi'(\alpha_i) - \phi'(\alpha_i) \cdot \exp(\alpha_i + j\Delta x_i)\,\Delta t_i$$

As we can see from the previous equation, $\alpha$is time dependent and not state dependent. Thus in order to obtain the value of the forward short rate at each node (i, j), we take $x_{i,j}$ and calculate $r_{i,j}$ using the formula,

$$r_{i,j} = e^{\alpha_i + x_{i,j}}$$

The BK model does not yield any analytical formulae to price a zero-coupon bond or any European bond option. Therefore, the only way to price interest rate derivatives, both vanilla and exotic, is through numerical procedures such as a lattice approximation or Monte Carlo simulation. However, by assuming that interest rates are lognormal distributed, it prevents the forward short rate from becoming negative. These negative rates could skew the price of a derivative that is priced using a normally distributed short-rate tree.

15.1.7 Two-Factor Models

The previously defined one-factor models assume is that at every instant, interest rates for all maturities are perfectly correlated with each other. For example, a long interest rate, such as a three year rate, is perfectly correlated with a short interest rate. The perfect correlation between interest rates along a yield curve implies that changes to the interest rate in the short end will be equally transmitted along the yield curve to the long end. In order to incorporate correlation between yields of different maturities, one can introduce more factors to improve the description of the evolution of the short rate.

A two-factor model will imply more precision in the modelling but at a higher cost of implementation. A common assumption amongst all multiple factor models is that the market must be complete; there must be as many tradable assets in the market as there are sources of uncertainty.

Brennan and Swartz model is an early (1982) and well-known two-factor model that is mainly used when pricing options on bonds. Instead of focusing solely on the short rate, this model incorporates the longest and shortest maturity default-free instruments. Brennan and Swartz goes on to assume that the two yields follow a Gaussian process and that each yield is driven by its own source of uncertainty. This model is believed to improve the precision of pricing derivatives that depend on the behaviour of both a short and long-dated instrument. For example, an option that allows one to swap a short-dated bond for a long-dated bond. The processes are given by

$$dr = [a_1 + b_1(\rho - r)]\, dt + \sigma_1 r dz$$

$$d\rho = \rho \cdot [a_2 + b_2 r + c_2 \rho]\, dt + \sigma_2 \rho dw$$

where $dwdz = \alpha dt$. σ_1 and σ_1 are the volatilities for the short-rate r and the long rate ρ. α is the correlation between the two rates.

Longstaff and Schwartz used the general equilibrium framework of Cox, Ingersoll and Ross with an extension by assuming two independent unspecified state variables which follow stochastic processes of the form

$$dx = a_x(b_x - x)dt + \sigma_x \sqrt{x} dz$$

$$dy = a_y(b_y - y)dt + \sigma_y \sqrt{y} dz$$

Both factors assumes to affect the mean of the instantaneous rate of return, but only the last factor is assumed to affect the variance. Therefore, risk is only priced using the second factor. By using the term-structure equation it is possible to determine the short-rate r and its instantaneous variance V as part of the equilibrium

$$r = x + y$$
$$V = \sigma_x^2 x + \sigma_y^2 y$$

Now, it is possible to change variables and express the valuation equation in terms of the new and observable variables r and V.

15.1.7.1 The Two-Factor HW Model

Hull and White proposed a two-factor model of a single term structure. This model provides a more accurate estimation for the changes in the term structure and volatility structure than their one-factor model. HW validates the use of this two-factor model for volatility structures that are humped in nature.

In the HW two-factor model, the risk-neutral process for the short rate, r, is

$$df(r) = \left[\theta(t) + u - af(r)\right] dt + \sigma_1 dz_1$$

where u has an initial value of zero and follows the process

$$du = -budt + \sigma_2 dz_2$$

The parameter $\theta(t)$ is a deterministic function of time. The stochastic variable u is a component of the reversion level of $r$and itself reverts to a level of zero at rate b. The parameters a, b, σ_1, and σ_2 are constants and dz_1 and dz_2 are Wiener processes with instantaneous correlation ρ.

This model provides a richer pattern of term-structure movements and a richer pattern of volatility structures than the one-factor model. For example, when $f(r) = r$, $a = 1$, $b = 0.1$, $\sigma_1 = 0.01$, $\sigma_2 = 0.0165$ and $\rho = 0.6$ the model exhibits, at all times, a "humped" volatility structure similar to that observed in the market.

When $f(r) = r$ the model is analytically tractable. The price at time t of a zero-coupon bond that provides a payoff of 1 CU at time T is

$$P(t, T) = A(t, T) \exp\left[-B(t, T) r - C(t, T) u\right]$$

where

$$B(t,\ T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

$$C(t,\ T) = \frac{1}{a(a-b)}e^{-a(T-t)} - \frac{1}{b(a-b)}e^{-b(T-t)} + \frac{1}{ab}$$

and $A(t,\ T)$is as given in the following.

The prices, c and p, at time zero of European call and put options on a zero-coupon bond are given by

$$c = LP(0,\ s)N(h) - KP(0,\ T)N\left(h - \sigma_p\right)$$

$$p = KP(0,\ T)N\left(-h - \sigma_p\right) - LP(0,\ s)N(-h)$$

where T is the maturity of the option, s is the maturity of the bond, K is the strike price and L is the bond's principal

$$h = \frac{1}{\sigma_p}\ln\frac{LP(0,\ s)}{P(0,\ T)K} + \frac{\sigma_p}{2}$$

and σ_p is as given in the following.

The $A(t,\ T)$, σ_p, and $\theta(t)$ Functions in the Two-Factor HW Model

In this part, we provide some of the analytic results for the two-factor HW model when $f(r) = r$.

The $A(t,\ T)$ function is

$$\ln A(t,\ T) = \ln\frac{P(0,\ T)}{P(0,\ t)} + B(t,\ T)F(0,\ t) - \eta$$

where

$$\eta = \frac{\sigma_1^2}{4a}(1 - e^{-2at})B(t,\ T)^2 - \rho\sigma_1\sigma_2[B(0,\ t)C(0,\ t)B(t,\ T) + \gamma_4 - \gamma_2]$$
$$-\frac{1}{2}\sigma_2^2[C(0,\ t)^2B(t,\ T) + \gamma_6 - \gamma_5]$$

and

$$\gamma_1 = \frac{e^{-(a+b)T}(e^{(a+b)t} - 1)}{(a+b)(a-b)} - \frac{e^{-2aT}(e^{2\,at} - 1)}{2a(a-b)}$$

$$\gamma_2 = \frac{1}{ab}(\gamma_1 + C(t,\, T) - C(0,\, T) + \frac{1}{2}B(t,\, T)^2 - \frac{1}{2}B(0,\, T)^2$$

$$+ \frac{t}{a} - \frac{e^{-a(T-t)} - e^{-aT}}{a^2})$$

$$\gamma_3 = -\frac{e^{-(a+b)t} - 1}{(a-b)(a+b)} + \frac{e^{-2at} - 1}{2a(a-b)}$$

$$\gamma_4 = \frac{1}{ab}(\gamma_3 - C(0,\, t) - \frac{1}{2}B(0, t)^2 + \frac{t}{a} + \frac{e^{-at} - 1}{a^2})$$

$$\gamma_5 = \frac{1}{b}[\frac{1}{2}C(t,\, T)^2 - \frac{1}{2}C(0,\, T)^2 + \gamma_2]$$

$$\gamma_6 = \frac{1}{b}[\gamma_4 - \frac{1}{2}C(0,\, t)^2]$$

where $B(t,\, T)$ and $C(t,\, T)$ functions are as we mentioned before and $F(t,\, T)$ is the instantaneous forward rate at time t for maturity T.

The volatility function, σ_p, is

$$\sigma_p^2 = \int_0^t \begin{aligned} &\{\sigma_1^2[B(\tau,\, T) - B(\tau,\, t)]^2 + \sigma_2^2[C(\tau,\, T) - C(\tau,\, t)]^2 \\ &+2\rho\sigma_1\sigma_2[B(\tau,\, T) - B(\tau,\, t)][C(\tau,\, T) - C(\tau,\, t)]\}d\tau \end{aligned}$$

This shows that σ_p^2 has three components. Define

$$U = \frac{1}{a(a-b)}(e^{-aT} - e^{-at})$$
$$V = \frac{1}{b(a-b)}(e^{-bT} - e^{-bt})$$

The first component of σ_p^2 is

$$\frac{\sigma_1^2}{2a}B(t,\, T)^2(1 - e^{-2at})$$

the second

$$\sigma_2^2(\frac{U^2}{2a}(e^{2at}-1)+\frac{V^2}{2b}(e^{2bt}-1)-2\frac{UV}{a+b}(e^{(a+b)t}-1))$$

and the third

$$\frac{2\rho\sigma_1\sigma_2}{a}(e^{-at}-e^{-aT})(\frac{U}{2a}(e^{2at}-1)-\frac{V}{a+b}(e^{(a+b)t}-1))$$

Finally, the $\theta(t)$ function is

$$\theta(t)=F_t(0,\,t)+aF(0,\,t)+\phi_t(0,\,t)+a\phi(0,\,t)$$

where the subscript denotes a partial derivative and

$$\phi(t,\,T)=\frac{1}{2}\sigma_1^2B(t,\,T)^2+\frac{1}{2}\sigma_2^2C(t,\,T)^2+\rho\sigma_1\sigma_2B(t,\,T)C(t,\,T)$$

15.1.8 Three-Factor Models

In recent years, researches have come up with some yield-based term-structure models which specify three factors driving the future from the term structure. Such models are assumed to follow stochastic processes which can take on different forms. For example

$$\begin{aligned}dr&=\kappa(\theta-r)dt+\sqrt{V}dz\\d\theta&=\alpha(\beta-\theta)dt+\eta dw\\dV&=a(b-V)dt+\phi\sqrt{V}dy\end{aligned}$$

where r is the short rate, θ denotes the long run mean of r and V the variance of the short rate. This set of factors are designed to give the term-structure evolution more flexibility in that it allows not only for parallel shifts, but also for twists and not perfectly correlated bond prices. This advantage comes at the price of higher computational demands and theoretical sophistication.

15.1.9 Fitting Yield Curves with Maximum Smoothness

Single-factor term-structure models, such as HW, can be used to fit yield curves and forward rate curves with maximum smoothness. Such a method will generally match the observable yield-curve data very well but between observable data points, yield-curve smoothing technique is necessary. Kenneth J. Adams and Donald R. Van Deventer provide an approach to yield-curve fitting by introducing the "maximum smoothness criterion".

The objective is to fit observable points on the yield curve with the function of time that produces the smoothest possible forward rate curve. To do this, a technique from numerical analyses is used. The smoothest possible forward rate curve on an interval (0, T) is defined as one that minimizes the functional

$$Z = \int_0^T \left[f''(0, s)\right]^2 ds$$

subject to

$$\exp\left\{-\int_0^{t_i} f(0, s)ds\right\} = P(0, t_i) i = 1, 2, \ldots, m$$

where $P(0, t_i)$ represent the observed prices of zero-coupon bonds with maturities t_i. Expressing the forward rate curve as a function of a specified form with a finite number of parameters may approach this problem. The maximum smoothness term structure can then be found within this parametric family, that is, it will be smoother than that given by any other mathematical expression of the same degree and same functional form.

However, it would be more useful to determine the maximum smoothness term structure within all possible functional forms. This is possible due to the theorem provided by Oldrich Vasicek and stated in an article by Adams and Van Deventer in The Journal of fixed income, pp. 53–62. June 1994.

Theorem 15.1.8. *The term-structure* $f(0, t)$, $0 \leq t \leq T$ *of forward rates that satisfies the previous equations is a fourth order spline with the cubic term absent given by*

$$f(0, t) = c_i t^4 + b_i t + a_i \quad t_{i-1} \leq t < t_i \quad i = 1, 2, \ldots, m+1$$

where the maturities satisfy $0 = t_0 < t_1 < \ldots < t_{m+1} < T$. *The coefficients* a_i, b_i, c_i, *satisfy the equations*

$$c_i t_i^4 + b_i t_i + a_i = c_{i+1} t_i^4 + b_{i+1} t_i + a_{i+1}$$

$$4c_i t_i^3 + b_i = 4c_{i+1} t_i^3 + b_{i+1}$$

$$c_{m+1} = 0$$

$$\frac{1}{5} c_i \left(t_i^5 - t_{i-1}^5 \right) + \frac{1}{2} b_i \left(t_i^2 - t_{i-1}^2 \right) + a_i (t_i - t_{i-1}) = -\ln \left\{ \frac{P(0,\ t_i)}{P(0,\ t_{i-1})} \right\}$$

For a proof of this theorem, we refer to the article written by Kenneth J. Adams and Donald R. Van Deventer. It is seen that the theorem specifies $3m + 1$ equations for the $3m + 3$ unknown parameters a_i, b_i, c_i, $i = 1, 2, .., m + 1$. The maximum smoothness solution is unique and can be obtained analytically as follows.

The objective function is proportional to

$$Z = \sum_{i=1}^{m} c_i^2 \left(t_i^5 - t_{i-1}^5 \right)$$

according to the term-structure $f(t)$. This function is quadratic in the parameters while the previously mentioned four conditions are all linear in the parameters. We have an unconstrained quadratic problem of the form

$$\min \mathbf{x}^{\mathbf{T}} \mathbf{D} \mathbf{x}$$

subject to

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

with the solution

$$\left(\mathbf{I} - \mathbf{A}^{\mathbf{T}} \left(\mathbf{A}\mathbf{A}^{\mathbf{T}} \right)^{-1} \mathbf{A} \right) \mathbf{D}\mathbf{x} = \mathbf{0}$$

Any two of these equations provide the remaining conditions on the parameters a_i, b_i, $c_{i.}$ Two additional requirements may be stated

1. $f'(T) = 0$ for the asymptotic behaviour of the term structure.
2. $a_0 = r$ which means that the instantaneous forward rate at time zero is equal to an observable rate r.

If both of the additional requirements are used, no equation from the previous equation is needed.

16

Heath-Jarrow-Morton

16.1 The Heath-Jarrow-Morton (HJM) Framework

Up to this point we have studied interest models where the short-rate r is the only explanatory variable. The main advantages with such models are as follows.

1. Specifying r as the solution of an Stochastic Differential Equation (SDE) allows us to use Markov process theory, so we may work within a partial differential equation (PDE) framework.
2. In particular it is often possible to obtain analytical formulas for bond prices and derivatives.

The main drawbacks of short-rate models are as follows.

1. From an economic point of view it seems unreasonable to assume that the entire money market is governed by only one explanatory variable.
2. It is hard to obtain a realistic volatility structure for the forward rates without introducing a very complicated short-rate model.
3. As the short-rate model becomes more realistic, the inversion of the yield curve described earlier becomes increasingly more difficult.

These, and other considerations, have led various authors to propose models that use more than one state variable. One obvious idea would, for example, be to present an a priori model for the short rate as well as for some long rate, and one could of course also model one or several intermediary interest rates. The method proposed by Heath-Jarrow-Morton (HJM) is at the far end of this spectrum — they choose the entire forward rate curve as their (infinite dimensional) state variable.

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We will now specify the HJM framework, and we start by specifying everything under a given objective measure P.

Assumption: We assume that, for every fixed $T > 0$, the forward rate $f(t,T)$ has a stochastic differential which under the objective measure P is given by

$$\begin{cases} df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW^P(t,T), \\ f(0,T) = f^*(0,T) \end{cases}$$

where W^P is a (d-dimensional) P-Wiener process whereas $\alpha(t,T)$ and $\sigma(t,T)$ is adapted processes.

Note that conceptually this is one stochastic differential in the t-variable for each fixed choice of T. The index T thus only serves as a "mark" or "parameter" in order to indicate which maturity we are looking at. Also note that we use the observed forward rated curve $\{f^*(0,T); T \geq 0\}$ as the initial condition. This will automatically give us a perfect fit between observed and theoretical bond prices at $t = 0$, thus relieving us of the task of inverting the yield curve.

It is important to observe that the HJM approach is not a proposal for a specific model, like for example, the Vasicek model. It is instead a framework to be used for analysing interest rates models. Every short-rate model can be equivalently formulated in forward rate terms, and for every forward rate model, the arbitrage-free price of a contingent T-claim X will still be given by the pricing formula

$$\Pi\left[0,X\right] = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}.X\right]$$

where the spot rate is as usual given by $r(t) = f(t,t)$.

Suppose now that we have specified α, σ and $\{f^*(0,T\}; T \geq 0\}$. Then we have specified the entire forward rate structure and thus, by the relation

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$$

we have in fact specified the entire term-structure $\{p(t,T); T > 0, 0 \leq t \leq T\}$. Since we have d sources of randomness (one for every Wiener process), and an infinite number of traded assets (one bond for each maturity T), we run a clear risk of having introduced arbitrage possibilities into the bond market. The first question we pose is thus very natural: How must the processes α and σ be related in order that the induced system of bond prices admits no-arbitrage possibilities? The answer is given by the HJM drift condition that relates α to σ.

We start as usual with the assumptions that there exists a local risk-free security with the price process B given by

$$\begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1 \end{cases}$$

where the spot rate is given by $r(t) = f(t,t)$.

We also assume that an equivalent probability measure $Q \sim P$ such as each Z^T-process is a Q-martingale on $[0,T]$, where the discounted bond prices Z^T is defined as

$$Z^T(t) = \frac{p(t,T)}{B(t)}$$

We also know that the dynamic of the forward rates imply the following dynamic for the bond prices:

$$dp(t,T) = p(t,T)\{r(t) + b(t,T)\}\,dt + p(t,T)a(t,T)dW(t)$$

where

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

and $a(t,T)$ and $b(t,T)$ are given by:

$$\begin{cases} a(t,T) = -\displaystyle\int_t^T \sigma(t,u)du \\ b(t,T) = -\displaystyle\int_t^T \alpha(t,u)du + \frac{1}{2}a(t,T)^2 \end{cases}$$

The dynamic of Z^T is then given by:

$$dZ(t,T) = b(t,T)Z(t,T)dt + a(t,T)Z(t,T)dW(t)$$

Therefore we have to find out if there exists a Girsanov transformation that for all T at the same time removes the drift.

$$dQ = L(T)dP \quad \text{on } \mathcal{F}_T$$

where

$$\begin{cases} dL(t) = g(t)L(t)dW(t) \\ L(0) = 1 \end{cases}$$

for some process $g(t)$. From Girsanov theorem we get

$$dW(t) = g(t)dt + dV(t)$$

where V is a Q-Wiener process. We then have

$$\begin{aligned}
dZ(t,T) &= \frac{\partial Z(t,T)}{\partial p(t,T)}dp(t,T) + \frac{\partial Z(t,T)}{\partial B(t)}dB(t) = \frac{1}{B(t)}dp(t,T) - \frac{p(t,T)}{B^2(t)}dB(t) \\
&= Z(t,T)\left\{r(t) + b(t,T)\right\}dt + Z(t,T)a(t,T)dW(t) - Z(t,T)r(t)dt \\
&= b(t,T)Z(t,T)dt + a(t,T)Z(t,T)dW(t) \\
&= \left\{b(t,T) + g(t)a(t,T)\right\}Z(t,T)dt + a(t,T)Z(t,T)dv(t)
\end{aligned}$$

We must have

$$g(t,T) = -\frac{b(t,T)}{a(t,T)}$$

This Girsanov kernel, $g(t,T)$, holds for a given T. Therefore a martingale measure Q^T will be generated such as Z^T becomes martingale. Remark! This depends on our choice of T, so there is no guarantee that Q^S for $S \neq T$ is Q^T-martingale. If there exists a Girsanov transformation that make all Z^T-processes martingale at the same time, then $g(t,T)$ must be independent of the choice of T.

Theorem 16.1. *The following statements are equivalent*

- *There exists a measure Q^T that makes all Z^T processes martingales.*
- *For all T and S we have*

  $$\frac{b(t,T)}{a(t,T)} = \frac{b(t,S)}{a(t,S)}$$

  for all $t \leq min(T,S)$.
- *The Girsanov kernel $g(t,T)$ is independent of T.*
- *For each S and T we have*

  $$\alpha(t,T) = -\sigma(t,T)\left\{g(t,S) - \int_t^T \sigma(t,s)ds\right\}$$

Proof: We only prove the last statement since the others are obvious. We have

$$g(t,S) = -\frac{b(t,T)}{a(t,T)}$$

which in detail gives

$$g(t,S)a(t,T) = -b(t,T)$$

$$g(t,S)\int_t^T \sigma(t,u)du = -\int_t^T \alpha(t,u)du + \frac{1}{2}a(t,T)^2$$

$$g(t,S)\int_t^T \sigma(t,s)ds = -\int_t^T \alpha(t,s)ds + \frac{1}{2}\left\{\int_t^T \sigma(t,s)ds\right\}^2$$

so

$$\int_t^T \alpha(t,s)ds = -g(t,S)\int_t^T \sigma(t,s)ds + \frac{1}{2}\left\{\int_t^T \sigma(t,s)ds\right\}^2$$

Take the derivative with respect to T and we are finished.

Theorem 16.2. *If one of the previous statements holds, then the market is free of arbitrage.*

It is natural to call the function $g(t,T)$, the market price of risk. We remember that on a market free of arbitrage, the market price of risk is the same for all securities.

Suppose one of the aforementioned statements hold. Then we can define a unique measure Q that makes all discounted bond prices martingales. The question we ask us is, how does the forward process look like under this measure? The answer is surprisingly simple.

Theorem 16.3. *Let the forward dynamic under P be given by:*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

Then, if any of the previous statements holds, the forward rates under Q are given by

$$df(t,T) = \alpha^*(t,T)dt + \sigma(t,T)dV(t)$$

where

$$\alpha^*(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds$$

Proof: After Girsanov transformation, the Q-dynamic is given by

$$df(t,T) = \{\alpha(t,T) + g(t)\sigma(t,T)\}\,dt + \sigma(t,T)dV(t)$$

Using

$$\alpha(t,T) = -\sigma(t,T)\left\{g(t,S) - \int_t^T \sigma(t,s)ds\right\}$$

$\Rightarrow$

$$\begin{aligned} df(t,T) &= \left\{-\sigma(t,T)\left\{g(t) - \int_t^T \sigma(t,s)ds\right\} + g(t)\sigma(t,T)\right\} dt + \sigma(t,T)dV(t) \\ &= \left\{\sigma(t,T)\int_t^T \sigma(t,s)ds\right\} dt + \sigma(t,T)dV(t) = \alpha^*(t,T)dt + \sigma(t,T)dV(t) \end{aligned}$$

The result is a little bit surprising, since the Q-dynamic is completely determined by the diffusion function $\sigma(t,T)$. Therefore, if the process $\sigma(t,T)$ is deterministic, the forward rates are independent of the market price of risk. This is also true in a more complex situation where the forward rate solves a system of SDEs.

If we remember what we found when we went from df^T to dp^T and if we use the super index * in the drift and diffusion under Q we have:

$$df(t,T) = \alpha^*(t,T)dt + \sigma^*(t,T)dV(t)$$
$$dp(t,T) = p(t,T)\left\{r(t) + b^*(t,T)\right\} dt + p(t,T)a^*(t,T)dW(t)$$
$$\begin{cases} a^*(t,T) = -\displaystyle\int_t^T \sigma^*(t,u)du \\ b^*(t,T) = -\displaystyle\int_t^T \alpha^*(t,u)du + \frac{1}{2}a^*(t,T)^2 \end{cases}$$

We know that a Girsanov transformation does not change the drift, so we must have $\sigma^*(t,T) = \sigma(t,T)$. Then we know that under Q the yield is given by the short rate. Therefore we must have $b^*(t,T) = 0$.

16.1.1 The HJM Program

To use the HJM framework we will use the following program:

1. Fix a filtrated probability space $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ and a Wiener process V. The filtration is the natural, generated by V.
2. Specify the choice of volatility structure for the forward rates for each $T > 0$, explicitly giving the process $\sigma(t, T)$.
3. Define the drift of the forward rates by

$$\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, s) ds$$

4. Observe on the market, the initial forward structure $\{f^*(0, T); T > 0\}$.
5. Integrate the forward rates with the equations

$$f(t, T) = f^*(0, T) + \int_0^t \alpha(u, T) du + \int_0^t \sigma(u, T) dV(u)$$

6. Calculate the bond prices as

$$p(t, T) = exp \left\{ - \int_t^T f(t, u) du \right\}$$

7. Calculate derivative prices based on $p(t, T)$.

16.1.1.1 Ho-Lee Model

To see how this works, we use the simplest we can think of, a constant volatility $\sigma(t, T) = \sigma > 0$. If we use the HJM equation

$$df(t, T) = \alpha^*(t, T) dt + \sigma(t, T) dV(t)$$

where

$$\alpha^*(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds$$

We then get

$$df(t,T) = \left(\sigma \int_t^T \sigma ds\right) dt + \sigma dV(t) = \sigma^2(T-t)dt + \sigma dV(t)$$

We see that the drift is then given by: $\alpha(t,T) = \sigma^2(T-t)$. We then get

$$f(t,T) = f^*(0,T) + \int_0^t \sigma^2(T-u)du + \int_0^t \sigma dV(u) = f^*(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma V(t)$$

We remember that a Wiener process at time $t = 0$ is zero: $V(0) = 0$. We recognize these rates as the one we got from the Ho-Lee model. Remark how easy we get them in the HJM framework. We also get the bond prices

$$p(t,T) = \exp\left\{-\int_t^T f^*(0,u)du - \frac{\sigma^2 Tt}{2}(T-t) - \sigma(T-t)V(t)\right\}$$

i.e.,

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\left\{-\frac{\sigma^2 Tt}{2}(T-t) - \sigma(T-t)V(t)\right\}$$

16.1.2 Hull-White Model

If we use a Gaussian forward rates with volatility given by

$$\sigma(t,T) = \sigma e^{-a(T-t)}$$

We get

$$df(t,T) = \alpha^*(t,T)dt + \sigma(t,T)dV(t)$$

where

$$\alpha^*(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds$$

Therefore

$$df(t,T) = \left(\sigma e^{-a(T-t)}\int_t^T \sigma e^{-a(T-s)}ds\right)dt + \sigma e^{-a(T-t)}dV(t)$$
$$= \frac{\sigma^2}{a}\left(e^{-a(T-t)} - e^{-2a(T-t)}\right)dt + \sigma e^{-a(T-t)}dv(t)$$

Integrating with respect to t, we obtain

$$f(t,T) = f^*(0,T) + \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 - \frac{\sigma^2}{2a^2}\left(1-e^{-a(T-t)}\right)^2 + \sigma\int_0^t e^{-a(T-s)}dV(s)$$

Introducing the notation

$$X(t) = \int_0^t e^{-a(t-s)}dV(s)$$

And using the fact that

$$\int_0^t e^{-a(T-s)}dV(s) = e^{-a(T-t)}X(t)$$

We obtain the following formulas for $f(t,T)$ and $r(t)$

$$f(t,T) = f^*(0,T) + \sigma e^{-a(T-t)}X(t) + \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 - \frac{\sigma^2}{2a^2}\left(1-e^{-a(T-t)}\right)^2$$

and

$$r(t) = f^*(0,t) + \sigma X(t) + \frac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2$$

We recognize these rates as the Hull-White (extended Vasicek) model. The formula shows that the forward rates and the instantaneous short rate are linear functions of the same Gaussian process $X(t)$, so we observe a perfect correlation of the forward rates.

The asymptotical behaviour of the short rate is given by

$$r(t) = f(0,t) + \sigma X(t) + \frac{\sigma^2}{2a^2}$$

which is a Gaussian random variable with mean

$$\mu_\infty(t) = f(0,t) + \frac{\sigma^2}{2a^2}$$

and variance

$$\sigma_\infty^2 = \sigma^2 E\left[\left(\int_0^t e^{-a(t-s)}dV(s)\right)^2\right] = \sigma^2 \int_0^t e^{-2a(t-s)}ds = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

We see that the short-rate fluctuation have a non-trivial asymptotic probability distribution. This fact is known as **mean-reversion** of the spot rate and a is called the **rate of mean reversion**.

16.1.2.1 The General Situation

In a mode general situation we can let the volatility depend on the forward rates and then solve a system of SDEs under Q. In more detail

1. Specify σ as function of three variables: t, T and $f(t,T)$.
2. Solve

$$\begin{cases} df(t,T) = \alpha(t,T)dt + \sigma\,(t,T,f\,(t,T))\,dV(t) \\ f(0,T) = f^*(0,T) \end{cases}$$

where

$$\alpha(t,T) = \sigma(t,T,f(t,T)) \int_t^T \sigma(t,s,f(t,s))ds$$

The question to ask at this point is under what condition on σ we can solve the previous equations. The situation is complex since this is infinite number of coupled equations where $\alpha(t,T)$ at time t not only depends on the actual forward rates, $f(t,T)$, but also all forward rates $f(t,s)$ with t $\leq s \leq T$. But more difficult is the problem with σ. If we do not specify σ well enough, α which is quadratic in σ can explode and give infinite forward rates. This gives bond prices of zero and possible arbitrage situations. If σ is Lipschitz continuous in $f(t,s)$, positive and uniformed limited, then there exists a solution to the system for all initial forward rates.

16.1.3 A Change of Perspective

The main result of the HJM approach consists in providing the extension of the Black and Scholes (1973) reasoning to the fixed income sector using forward rates. This can be done as there exists a one-to-one correspondence between instantaneous forward rates and bond prices. Bonds are traded assets, so we can apply the procedure of replacing the drift coefficient with the short rate under a risk-neutral probability measure. Passing from spot rates to forward rates, thus, allows us to incorporate directly arbitrage restrictions without specifying in advance the market price of risk.

The noticeable fact is that the drift, determined by arbitrage arguments, depends only on the volatility parameters, and this resembles the Black and Scholes (1973) results. In this sense it can be said that the HJM can be considered the true extension of their methodology to the fixed income sector. Up to now it might seem that HJM comes in at no cost, but this is not the case. Switching to forward rates and relying only on volatility calibration, has two main drawbacks: first, under the risk-free measure, the forward rates are biased estimators of the future spot rates; second there may be cases in which the spot rate does not follow a Markov process. This is a somewhat unpleasant feature of the HJM approach because of the heavy computational difficulties arising in non-Markovian contexts.

Jeffrey in a 1995 article derived general conditions on the volatility structures in HJM under which markovness is still retained. To be more specific, he provides necessary and sufficient conditions such that one can determine which volatility structures are allowable in a Markovian spot interest rate context and the set of allowable initial term structures corresponding to a given volatility structure. We will not discuss this here, because this is out of the scope at this point.

Mari (2003) has proposed a perturbative extension of a model for Bond Prices within the affine class. This model is set up with the property of consistency with arbitrary initial term structures. Affine structures possess very interesting properties: first of all they are mathematically very tractable; as a consequence they allow for risk analysis and estimation via closed-form solutions of PDE or via solutions of Ordinary Differential Equation (ODE) of the first order; moreover they can be estimated using maximum likelihood techniques. In the model it is assumed that the discount factor is a smooth function of the spot rate and of maturity T.

Under the risk-neutral measure, the stochastic dynamics of the term structure is given by

$$\begin{cases} \dfrac{dp(t,T,r(t))}{p} = r(t)dt + \sigma_p\,(t,T,r\,(t))\,dw^*(t) \\ p(0,T,r(0)) = p^*(0,T) \end{cases}$$

Mari (2003) proved that the model can be fitted consistently with arbitrary initial term structures and the implied spot rate follows a Markov process if and only the following condition holds

$$\sigma_p(t,T,r(t)) = \sqrt{h(t)+k(t)r(t)}B(t,T)$$

with

$$B(t,T) = 2\frac{C'(t)-A(t)}{k(t)}\left[\frac{1}{C(t)} - \frac{1}{\int_t^T A(u)du + C(t)}\right]$$

and

$$A(t) = \frac{1}{2}\left\{C'(t) - \sqrt{C'^2(t) - 2k(t)C^2(t)}\right\}$$

where $h(t)$, $k(t)$ and $C(t)$ are functions that can be arbitrarily chosen which must satisfy the condition $C'(t)^2 \geq 2k(t)C^2(t)$. Under this condition Mari shows that the solution of the term structure is:

$$p(t,T,r(t)) = \frac{p^*(0,T)}{p^*(0,t)} \exp \left\{ \begin{array}{l} f^*(0,t)B(t,T) - \int_0^t H(u)B(u,T)du + \\ \frac{1}{2} \int_0^t \sigma^2(u,T,f^*(0,t))B^2(u,T)du \end{array} \right\} e^{-r(t)B(t,T)}$$

where $H(t)$ is the solution of the Volterra integral equation of the first kind:

$$\int_0^t H(u)B(u,T)du = G(t)$$

with

$$g(t) = -\frac{1}{2} \int_0^t \sigma^2(u,T,f^*(0,t))B^2(u,T)du$$

and f^* as usual is the initial forward rate curve. As a corollary, Mari proves that the dynamics of the spot rate is described by

$$dr(t) = \{a(t) - b(t)r(t)\}\, dt + \sqrt{h(t) + k(t)r(t)}dW(t)$$

where

$$\begin{cases} a(t) = \dfrac{\partial f^*(0,t)}{\partial t} + b(t)f^*(0,t) - H(t) \\ b(t) = \dfrac{\partial B(t,T)}{\partial T^2}|_{T=t} \end{cases}$$

The innovation of the paper consists in the explicit determination of the function $a(t)$ which is the term accounting for the initial term structure, and in bringing to the forefront the Volterra equation, as a device to overcome the obstacle met by Hull and White (1992). In the following, Mari goes on considering some applications.

Gaussian Models, the CIR volatility structure and the generalized CIR volatility structure, in which $h(t) \neq 0$. The problem is that, in general, Volterra equation does not admit closed-form solution, except for simple case, as the Vasicek one. In fact for the extended Vasicek model, this method gives the same solution of the Hull and White (1992). As for the generalized CIR model, a perturbative solution of the Volterra equation is proposed.

17

A New Measure – The Forward Measure

17.1 Forward Measures

In previous sections, we have used two probability measures: **the objective (real) probability measure *P***, and the **"risk-neutral" martingale measure *Q***. In this section we will introduce a whole new class of probability measures, so-called forward measures, including Q as a member of that class. These probability measures are connected to a technique called **change of numeraire**. They are of great importance both in the understanding and for practical calculations since the amount of computational work needed in order to obtain a pricing formula can be drastically reduced by a suitable choice of numeraire. Especially the forward measures simplify the calculations of prices on bond options.

To get some feeling for where we are heading, let us consider a pricing problem. But first we remember that a martingale is a zero-drift stochastic process. We will also in general think of measures as units in which we value other securities. If we use the price of a traded security as such a unit measure, then there is some market price of risk for which all other security prices are martingales.

Suppose that $p_1(t,T)$ and $p_2(t,T)$ are prices of two traded securities that depend on a single source of uncertainty. Define the relative price of $p_1(t,T)$ with respect to $p_2(t,T)$ as

$$\gamma = \frac{p_1(t,T)}{p_2(t,T)}$$

We refer $p_2(t,T)$ as the **numeraire**. The equivalent martingale measure states that, if there are no arbitrage opportunities, γ is martingale

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for some market price of risk. What is more, for a given numeraire security $p_2(t,T)$, the same choice of the market price of risk makes γ martingale for all securities $p_1(t,T)$. This choice of market price of risk is the volatility of $p_2(t,T)$. We can state this as a **theorem** and we now give a proof.

Proof: Suppose the volatilities of $p_1(t,T)$ and $p_2(t,T)$ are σ_1 and σ_2. In general, when we introduce the market price of risk, λ we have:

$$dp = (r + \lambda\sigma)pdt + \sigma pdW$$

Therefore

$$\begin{cases} dp_1 = (r + \sigma_1\sigma_2)p_1dt + \sigma_1 p_1 dW \\ dp_2 = (r + \sigma_2^2)p_2dt + \sigma_2 p_2 dW \end{cases}$$

Using Itô's lemma we get

$$\begin{aligned}
d\left(\frac{p_1}{p_2}\right) &= \frac{\partial}{\partial p_1}\left(\frac{p_1}{p_2}\right)dp_1 + \frac{\partial}{\partial p_2}\left(\frac{p_1}{p_2}\right)dp_2 + \frac{1}{2}\frac{\partial^2}{\partial p_2}\left(\frac{p_1}{p_2}\right)(dp_2)^2 \\
&\quad + \frac{\partial^2}{\partial p_1 \partial p_2}\left(\frac{p_1}{p_2}\right)dp_1dp_2 \\
&= \frac{1}{p_2}\{(r + \sigma_1\sigma_2)p_1dt + \sigma_1p_1dW\} - \frac{p_1}{p_2^2}\{(r + \sigma_2^2)p_2dt + \sigma_2p_2dW\} \\
&\quad + \frac{1}{2}2\frac{p_1}{p_2^3}\sigma_2^2p_2^2dt - \frac{1}{p_2^2}\sigma_1\sigma_2p_1p_2dt \\
&= \frac{p_1}{p_2}\{(r + \sigma_1\sigma_2)dt + \sigma_1dW\} - \frac{p_1}{p_2}\{(r + \sigma_2^2)dt + \sigma_2dW\} \\
&= \frac{p_1}{p_2}\sigma_2^2dt - \frac{p_1}{p_2}\sigma_1\sigma_2dt \\
&\quad + \frac{p_1}{p_2}\sigma_1dW - \frac{p_1}{p_2}\sigma_2dW + \frac{p_1}{p_2}(\sigma_1 - \sigma_2)dW
\end{aligned}$$

We then see that p_1/p_2 is martingale. We say that in a world where the market price of risk is σ_2 the world is **forward risk neutral** with respect to p_2. Therefore we call this measure (in terms of p_2) as a **forward measure**.

This very simple analysis shows that we can change to any numeraire security, and use that as a forward measure where the market price of risk is the volatility of the numeraire security. Then in terms of this security all processes become martingales.

We have used exactly this in option pricing on equities, where we used $B(t)$ as the numeraire security. Then we found that the discounted stock price was martingale. If we remember the "rollover strategy" in the bond pricing section, we called this the **money market account**.

Because f/g, where f and g are any securities, is martingale in a world that is forward risk neutral with respect to g, it follows that

$$\frac{f(t)}{g(t)} = E^g\left[\left.\frac{f(s)}{g(s)}\right|F_t\right]$$

or

$$f(t) = g(t)\cdot E^g\left[\left.\frac{f(s)}{g(s)}\right|F_t\right]$$

Let us now consider the pricing problem for a contingent claim X, in a model with a stochastic short rate of interest $r(t)$. From the general theory we know that the price at $t = 0$ of X is given by the formula

$$\Pi[0,X] = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\cdot X\right]$$

The problem with this formula from a computational point of view is that, in order to compute the expected value we have to get hold of the joint distribution (under Q) of the two stochastic variables (the integral of $r(s)$ and X) and finally integrate with respect to that distribution. Thus we have to compute a double integral, and in most cases this turns out to be rather hard work.

Let us now make the (extremely unrealistic) assumption that r and X are independent under Q. Then the previously mentioned expectation splits, and we have the formula

$$\Pi[0,X] = E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\right]E^Q[X]$$

which we may write as

$$\Pi[0,X] = p(0,T)\cdot E^Q[X]$$

We now note that this is a much nicer formula since:

- We only have to compute the single integral $E^Q[X]$ instead of the double integral.
- The bond price $p(0, T)$ does not have to be computed theoretically at all. We can **observe** it (at $t = 0$) directly on the bond market.

The drawback with the previous argument is that, in most concrete cases, r and X are not independent under Q, and if X is a contingent claim on an underlying bond, this is of course obvious. What may be less obvious is that even if X is a claim on an underlying stock that is P-independent of r, it will still be the case that X and r will be dependent (generically) under Q. The reason is that under Q the stock will have r as its local rate of return, thus introducing a Q-dependence.

This is the bad news. The good news is that there exists a **general** pricing formula, a special case of which reads as

$$\Pi[0, X] = p(0, T) \cdot E^T[X]$$

Here E^T denotes expectation with respect to the so-called **forward-neutral measure**Q^T, which we will discuss later. We see from this formula that we do indeed have the multiplicative structure, but the price we have to pay for generality is that the measure Q^T depends upon the choice of maturity date T. We define, on the bond market, the forward measure Q^T on $\mathcal{F}^T$ as:

Definition 17.1 Let T be a fixed time. Then *the forward measure* Q^{T} *on* $\mathcal{F}^{\mathrm{T}}$ is defined by

$$\frac{1\mathrm{d}Q^{\mathrm{T}}}{\mathrm{d}Q} = \frac{\exp\left\{-\int_0^{\mathrm{T}} r(s)\mathrm{d}s\right\}}{E^Q\left[\exp\left\{-\int_0^{\mathrm{T}} r(s)\mathrm{d}s\right\}\right]}$$

That is, the Radon-Nikodym derivative R^{T} is given by

$$\mathrm{R}^{\mathrm{T}} = \frac{1}{p(0, \mathrm{T})} \exp\left\{-\int_0^{\mathrm{T}} r(s)\mathrm{d}s\right\}$$

It is very important to notice that we get different measures Q^{T} for different choices of T.

We will now prove a stronger pricing formula:

Theorem 17.2. *Let X be a given T-claim. Then the arbitrage-free price of X is given by*

$$\Pi[t, X] = p(t, T) \cdot E^T[X|\mathcal{F}_t]$$

where E^T quote integrations with respect to Q^T.

Proof: If we use

$$E^T[X|\mathcal{F}_t] = \frac{E^Q\left[R^T X|\mathcal{F}_t\right]}{E^Q\left[R^T|\mathcal{F}_t\right]}$$

We then have

$$\begin{aligned}
E^Q\left[R^T X|\mathcal{F}_t\right] &= \frac{1}{p(0,T)} E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\} X|\mathcal{F}_t\right] \\
&= \frac{1}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\} E^Q\left[\exp\left\{-\int_t^T r(s)ds\right\} X|\mathcal{F}_t\right] \\
&= \frac{\Pi[t,x]}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\}
\end{aligned}$$

and

$$\begin{aligned}
E^Q\left[R^T|\mathcal{F}_t\right] &= \frac{1}{p(0,T)} E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\} |\mathcal{F}_t\right] \\
&= \frac{p(t,T)}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\}
\end{aligned}$$

These two give the result.

Theorem 17.3. *The likelihood process L^T is given by:*

$$L^T = \frac{p(t,T)}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\} = \frac{p(t,T)}{p(0,T)B(t)}$$

Proof: Since L^T is Q^T martingale

$$L^T(t)E^Q\left[L^T(t)|\mathcal{F}_t\right] = E^Q\left[R^T|\mathcal{F}_t\right]\frac{p(t,T)}{p(0,T)} \exp\left\{-\int_0^t r(s)ds\right\} = \frac{p(t,T)}{p(0,T)B(t)}$$

If we are using one-factor models where all uncertainty is generated by the Q-Wiener process V, we know that all absolute continuous transformations of measure are given by a Girsanov kernel. Especially, there must exist a Girsanov transformation between Q and Q^T. We are curious of how this Girsanov transformation looks like.

We know that the dynamic of a T-bond under Q is given by

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dV(t)$$

With Itô we get

$$\begin{aligned}
dL^T(t) &= \frac{\partial L^T}{\partial p}dp + \frac{\partial L^T}{\partial B}dB = \frac{1}{p(0,T)B(t)}dp - \frac{p(t,T)}{p(0,T)B^2(t)}dB \\
&= \frac{1}{p(0,T)B(t)}\{r(t)p(t,T)dt + v(t,T)p(t,T)dv(t)\} \\
&\quad - \frac{p(t,T)}{p(0,T)B^2(t)}r(t)B(t)dt \\
&= \frac{p(t,T)}{p(0,T)B(t)}v(t,T)dV(t) = v(t,T)L^T(t)dV(t)
\end{aligned}$$

Therefore we have proved the following result:

Theorem 17.4. *For a given T, the Girsanov transformation from Q to Q^T is given by a likelihood process given by:*

$$dL^T(t) = v(t,T) = L^T(t)dV(t)$$

The Girsanov kernel is given by the process $v(t, T)$ and the likelihood process L^T have the representation given by

$$L^T(t) = \exp\left\{\int_0^t v(s,T)dV(s) - \frac{1}{2}\int_0^t v^2(s,T)ds\right\}$$

Furthermore

$$dV(t) = v(t,T)dt + dW^T(t)$$

where $dW^T(t)$ is a Q^T-Wiener process on the interval [0, T].

The forward measures have an important economical interpretation as well. In the aforementioned discussion we have used a T-bond as numeraire. But generally we can use any security as the numeraire process.

In order to understand why formulas of this type have anything to do with the choice of numeraire, let us give a very brief and informal argument.

We start by recalling that the risk-neutral martingale measure Q has the property that for every choice of a price process $\Pi(t)$ for a traded asset, the quote

$$\frac{\Pi(t)}{B(t)}$$

is a Q-martingale. The point here is that we have divided the asset price $\Pi(t)$ by the **numeraire asset price** $B(t)$. It is now natural to investigate whether this martingale property can be generalized to other choices of numeraire, and we are led to the following conjecture:

Consider a fixed financial market, and a fixed "numeraire" asset price process $S_0(t)$ on the market. Then there exists a probability measure, denoted Q^0, such that

$$\frac{\Pi(t)}{S_0(t)}$$

is a Q^0-martingale for every asset price process $\Pi(t)$.

Let us for the moment assume that the conjecture is true. We then fix a certain date of maturity T, and we choose the bond price process $p(t, T)$ (for this fixed T) as numeraire. According to the conjecture there should then exist a probability measure, which we denote by Q^T, such that the quotient

$$\frac{\Pi(t)}{p(t,T)}$$

is a martingale under the measure Q^T, for every $\Pi(t)$ which is the price process of a traded asset. In particular we have (using the relation $p(T,T)=1$)

$$\frac{\Pi(0)}{p(0,T)} = E^T\left[\frac{\Pi(T)}{p(T,T)}\right] = E^T\left[\Pi(T)\right]$$

where E^T denotes expectation under Q^T. Let us now choose the derivative price process $\Pi(t,X)$ as $\Pi(t)$ in the previous equation. Then we have $\Pi(T)=\Pi(t,X)=X$:

$$\frac{\Pi(0,X)}{p(0,T)} = E^T[X]$$

$\Rightarrow$

$$\Pi(0,X) = p(0,T)\cdot E^T[X]$$

An alternative is a try to construct a measure Q^T with $p(t,T)$ as numeraire that makes all prices to martingales. Especially this measure should make

$$Z^T(t)\frac{B(t)}{p(t,T)}$$

to a martingale. If we use the dynamics of $p(t,T)$ under Q:

$$dp(t,T) = r(t)p(t,T)dt + v(t,T)p(t,T)dV(t)$$

and use Itô's formula on Z^T

$$\begin{aligned}
dZ^T(t) &= \frac{\partial Z^T}{\partial B}dB + \frac{\partial Z^T}{\partial p}dp + \frac{1}{2}\frac{\partial^2 Z^T}{\partial p^2}(dp)^2\\
&= \frac{1}{p(t,T)}dB - \frac{B(t)}{p^2(t,T)}dp + \frac{B(t)}{p^3(t,T)}(dp)^2\\
&= \frac{B(t)}{p(t,T)}rdt - \frac{B(t)}{p^2(t,T)}\left(rp(t,T)dt + v(t,T)p(t,T)dV(t)\right)\\
&\quad + \frac{B(t)}{p^3(t,T)}v^2(t,T)p^2(t,T)dt\\
&= Z^T(t)rdt - Z^T(t)\left(rdt + v(t,T)dV(t)\right) + Z^T(t)v^2(t,T)dt\\
&= Z^T(t)v^2(t,T)dt - Z^T(t)v(t,T)dV(t)
\end{aligned}$$

If we now make a Girsanov transformation from Q to Q^T with the kernel g(t) we get

$$dZ^T(t) = Z^T(t)\left\{v^2(t,T) - g(t)v(t,T)\right\} dt - Z^T(t)v(t,T)dV^T(t)$$

where V^T is a Q^T-Wiener process. We see that Z^T becomes a Q^T-martingale if $g(t) = v(t,T)$. This Girsanov kernel is the same as the one in the previous theorem. Thus Q^T makes all prices martingales with $p(t,T)$ as numeraire.

Theorem 17.5. *If S is a process such as S(t)/B(t) is a Q-martingale. Then, the process*

$$Z^T(t) = \frac{S(t)}{p(t,T)}$$

is a Q^T-martingale on the interval $[0,T]$.

Under Q^T, also

$$\frac{\Pi(t,X)}{p(t,T)}$$

is a Q^T-martingale. By using $p(T,T) = 1$ *and* $\Pi(T,X) = X$ *we get*

$$\frac{\Pi(t,X)}{p(t,T)} = E^T\left[\frac{\Pi(T,X)}{p(T,T)}|\mathcal{F}_t\right] = E^T\left[X|\mathcal{F}_t\right]$$

*This is the **forward price** at time t for the contract X.*

17.1.1 Forwards and Futures

Let us study a simple example where we need the forward measure, to value a forward or a future contract on underlying equity.

We suppose there exists a martingale measure Q. If we buy a T-contract today, at time t we know that the price is given by

$$\pi_t[X] = E^Q\left[X \cdot \exp\left\{-\int_t^T r(s)ds\right\}|\mathcal{F}_t\right]$$

The cash flows are:

1. At time t we pay the amount $\pi_t[X]$.
2. At time T we receive the stochastic amount X.

A forward and a future contract are variants of the aforementioned contract, but they differ on how the cash flows are paid. We start with the simplest, the forward contract.

Definition 17.2 Let X be a contingent T-claim. With a **forward contract** on X contracted at time t, with the delivery at T and the **forward price** $\phi(\mathrm{t},\mathrm{T},\mathrm{X})$ we mean the following construction:

(i) The holder of the forward contract receives at time T the stochastic amount X cash units.

(ii) The holder of the forward contract pays at time T the amount $\phi(\mathrm{t},\mathrm{T},\mathrm{X})$ cash units.

(iii) The forward price $\phi(\mathrm{t},\mathrm{T},\mathrm{X})$ is determined when we sign the contract at time t.

(iv) The forward price is determined so that the arbitrage-free price at the contract is equal zero when we sign the contract at time t.

The forward contract defined earlier are traded over-the-counter (OTC) and not at exchanges. An important characteristic of the contract is the value of zero when the contract is signed at time t. Our problem is to find the mathematical price of the contract X of the previous construction. This is quite obvious:

$$
\begin{aligned}
0 = \pi_{\mathrm{t}}[\mathrm{X}-\phi(\mathrm{t},\mathrm{T},\mathrm{X})] &= \mathrm{E}^{\mathrm{Q}}\left[[\mathrm{X}-\phi(\mathrm{t},\mathrm{T},\mathrm{X})]\cdot\exp\left\{-\int_{\mathrm{t}}^{\mathrm{T}}\mathrm{r}(\mathrm{s})\mathrm{ds}\right\}|\mathrm{F}_{\mathrm{t}}\right] \\
&= \mathrm{E}^{\mathrm{Q}}\left[\mathrm{X}\cdot\exp\left\{-\int_{\mathrm{t}}^{\mathrm{T}}\mathrm{r}(\mathrm{s})\mathrm{ds}\right\}|\mathrm{F}_{\mathrm{t}}\right] \\
&\quad -\phi(\mathrm{t},\mathrm{T},\mathrm{X})\cdot\mathrm{E}^{\mathrm{Q}}\left[\exp\left\{-\int_{\mathrm{t}}^{\mathrm{T}}\mathrm{r}(\mathrm{s})\mathrm{ds}\right\}|\mathrm{F}_{\mathrm{t}}\right] \\
&= \pi_{\mathrm{t}}[\mathrm{X}]-\phi(\mathrm{t},\mathrm{T},\mathrm{X})\mathrm{p}(\mathrm{t},\mathrm{T})
\end{aligned}
$$

where $p(t,T)$ is the price at time t of a zero-coupon bond paying one cash unit at maturity T. We can summarize this and write down the price of the forward contract.

$$\phi(t,T,X)=\frac{\pi_t[X]}{p(t,T)}=\frac{1}{p(t,T)}\cdot E^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}|F_t\right]$$

We write this as

$$\phi(t,T,X)=E^T[X|F_t]$$

where E^T means integration with respect to the **forward measure** Q^T.

The reason that this contract is not traded at exchanges is credit risk (i.e. that the other party cannot fulfil the obligation). Therefore, another contract, which can be traded, standardized on exchanges, has been created. This is the future contract.

Definition 17.3 Let X be a contingent T-claim. With a **future contract** on X contracted at time t, with the **future price** $\Phi(t,T,X)$ we mean the following construction:

(i) For each time t there exists a price, $\Phi(t,T,X)$ called the **future price** for X with delivery at T.

(ii) At time T the holder of the contract pays $\Phi(t,T,X)$ and receives X cash units.

(iii) During each time interval $(t,t+dt]$ the holder receives $\Phi(t+dt,T,X)-\Phi(t,T,X)$ cash units.

(iv) The market price at each time is equal to zero.

In practical situations, the time *dt* is one bank day but can also be a week or a month. Step (iii) means that there are continuous cash flows between the buyer and the seller of the contract. In such construction, the credit risk is minimized to the change of the price during a period of *dt*. The buyer and the seller also have to hold a margin requirement on an account on the market place. This margin can be used to close the contract at time T.

We can notice the following about the future contract.

1. $\Phi(T,T,X)=X$ so at time T there are no reason to deliver anything. This is also true in real situations.

2. A future contract is not a contract where you have to deliver any security at T with a predefined price.
3. There are no costs to enter or leave a future contract.
4. The only part of the contract is the cash flows during the lifetime calculated as the price difference during a period *dt* in time.

We will now give a mathematical definition of the future contract.

Definition 17.4 A **future contract** on a contingent T-claim X is a security with an adapted price-and-dividend process $[\pi, \Phi]$ given by the following conditions:

$$\begin{aligned} &\Phi(\mathrm{t},\mathrm{T},\mathrm{X}) = \mathrm{X}, \mathrm{P} - \text{almost true} \\ &\pi_{\mathrm{t}} = 0 \qquad\qquad \mathrm{P} - \text{almost true}, \forall \mathrm{t} \leqslant \mathrm{T} \end{aligned}$$

Theorem 17.9. *The price of a future contract is given by*

$$\Phi(t,T,X) = E^{Q}[X|\mathcal{F}_t]$$

If the short-rate r and X are independent, the price of the forward and future contract coincide. That is,

$$\Phi(t,T,X) = \phi(t,T,X) = E^{Q}[X|\mathcal{F}_t]$$

17.1.2 A General Option Pricing Formula

As we have seen earlier, that for a forward measure Q^T "takes care of the stochasticity" on the interval $[0,T]$. This can be seen in the previous theorem that states the pricing formula:

$$\Pi(t,X) = p(t,T) \cdot E^T[X|\mathcal{F}_t]$$

We will now show how the calculations of prices of interest rate options can be simplified by using forward measures. But first we give the following lemma:

Lemma 17.10 *For a fixed T, then the forward rate process $f(t,T)$ is a Q^T-martingale. Especially, for all $t \leqslant T$ we have:*

$$E^T[r(T)|\mathcal{F}_t] = f(t,T)$$

Proof: From the previous theorem we have

$$\begin{aligned}
E^T[r(T)|\mathcal{F}_t] &= \frac{1}{p(t,T)}E^Q\left[r(T)\exp\left\{-\int_t^T r(s)ds\right\}|\mathcal{F}_t\right] \\
&= -\frac{1}{p(t,T)}E^Q\left[\frac{\partial}{\partial T}\exp\left\{-\int_t^T r(s)ds\right\}|\mathcal{F}_t\right] \\
&= -\frac{1}{p(t,T)}\frac{\partial}{\partial T}E^Q\left[\exp\left\{-\int_t^T r(s)ds\right\}|\mathcal{F}_t\right] \\
&= -\frac{p_T(t,T)}{p(t,T)} = f(t,T)
\end{aligned}$$

To simplify the calculations of options, we suppose that the short-rate $r(t)$ under Q is given by

$$dr(t) = \{a(t) + b(t)r(t)\}\,dt + \sigma(t)dV(t)$$

Then we know

1. $r(t)$ is a normal distributed process.
2. The bond prices have the form of $p(t,T) = \exp\{A(t,T) - B(t,T)r(t)\}$
3. The price of a European call option on a S-bond, with maturity T and strike K is given by:

$$\Pi(t,X) = E^Q\left[g\{r(T)\}\exp\left\{-\int_0^T r(s)ds\right\}|\mathcal{F}_t\right]$$

$$X = g(r) = \max(\exp\{A(T,S) - B(T,S)r(T)\} - K, 0)$$

With the aforementioned theorem, the price of this contract can at $t = 0$ be written as:

$$\Pi(0,X) = p(0,T)\cdot E^T\left[g(r(T))\right]$$

This is a much simpler formula than before, since:

- We do not have to calculate $p(0,T)$. This value is given on the market!
- The expectation value is a simple integral instead of an double integral.

To use the pricing formula, we have to find the distribution of $r(T)$ under Q^T. But this is a simple procedure.

Theorem 17.11. *Suppose that the dynamics of r under Q is given by*

$$dr(t) = \{a(t) + b(t)r(t)\}\, dt + \sigma(t)dV(t)$$

Then r(T) is normal distributed under Q^T with:

$$E^T[r(T)] = f(0,T)$$

$$Var^T[r(T)] = Var^Q[r(T)] = E^T\left[\left(\int_0^T \sigma(s)\cdot \exp\left\{\int_s^T b(\tau)d\tau\right\} dV(s)\right)^2\right]$$

$$= \int_0^T \sigma^2(s)e^{2H(T,s)}ds$$

where H is defined by

$$H(t,T) = \int_t^T b(u)du$$

We show this by usung a integrating factor $\exp\left\{-\int_0^t b(u)du\right\}$ and calculate

$$d\left(\exp\left\{-\int_0^t b(u)du\right\} r(t)\right) = r(t)\frac{\partial}{\partial t}\left(\exp\left\{-\int_0^t b(u)du\right\}\right) dt$$
$$+ \exp\left\{-\int_0^t b(u)du\right\} \frac{\partial}{\partial r}(r(t))\, dr$$

The first term is calculated as

$$-b(t)r(t)\cdot\left(\exp\left\{-\int_0^t b(u)du\right\}\right) dt$$

and the second as

$$\exp\left\{-\int_0^t b(u)du\right\} [\{a(t) + b(t)r(t)\}\, dt + \sigma(t)dV(t)]$$

We finally have

$$d\left(\exp\left\{-\int_0^t b(u)du\right\} r(t)\right) = \exp\left\{-\int_0^t b(u)du\right\}[a(t)dt + \sigma(t)dV(t)]$$

With the definition of $H(t,T)$ we can write this as

$$d\left(e^{-H(0,T)}r(t)\right) = e^{-H(0,T)}[a(t)dt + \sigma(t)dV(t)]$$

To calculate the variance of $r(T)$ under Q^T, we first integrate the process of r, giving

$$r(T) = r(0)\exp\left\{\int_0^T b(u)du\right\} + \int_0^T \exp\left\{\int_s^T b(u)du\right\} a(s)ds$$
$$+ \int_0^T \exp\left\{\int_s^T b(u)du\right\} \sigma(s)ds$$

or

$$r(T) = r(0)e^{H(0,T)} + \int_0^T e^{H(s,T)}a(s)ds + \int_0^T e^{H(s,T)}\sigma(s)dV(s)$$

We know that from Section 13.1 that under Q

$$dp(t,T) = r(t)p(t,T)dt + \nu(t,T)p(t,T)dV(t)$$

have the kernel $g(t) = \nu(t,T)$. If we change the measure to Q^T with this Girsanov transformation and use the process for an affine term structure

$$dp(t,T) = r(t)p(t,T)dt - \sigma(t,r(t))B(t,T)p(t,T)dV(t)$$

we get

$$\nu(t,T) = -\sigma(t)B(t,T)$$

where $\nu(t, T)$ is deterministic. After the Girsanov transformation we have

$$r(T) = r(0)e^{H(0,T)} + \int_0^T e^{H(s,T)} \{a(s) + \nu(s, T)\} \, ds + \int_0^T e^{H(s,T)} \sigma(s) dW^T(s)$$

We can sum up this in the following theorem.

Theorem 17.12. *Suppose that the dynamics of r under Q is given by*

$$dr(t) = \{a(t) + b(t)r(t)\} \, dt + \sigma(t) dV(t)$$

Then the price of a European call option is given by

$$\Pi(t, X) = p(t, T) \int_{-\infty}^{\infty} \max \left(\exp \{A(T, S) - B(T, S)z\} - K, 0 \right) \phi(z) dz$$

where ϕ is the density of a normal distribution with the expectation value

$$m = E^T \left[r(T) | \mathcal{F}_t \right] = f(t, T)$$

and variance

$$v = var^T [r(T)] = \int_t^T e^{2H(T,s)} \sigma^2(s) ds$$

The price is given by

$$\Pi [t, X] = \frac{p(t, T)}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} g(r(T)) \exp \left\{ -\frac{(r(T) - m)^2}{2v^2} \right\} dr$$

We remember from the end of lecture notes I that a general payoff for a European call option is given by:

$$C_T = \max (S(T) - K, 0) = (S(T) - K) \, I_{\{S(T) > K\}}$$

where $I_{\{S(T)>k\}}$ is an indicator function equal to 1, if $S(T) > K$ and 0 else. We use $S(t)$ as any underlying security. We then have the arbitrage-free price as

$$\begin{aligned}\Pi(0,X) &= E^{Q}\left[\frac{1}{B(T)}\left(S(T)-K\right)I_{\{S(T)\geqslant K\}}\right]\\ &= E^{Q}\left[\exp\left\{-\int_0^T r(s)ds\right\}S(T)I_{\{S(T)\geqslant K\}}\right]\\ &\quad - K\cdot E^{Q}\left[\exp\left\{-\int_0^T r(s)ds\right\}I_{\{S(T)\geqslant K\}}\right]\\ &= A - B\end{aligned}$$

The first expectation value is, if we use $S(t)$ as numeraire, $S(T)$ discounted to a present value $S(0)$

$$\begin{aligned}A &= E^{Q}\left[\exp\left\{-\int_0^T r(s)ds\right\}S(T)I_{\{S(T)\geqslant K\}}\right] = S(0)E^{S}\left[I_{\{S(T)\geqslant K\}}\right]\\ &= S(0)Q^{S}\left(S(T)\geqslant K\right)\end{aligned}$$

The second expectation value is, if we use $p(t,T)$ as numeraire, the price of a discount bond with maturity T

$$\begin{aligned}B &= K.E^{Q}\left[\exp\left\{-\int_0^T r(s)ds\right\}I_{\{S(T)\geqslant K\}}\right] = K\cdot p(0,T)E^{T}\left[I_{\{S(T)\geqslant K\}}\right]\\ &= K\cdot p(0,T)Q^{T}\left(S(T)\geqslant K\right)\end{aligned}$$

Then we can write

$$\Pi(0,X) = S(0)Q^{S}\left(S(T)\geqslant K\right) - K\cdot p(0,T)Q^{T}\left(S(T)\geqslant K\right)$$

where Q^T denotes the T-forward measure and Q^S the martingale measure for the numeraire process $S(t)$.

In order to use this formula in real situation we have to be able to calculate the previously mentioned probabilities. Before we do the calculation we will repeat some parts discussed in earlier sections. We know that $S(t)/B(t)$ is Q-martingale, where

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}$$

Therefore

$$d\left(\frac{S(t)}{B(t)}\right) = \frac{S(t)}{B(t)}\sigma(t)dW(t).$$

The zero-coupon bond price is given by

$$p(t,T) = E^Q\left[\exp\left\{-\int_t^T r(u)du\right\}|\mathcal{F}_t\right] = E^Q\left[\frac{B(t)}{B(T)}|\mathcal{F}_t\right]$$

so

$$\frac{p(t,T)}{B(t)} = E^Q\left[\frac{1}{B(T)}|\mathcal{F}_t\right]$$

is also a martingale. The T-forward price $F(t,T)$ of S is the price set at time t for delivery of S at time T with payment at time T. The value of the forward contract at t is zero, so

$$\begin{aligned}
0 &= E^Q\left[\frac{B(t)}{B(T)}\{S(T) - F(t,T)\}|\mathcal{F}_t\right] \\
&= B(t)E^Q\left[\frac{S(T)}{B(T)}|\mathcal{F}_t\right] - F(t,T)E^Q\left[\frac{B(t)}{B(T)}|\mathcal{F}_t\right] \\
&= B(t)\frac{S(t)}{B(t)} - F(t,T)p(t,T) = S(t) - F(t,T)p(t,T)
\end{aligned}$$

Therefore,

$$F(t,T) = \frac{S(t)}{p(t,T)}$$

Definition 17.5 Any asset in the model whose price is always strictly positive can be taken as *numeraire*. We can denominate all other assets in units of this numeraire.

Example 17.1.14

Money market account as numeraire. At time t, a stock S is worth $S(t)/B(t)$ units of money market and a T-bond is worth $p(t,T)/B(t)$ units of money market.

Example 17.1.15

Bond as numeraire. At time $t < T$, a stock S is worth $F(t, T)$ units of a T-maturity bond and the T-maturity bond is worth 1 unit.

Theorem 17.16. *Let N be a numeraire, that is, the price process for some asset whose price is always positive. Then Q^N defined by*

$$Q^N(A) = \frac{1}{N(0)} \int_A \frac{N(T)}{B(T)} dQ, \quad \forall A \in \mathcal{F}_T$$

is risk-neutral for N. Q^N is called the risk-neutral measure for the numeraire N.
Note: Q and Q^N are equivalent, that is, they have the same probability zero set, and

$$Q(A) = N(0) \int_A \frac{B(T)}{N(T)} dQ^N, \qquad \forall A \in \mathcal{F}_T$$

Proof: Because N is the price process of some asset, $N/B(t)$ is martingale under Q. Therefore

$$Q^N(\Omega) = \frac{1}{N(0)} \int_\Omega \frac{N(T)}{B(T)} dQ = \frac{1}{N(0)} E^Q\left[\frac{N(T)}{B(T)}\right] = \frac{1}{N(0)} \frac{N(0)}{B(0)} = 1$$

and we see that Q^N is a probability measure. Let Y be a traded asset price. Under $Q, Y/B(t)$ is a martingale. We must show that under $Q^N, Y/N$ is a martingale. Using

$$E^Q[X] = \int_\Omega X dQ = \int_\Omega X \frac{dQ}{dP} dP = E^P\left[\frac{dQ}{dP} X\right]$$

$$E^{Q^N}\left[\frac{Y(T)}{N(T)} | \mathcal{F}_t\right] = \frac{B(t)}{N(t)} E^Q\left[\frac{N(T)}{B(T)} \frac{Y(T)}{N(T)} | \mathcal{F}_t\right] = \frac{B(t)}{N(t)} E^Q\left[\frac{Y(T)}{B(T)} | \mathcal{F}_t\right]$$
$$= \frac{B(t)}{N(t)} \frac{Y(t)}{B(t)} = \frac{Y(t)}{N(t)}$$

which is the martingale property for Y/N under the probability measure Q^N.

17.1.2.1 The Bond Price as Numeraire

Fix $T \in [0, T]$ and let $p(t, T)$ be the numeraire. The risk-neutral measure for this numeraire is

$$Q^T(A) = \frac{1}{p(0,T)} \int_A \frac{p(T,T)}{B,T} \, dQ = \frac{1}{p(0,T)} \int_A \frac{1}{B(T)} dQ, \quad \forall A \in \mathcal{F}_T$$

Because the bond is not defined after the time T, we change measure only "up to time T", that is, using

$$\frac{1}{p(0,T)} \frac{p(T,T)}{B(T)} \text{ and only for } A \in \mathcal{F}_T$$

Q^T is called the ***T*-forward measure**. Denominated in units of the T-maturity bond the value of the security S is

$$F(t,T) = \frac{S(t)}{p(t,T)}, 0 \leqslant t \leqslant T$$

This is a martingale under Q^T and has the differential form:

$$dF(t,T) = \sigma_F(t,T) F(t,T) dW^T(t), \quad 0 \leqslant t \leqslant T$$

That is, a differential without a drift term dt. The process $\{W^T; 0 \leqslant t \leqslant T\}$is a Brownian motion under Q^T, and we may assume without loss of generality that $\sigma_F(t,T) \geqslant 0$.
Remark: The numeraire $p(t, T)$ is the price of the bond with maturity, T. Therefore, different forward-neutral measures are not compatible against each other's. The value of \$1 on maturity T cannot be equal to another measure.

17.1.2.2 The Stock Price as Numeraire

Let $S(t)$ be the numeraire. In terms of this numeraire, the stock price is identical to 1. The risk-neutral measure under this numeraire is

$$Q^S(A) = \frac{1}{S(0)} \int_A \frac{S(T)}{B(T)} dQ, \quad \forall A \in \mathcal{F}_T$$

Denominated in shares of the stock, the value of the T-maturity bond is:

$$\frac{p(t,T)}{S(t)} = \frac{1}{F(t,T)}, \quad 0 \leqslant t \leqslant T$$

This is a martingale under Q^S and so has the differential form:

$$d\left(\frac{1}{F(t,T)}\right) = \gamma(t,T)\left(\frac{1}{F(t,T)}\right) dW^S(t)$$

Where $\{W^S; 0 \leqslant t \leqslant T\}$ is a Brownian motion under Q^S, and we may assume without loss of generality that $\gamma(t,T) \geqslant 0$.

Theorem 17.17. *The volatility $\gamma(t,T)$ is equal to the volatility $\sigma_F(t,T)$. In other words we have*

$$d\left(\frac{1}{F(t,T)}\right) = \sigma_F(t,T)\left(\frac{1}{F(t,T)}\right) dW^S(t)$$

Proof: Let $g(x) = 1/x$, so $g'(x) = -1/x^2, g''(x) = 2/x^3$. Then

$$\begin{aligned}
d\left(\frac{1}{F(t,T)}\right) &= dg\left(F(t,T)\right) = g'\left(F(t,T)\right) dF(t,T) + \frac{1}{2}g''\left(F(t,T)\right)\left(dF(t,T)\right)^2 \\
&= -\frac{1}{F^2(t,T)}\sigma_F(t,T)F(t,T)dW^T(t) + \frac{1}{F^3(t,T)}\sigma_F^2(t,T)F^2(t,T)dt \\
&= \frac{1}{F(t,T)}\left[-\sigma_F(t,T)dW^T(t) + \sigma_F^2(t,T)dt\right] \\
&= \sigma_F(t,T)\frac{1}{F(t,T)}\left[-dW^T(t) + \sigma_F(t,T)dt\right]
\end{aligned}$$

Under Q^T, $-W^T$ is a Brownian motion. Under this measure $1/F(t,T)$ has volatility $\sigma_F(t,T)$ and mean rate of return $\sigma_F^2(t,T)$. The change of measure from Q^T to Q^S makes $1/F(t,T)$ a martingale, that is, it changes the mean return to zero, but the change in measure does not affect the volatility. Therefore $\gamma(t,T)$ must be $\sigma_F(t,T)$ and W^S must be

$$W^S(t) = -W^T(t) \int_0^t \sigma_F(u,T)du$$

We now turn back to the general pricing formula for a call option

$$\Pi(0, X) = S(0)Q^S\left(S(T)K \geqslant K\right) - K \cdot p(0, T)Q^T\left(S(T) \geqslant K\right)$$

We assume that the process

$$Z(t) = \frac{S(t)}{p(t, T)}$$

has a stochastic differential of the form

$$dZ(t) = Z(t)m(t)dt + Z(t)\sigma(t)dW$$

We start to compute the second term:

$$Q^T\left(S(T) \geqslant K\right) = Q^T\left(\frac{S(T)}{p(T, T)} \geqslant K\right) = Q^T\left(Z(T) \geqslant K\right)$$

Since Z is an asset price, normalized by the price of a T-bond, it has zero drift under Q^T, so its Q^T-dynamics are given by

$$dZ(t) = Z(t)\sigma(t)dW^T$$

The solution to this is given by (use Itô's formula on $\ln(Z)$):

$$\begin{aligned} d\ln\left(Z(t)\right) &= \frac{\partial}{\partial Z}\ln\left(Z(t)\right)dZ(t) + \frac{1}{2}\frac{\partial^2}{\partial Z^2}\ln\left(Z(t)\right)\left(dZ(t)\right)^2 \\ &= -\frac{1}{2}\sigma^2(t)dt + \sigma dW^T \end{aligned}$$

and we get

$$\begin{aligned} Z(T) &= \frac{S(0)}{p(0, T)}\exp\left\{\int_0^T \sigma(t)dW^T(t) - \frac{1}{2}\int_0^T \sigma^2(t)dt\right\} \\ &= \frac{S(0)}{p(0, T)}\exp\left\{\Sigma(T) \cdot W^T - \frac{1}{2}\Sigma^2(T)\right\} \end{aligned}$$

We know from stochastic calculus that the previous stochastic integral has a normal distribution with mean zero and variance

$$\Sigma^2(T) = \int_0^T \sigma^2(s)ds$$

The entire exponent is thus normal distributed, and we can write the probability as

$$\begin{aligned}
Q^T(Z(T) \geqslant K) &= Q^T\left(\frac{S(0)}{p(0,T)}\exp\left\{\Sigma \cdot W^T - \frac{1}{2}\Sigma^2(T)\right\} \geqslant K\right)\\
&= Q^T\left(\Sigma(T)\cdot W^T - \frac{1}{2}\Sigma^2(T) \geqslant \ln\left(\frac{K\cdot p(0,T)}{S,(0)}\right)\right)\\
&= Q^T\left(\Sigma(T)\cdot W^T \geqslant \ln\left(\frac{K\cdot p(0,T)}{S0}\right) + \frac{1}{2}\Sigma^2(T)\right)\\
&= Q^T\left(-W^T \leqslant \frac{1}{\Sigma(T)}\left\{\ln\left(\frac{S(0)}{K\cdot p(0,T)}\right) - \frac{1}{2}\Sigma^2(T)\right\}\right)\\
&= N[d_2]
\end{aligned}$$

where

$$d_2 = \frac{\ln\left(\frac{S(0)}{K\cdot p(0,T)}\right) - \frac{1}{2}\Sigma^2(T)}{\Sigma(T)}$$

The first probability in the option formula is a Q^S-probability, which we write as

$$Q^S(S(T) \geqslant K) = Q^S\left(\frac{p(T,T)}{S(T)} \leqslant \frac{1}{K}\right) = Q^S\left(Y(T) \leqslant \frac{1}{K}\right)$$

where we defined $Y(t)$ as

$$Y(t) = \frac{p(t,T)}{S(t)} = \frac{1}{Z(t)} = \frac{p(0,T)}{S(0)}\exp\left\{\Sigma(T)\cdot W^S - \frac{1}{2}\Sigma^2(T)\right\}$$

Therefore

$$\begin{aligned}
&Q^S(S(T) \geqslant K)\\
&\quad = Q^S\left(Y(T) \leqslant \frac{1}{K}\right) = Q^S\left(\frac{p(0,T)}{S(0)}\exp\left\{\Sigma(T)\cdot W^S - \frac{1}{2}\Sigma^2(T)\right\} \leqslant \frac{1}{K}\right)\\
&\quad = Q^S\left(\Sigma(T)\cdot W^S - \frac{1}{2}\Sigma^2(T) \leqslant \ln\left(\frac{S(0)}{K\cdot p(0,T)}\right)\right)\\
&\quad = Q^S\left(W^S \leqslant \frac{1}{\Sigma(T)}\left\{\ln\left(\frac{S(0)}{K\cdot p(0,T)}\right) + \frac{1}{2}\Sigma^2(T)\right\}\right) = N[d_1]
\end{aligned}$$

where

$$d_1 = d_2 + \Sigma(T)$$

To summarize, we have a general formula for call options

$$\Pi(0, X) = S(0) \cdot N[d_1] - K \cdot p(0, T) \cdot N[d_2]$$

where

$$d_2 = \frac{\ln\left\{\frac{S(0)}{K \cdot p(0,T)}\right\} - \frac{1}{2}\Sigma^2(T)}{\Sigma(T)}$$

and

$$d_1 = d_2 + \Sigma(T)$$

Remark! If $r(t)$ is a constant, then $p(t, T) = e^{-rT}$ and we get the usual Black-Scholes formula.

17.1.2.3 The Hull-White Model

As a concrete application of the option pricing formula of the previous section, we will now consider the case of interest rate options in the simplified Hull-White model (the extended Vasicek model). To this end recall that in the Hull-White model the Q-dynamics of $r(t)$ are given by

$$dr = (\theta(t) - ar)\, dt + \sigma\, dV(t)$$

We recall that we have an affine term structure

$$p(t, T) = e^{A(t,T) - B(t,T) r(t)}$$

where $A(t, T)$ and $B(t, T)$ are deterministic functions, and where $B(t, T)$ is given by

$$B(t, T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

The project is to price a European call option with date of maturity T_1 and strike price K, on an underlying bond with date of maturity T_2, where $T_1 < T_2$. In the notation of the aforementioned general theory, this means that $T = T_1$ and that $S(t) = p(t, T_2)$. We start by checking if the volatility, σ_Z, of the process

$$Z^T(t) = \frac{p(t, T_2)}{p(t, T_1)}$$

is deterministic. Inserting $p(t, T)$ into this gives

$$Z(t) \exp \{A(t, T_2) - A(t, T_1) - [B(t, T_2) - B(t, T_1)]\, r(t)\}$$

Applying the Itô formula to this expression, we get the Q-dynamics:

$$\begin{aligned} dZ(t) &= \frac{\partial}{\partial t} Z(t)dt + \frac{\partial}{\partial r} Z(t)dr + \frac{1}{2}\frac{\partial^2}{\partial r^2} Z(t)\sigma_Z^2 dt \\ &= Z(t)\,\{\ldots\}\, dt - Z(t) \cdot \{B(t, T_2) - B(t, T_1)\}\, \sigma\, dV \\ &= Z(t)\,\{\ldots\}\, dt + Z(t) \cdot \sigma_Z(t) dV \end{aligned}$$

That is,

$$\begin{aligned} \sigma_Z(t) &= -\sigma\, \{B(t, T_2) - B(t, T_1)\} = -\frac{\sigma}{a}\left\{1 - e^{-a(T_2 - t)} - 1 + e^{-a(T_1 - t)}\right\} \\ &= \frac{\sigma}{a}\left\{e^{-a(T_2 - t)} - e^{-a(T_2 - t)}\right\} = \frac{\sigma}{a} e^{at}\left\{e^{-aT_2} - e^{-aT_1}\right\} \end{aligned}$$

Thus σ_Z is in fact deterministic, so we may apply the option formula. We obtain the following result, which also holds for the Vasicek model.

The Hull-White bond option: In the Hull-White model the price, at $t = 0$, of a European call with strike price K, and time of maturity T_1, on a bond maturing at T_2 is given by the formula

$$\Pi(0, X) = p(0, T_2) \cdot N[d_1] - K \cdot p(0, T_1) \cdot N[d_2]$$

where

$$d_2 = \frac{\ln\left\{\frac{p(0,T_2)}{K \cdot p(0,T_1)}\right\} - \frac{1}{2}\Sigma^2}{\Sigma}$$

and

$$d_1 = d_2 + \Sigma$$

where

$$\begin{aligned} \Sigma^2 &= \int_0^T \sigma_z^2(s) ds = \frac{\sigma^2}{a^2}\left\{e^{-aT_2} - e^{-aT_1}\right\}^2 \int_0^{T_1} e^{2as} ds \\ &= \frac{\sigma^2}{2a^3}\left\{e^{-2aT_2} + e^{-2aT_1} - 2e^{-aT_1} e^{-aT_2}\right\}\left\{1 - e^{-aT_1}\right\} \\ &= \frac{\sigma^2}{2a^3}\left\{e^{-2a(T_2 - T_1)} + 1 - 2e^{-a(T_2 - T_1)}\right\}\left\{1 - e^{2aT_1)}\right\} e^{-2aT_1} \\ &= \frac{\sigma^2}{2a^3}\left\{1 - e^{-2aT_1)}\right\}\left\{1 - e^{-a(T_2 - T_1)}\right\}^2 \end{aligned}$$

We have used from the previous equation the formula

$$\Pi(0, X) = S(0) \cdot N[d_1] - K \cdot p(0, T) \cdot N[d_2]$$

and replaced the stock price $S(0)$ with the discount underlying bond $p(0, T_2)$ with maturity at T_2 and T with time to maturity for the bond option, T_1.

We end the discussion of the Hull-White model, by studying the pricing problem for a claim of the form

$$Z = \Phi\left(r(T)\right)$$

Using the T-bond as numeraire, we have by using the forward measure

$$\Pi(t, Z) = p(t, T) E_{t,r}^T \left[\Phi\left(r(T)\right)\right],$$

so we must find the distribution of $r(T)$ under Q^T, and to this we will use the volatility of the T-bond, and we obtain bond prices (under Q) as

$$dp(t, T) = r(t)p(t, T)dt + v(t, T)p(t, T)dV(t)$$

where the volatility $v(t, T)$ is given by

$$v(t, T) = -\sigma(t)B(t, T)$$

Thus, the Q^T-dynamics of the short rate are given by

$$dr = \left(\theta(t) - ar - \sigma^2 v(t, T)\right) dt + \sigma dV^T(t)$$

where V^T is a Q^T-Wiener process. We observe that, since $v(t, T)$ and $\theta(t)$ are deterministic, r is a Gaussian process, so the distribution of $r(T)$ is completely determined by its mean and variance under Q^T. Solving the aforementioned linear SDE gives us

$$r(T) = e^{-a(T-t)} r(t) + \int_t^T e^{-a(T-s)} \left[\theta(s) - \sigma^2 v(s, T)\right] ds + \sigma \int_t^T e^{-a(T-s)} dV^T(s)$$

We can now compute the conditional Q^T-variance of $r(T)$, $\sigma_r^2(t, T)$, as

$$\sigma_r^2(t, T) = \sigma^2 \int_t^T e^{-a(T-s)} ds = \frac{\sigma^2}{2a} \left\{1 - e^{-2a(T-t)}\right\}$$

Note that the Q^T-mean of $r(T)$, does not have to be computed at all, since we have

$$m_r(t,T) = E^T_{t,r}\left[r(T)\right] = f(t,T)$$

which can be observed directly from market data. Under Q^T, the conditional distribution of $r(T)$ is thus the normal distribution $N[f(t,T), \sigma_r(t,T)]$, and performing the integration we have the final result.

Given the previous assumptions, the price of the claim

$$X = \Phi\left(r(T)\right)$$

is given by

$$\Pi(t,X) = p(t,T)\frac{1}{\sqrt{2\pi\sigma_r^2(t,T)}}\int_{-\infty}^{\infty}\Phi(z)\exp\left\{-\frac{(z-f(t,T))^2}{2\sigma_r^2(t,T)}\right\}dz$$

17.1.2.4 The General Gaussian Model

In this section we extend our earlier results, by computing prices of bond options in a general Gaussian forward rate model. We specify the model (under Q) as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dV(t)$$

where V is a d-dimensional Q-Wiener process. We assume that the volatility vector function

$$\sigma(t,T) = \left[\sigma_1(t,T),\ldots,\sigma_p(t,T)\right]$$

is a **deterministic** function of the variables t and T. Using the bond price dynamics under Q given by

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)v(t,T)dV(t)$$

where the volatility is given by

$$v(t,T) = -\int_t^T \sigma(t,s)ds$$

we consider a European call option, with expiration date T_0 and exercise price K, on an underlying bond with maturity T_1 (where of course $T_0 < T_1$). In order to compute the price of the bond, we use the previous pricing formula, which means that we first have to find the volatility σ_{T_1,T_0} of the process

$$Z(t) = \frac{p(t, T1)}{p(t, T_0)}$$

in easy calculation shows that in fact

$$\sigma_{T_1,T_0} = v(t, T_1) - v(t, T_0) = -\int_{T_0}^{T_1} \sigma(t, s) ds$$

This is clearly deterministic. We now have the following pricing formula for prices of Gaussian forward rates. The price, at $t = 0$, of the option

$$X = \max\{p(T_0, T_1) - K, 0\}$$

is given by

$$\Pi(0, X) = p(0, T_1) \cdot N[d_1] - K \cdot p(0, T_0) \cdot N[d_2]$$

where

$$d_2 = \frac{\ln\left\{\frac{p(0,T_1)}{K \cdot p(0,T_0)}\right\} - \frac{1}{2}\Sigma^2_{T_0,T_1}}{\sqrt{\Sigma^2_{T_0,T_1}}}$$

and

$$d_1 = d_2 + \sqrt{\Sigma^2_{T_0,T_1}}$$

and

$$\Sigma^2_{T_0,T_1} = \int_0^{T_0} \|\sigma_{T_0,T_1}(s)\|^2 ds$$

18

Exotic Instruments

18.1 Some Exotic Instruments

For some exotic instruments, we can use the forward measure pricing described in the previous chapter. We will now describe methods of how we can calculate prices for such kinds of derivatives.

18.1.1 Constant Maturity Contracts

Constant maturity contracts are instruments using a floating rate, based on a swap index (i.e. the par rate of a generic swap). They can be valued using the forward measure technology based on term-structure models. This requires the mapping/calibration of a volatility structure with a term-structure model.

Let the value of such a CMS contract be $g(R_f(T_1, T_2))$ at the payday T_p, where

$$R_f\left(T_1,T_2\right) = \frac{1}{\tau}\left[\frac{1 - p\left(T_1, T_N\right)}{\sum_{i=2}^{n} p\left(T_1, T_i\right)}\right]$$

is the swap rate, having $p(T_1, T_2)$ equal to a zero-coupon bond price at T_1 of bond maturing at T_2. Here t is the reset period with T_1 as reset day and T_2 as payday. We will primarily study the calculations using the Hull and White model.

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If the dynamics of the instantaneous rate under the measure Q is

$$dr = (q(t) - kr)dt + sdz,$$

and the forward measure Q^T is defined by

$$\frac{dQ^T}{dQ} = \frac{\exp\left\{-\int_0^T r(s)ds\right\}}{E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\right]}$$

then the present value of the contract $PV(t)$ can be expressed as

$$PV(t) = p\left(t, T_p\right) E^{Q_{T_p}}\left[g\left(R_f\left(T_p, T_s, T_e\right)\right) r(t) = r\right]$$

where the expectation value ends up in an integral in $r(T_p)$

$$PV(t) = p\left(t, T_p\right) \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} g(r) \exp\left(\frac{-(r-m)^2}{2v}\right) dr$$

with

$$\begin{aligned} m &= E^{Q_{T_p}}\left[r(T_r)\right] \\ &= -\frac{\partial}{\partial s} \ln p(0,s)\Big|_{s=T_r} + \frac{\sigma^2}{2\kappa^2} e^{-\kappa T_r}\left[e^{-}\kappa T_r - e^{\kappa T_r} + e^{-\kappa T_p} e^{2\kappa T_r} - e^{-\kappa T_p}\right] \end{aligned}$$

and

$$v = Var^{Q^{T_p}}\left[r(T_r)\right] \frac{\sigma^2}{2\kappa} e^{-\kappa T_r}\left(e^{\kappa T_r} - e^{-\kappa T_r}\right)$$

If instead using the Ho and Lee model, we have

$$m = -\frac{\partial}{\partial S} \ln p(0,s)\Big|_{s=T_r} + \sigma^2(T_r^2 - T_p T_r)$$

and

$$v = \sigma^2 T_r$$

We can here use Romberg method to calculate the integral.

18.1.2 Compound Options

By compound options, we mean options on option-style instruments. These include:

- options on options
- options on caps/floors
- options on free defined cash flows where there is at least one optional cash flow.

To make an accurate valuation possible, these contracts must also be mapped/calibrated to a volatility structure with a term-structure model. The underlying instrument should here use the same volatility structure as the compound option, regardless of how the instrument is underlying.

Compound options can be valued using the forward measure technique described under constant maturity contracts. With this technique, the valuation is carried out by integrating the product of the payout function at expiry and a density function and then discounting the result.

For compound options, no swap rate is involved and there is no need to calculate the expectation value of $r(T_r)$ under the forward measure Q^{Tp} (i.e. T_r here equals T_p), which simplifies the calculations. The present value $PV(t)$ of a compound option is

$$PV(t) = p\left(t, T_p\right) \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} g(r) \exp\left(\frac{-(r-m)^2}{2v}\right) dr$$

with T_p = option expiration day,

$$g(r) = g\left(r\left(T_p\right)\right)$$

is the boundary condition, including the value at $r = r(T_p)$ of the underlying option/cap/floor defined cash flow

$$m = E^{Q^{Tp}}\left[r\left(T_p\right)\right] = -\frac{\partial}{\partial s} \ln p(0, s)\bigg|_{s=T_p}$$

Using Hull and White

$$v = Var^{Q^{Tp}}\left[r\left(T_p\right)\right] = \frac{\sigma^2}{2\kappa}\left(1 - \exp\left(-2\kappa T_p\right)\right)$$

or Ho and Lee

$$v = \sigma^2 T_p$$

Romberg's method is used when calculating the integral.

18.1.3 Quanto Contracts

Quanto contracts have floating cash flows where the reference rate is a rate index in a currency other than the payout currency. Such quanto products are:

- Differential swaps
- Quanto caps/floors
- Quanto bond options
- Swaptions

These products can be valued according to term-structure models Ho and Lee or Hull and White.

The model is a multi-factor model in the sense that the domestic rate, $r(t)$, the foreign rate, $y(t)$, and the exchange rate are modelled as stochastic processes:

$$dr(t) = \alpha_r dt + \sigma_r dZ$$

and

$$dy(t) = \alpha_y dt + \sigma_y dW^F$$

For the exchange rate $S(t)$, we have the following differential equation in the domestic world

$$\frac{ds(t)}{S(t)} = (r - y)dt + \sigma_s dX$$

The T-forward exchange rate is

$$f_S(t) = \frac{S(t)q^T(t)}{p^T(t)}$$

For valuation, the correlation between the domestic interest rate and the foreign interest rate, ρ as well as the correlation between the foreign interest rate and the exchange rate, δ is needed.

In addition, a quanto option needs three volatility structures for the valuation: the volatility of the domestic rate of the option, the volatility of the foreign rate of the underlying asset and the volatility of the exchange rate of the underlying currency.

A quanto swaption is defined by selecting a different instrument currency for the option than the underlying swap currency. The type of quanto bond option that is valued as the difference between the price of the foreign bond, $q(T)$, and a fixed amount in the domestic currency (strike K):

$$payout = \max\left(q\left(T\right) - K, 0\right),$$

with the payout in the domestic currency.

The explicit formulas for each contract type are given next.

18.1.3.1 Differential Swaps

Differential swaps are valued using the following formula:

$$PV\left(t\right) = p^{t1}\left(t\right)\left[\frac{1}{q^{t1}\left(t_0\right)} - \frac{1}{p^{t1}\left(t_0\right)}\right] + \sum_{i=2}^{n}\left(D_{t_{i-1},t_i}\left(t\right) - p^{t_{i-1}}\left(t\right)\right)$$

where

- $p^T(t)$ is the value of a zero-coupon bond paying out 1 unit of the domestic currency at time T,
- $q^T(t)$ is the value of a zero-coupon bond paying out 1 unit of the foreign currency at time T

$$D_{T,\tau}(t) = \frac{p^{\tau}(t)q^{\tau}(t)}{q^{\tau}(t)}a(t)$$

$a(t) = e^{cov(t,T,\tau)}$ is the "correction factor" that takes the quanto effect into account and where cov(...) is model dependent.

In Ho and Lee we have

$$\text{cov}(t,T,\tau) = \sigma_y(\tau - T)(T - t)\left[[\delta\sigma_s - \sigma_y\tau + \rho\sigma_r T] + \frac{\sigma_y - \rho\sigma_r}{2}(T + t)\right]$$

and in Hull and White we have

$$\text{cov}(t,T,\tau) = C(T,\tau)\cdot[I_1 + I_2 + I_3]$$

where

$$C(T,\tau) = \frac{\sigma_y}{\kappa_y}\left[e^{-\kappa_y(\tau-T)} - 1\right]$$

$$I_1 = \frac{\delta\sigma_s}{\kappa_y}\left(1 - e^{-\kappa_y(T-t)}\right)$$

$$I_2 = \frac{\sigma_y}{\kappa_y}\left[\frac{1 - e^{-\kappa_y(T-t)}}{\kappa_y} - \frac{1 - e^{-2\kappa_y(T-t)}}{2\kappa_y}\right]$$

and

$$I_3 = \frac{\rho\sigma_r}{\kappa_r}\left[\frac{1 - e^{-\kappa_y(T-t)}}{\kappa_y} - \frac{1 - e^{-(\kappa_y+\kappa_r)(T-t)}}{\kappa_y + \kappa_r}\right]$$

δ is the correlation between the foreign interest rate and the exchanged rate and ρ is the correlation between the domestic and foreign short interest rates.

The time t_0 does not have to be the starting time of the contract, it can be any reset date.

Quanto Caps/Floors

Quanto caps/floors have a present value that equals the sum of the present value of each caplet. The caplet value is

$$Caplet(t) = p^T(t)\left[\frac{a(t).q^T(t)}{q^\tau(t)}\Phi(d_+) - (1 + (\tau - T)\cdot R_{CAP})\,\Phi(d_-)\right]$$

where

$$\Phi(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}\,dx,$$

$$d_\pm = \frac{\ln\left(\dfrac{a(t)\cdot q^T(t)}{q^\tau(t)\cdot(1 + (\tau - T)\cdot R_{CAP})}\right)}{v(t)} \pm \frac{v(t)}{2},$$

$$v^2(t) = \int_t^T var_s\left[\frac{df(s)}{f(s)}\right]$$

and

$$f = \frac{a \cdot q^T}{q^t}$$

The rest of the variables used are the same as for differential swaps.

Quanto Bond Options

According to a result obtained by Jamshidian, an option on a portfolio of zero-coupon bonds can be valued as a portfolio of options on zero-coupon bonds

$$C(t) = \max\left(0, q_{coupon}(T) - K\right) = \sum_{i=1}^{n} c_i \max\left(0, q^{T_i}(T) - D_i\right)$$

If all bond prices are continuously decreasing functions of the short interest rate y, there is a value of y where the value of the coupon bond equals the strike price K. Let us call this value $y*$.

The option will only be exercised if the value of the short rate is below $y*$. With $y*$ and the formula for bonds as a function of the interest rate, discount factors D_i can be calculated from the exercise date T to the different coupon dates T_i.

The value of the *ith* option on a zero-coupon bond is then:

$$c_i \max\left(0, q^{T_i}(T) - D_i\right) = p^T(t)\left[\frac{q^{T_i}(t)a(t)}{q^T(T)}\Phi(d_+) - D_i\Phi(d_-)\right]$$

where the parameters are the same as the previous equation and

$$\Phi(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}\,dx$$

$$d_{\pm}(t) \equiv \frac{\ln\left(f(t)/D_i\right)}{v(t)} \pm \frac{v(t)}{2}$$

$$v^2(t) = \int_t^T var_s\left[\frac{df(s)}{f(s)}\right]$$

and

$$f = \frac{q^{T_i} \cdot a}{q^T}$$

19

The Black Model

19.1 Pricing Interest Rate Options Using Black

The Black-76 modified Black-Scholes model has become the standard model for valuing over-the-counter (OTC) interest rate options, caps, floors and European swaptions. The formula was originally developed to price options on forwards and assumes that the underlying asset is lognormal distributed.

Black's formula is often recalled as a special case of the Black-Scholes one, but it is in reality a generalization: if one applies Black to an equity option, where $S(T) = S(0)e^{rT}$ one gets the Black-Scholes formula. But the Black-Scholes formula also holds when the spot has complex dynamics and there is no replication of the forward with the spot.

When used to price a cap, for example, the underlying forward rates of the cap are thus assumed to be lognormal. Similarly, when used to price a swaption (an option on a swap), the underlying swap rate is assumed to be lognormal.

The lognormality can be justified when pricing cap/floors and swaptions independently (Jamshidan (1996), Miltersen, Sandmann and Sondermann (1997)). Still, a simultaneous valuation of both a cap and a swaption with the Black formula is theoretically inconsistent. Both the forward rate of the cap and the swap rate cannot be lognormal simultaneously. However, the great popularity of this model for pricing both caps and swaptions indicates that any problems due to this inconsistency are negligible in an economic sense. Traders use to adjust this inconsistency by adjusting the volatility based on experience for the particular market in which they operate.

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_19

The same is true with bond prices and swap rates; they cannot be lognormal at the same time. For instance, if the bond price is assumed to be lognormal, the continuously compounded swap rate must be normally distributed.

19.1.1 Par and Forward Volatilities

Volatilities quoted on the market are par volatilities applied to some generic instruments. As an example, we see next the cap volatilities based on 3-month USD Libor.

Cap maturity	Volatility [%]
1 yr	10.37
2 yr	12.87
3 yr	14.12
4 yr	15.12
5 yr	15.25
7 yr	15.13
10 yr	14.88

The par or average volatility of 10.37 % would apply for all the three caplets in a 1-year cap (normally, there is no option on the first Libor fixing), the par volatility of 12.87 % would apply to all seven caplets in a 2-year cap and so on.

This makes the quotation in terms of volatility very easy. However, the first three caplets in the 2-year cap must be identical to the caplets in the 1-year cap. Therefore, it would seem sensible that they should always be priced using the same volatilities.

Let us define "forward" volatility[1] as the volatility that would apply to a single caplet. The forward volatility for the very first caplet would be the volatility of a 3-month rate, which will be fixed in 3 months' time. The forward volatility for the second caplet would be the volatility again of a 3-month rate, but this time fixed in 6 months' time, and so on.

[1] Some does not use the confusing, expression "forward volatility". Since volatility is not a traded asset, it can therefore be either present or future, but not forward. Similarly, one might not use the term "forward-forward volatility", which probably is used to mean the future volatility of a forward quantity.

In the cap market, forward volatilities are derived from quoted par volatilities. Let T denote the maturity of the Tth generic cap for which we have par volatilities. Define the price of a cap of maturity T using par volatility V_T as:

$$C_T = \sum_{t \le T} c_t(V_T)$$

where $c_t(V_T)$ is the price of a single caplet of maturity t. For arbitrage reasons, the same cap using the forward volatility curve should have the same price:

$$C_T = \sum_{t \le T} c_t(V_T) = \sum_{t \le T} c_t(v_t)$$

where v, is the single period forward volatility. Hence, we can define a recursive relationship:

$$C_T = C_{T-1} = \sum_{T-1 \le t \le T} c_t(v_t)$$

A crude but common assumption is to set v, equal to a constant for $T—1 < t < T$. Then we can calculate sequentially the forward volatilities. Also, remember that par volatilities are most appropriate for ATM options.

To estimate the forward volatility curve we use the following process:

1. Guess a forward piece-wise constant volatility curve.
2. Price each of the caps using this curve.
3. Adjust each segment of the volatility curve, starting at the short end in a bootstrapping fashion, so that the price of each cap based on the forward volatility curve matches the original price.

We end up with a curve like Fig. 19.1.

Whilst such a curve is arbitrage free, a smoother curve would be better. The approach may use an optimization technique using a smoothness criterion: $\sum (\sigma_t - \sigma_{t-1})^2$ which has to be minimized whilst still being arbitrage free.

The result might look like. (Fig. 19.2)

Notice the very typical "humped" structure over the 2- to 5-year region; this is likely because of the traditional high demand by end-users for interest rate protection over those maturities.

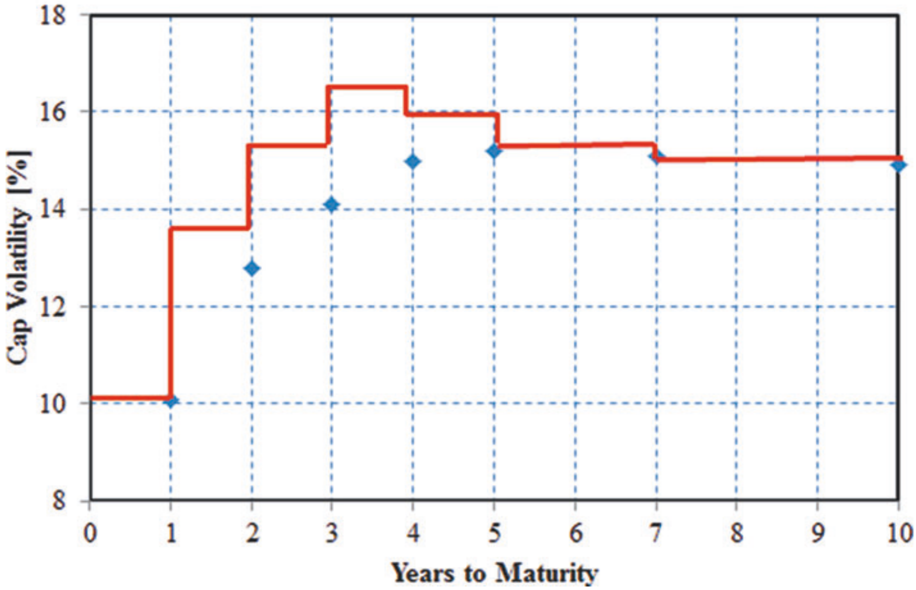

Fig. 19.1 The initial caplet volatility curve. The dots represent the cap volatility

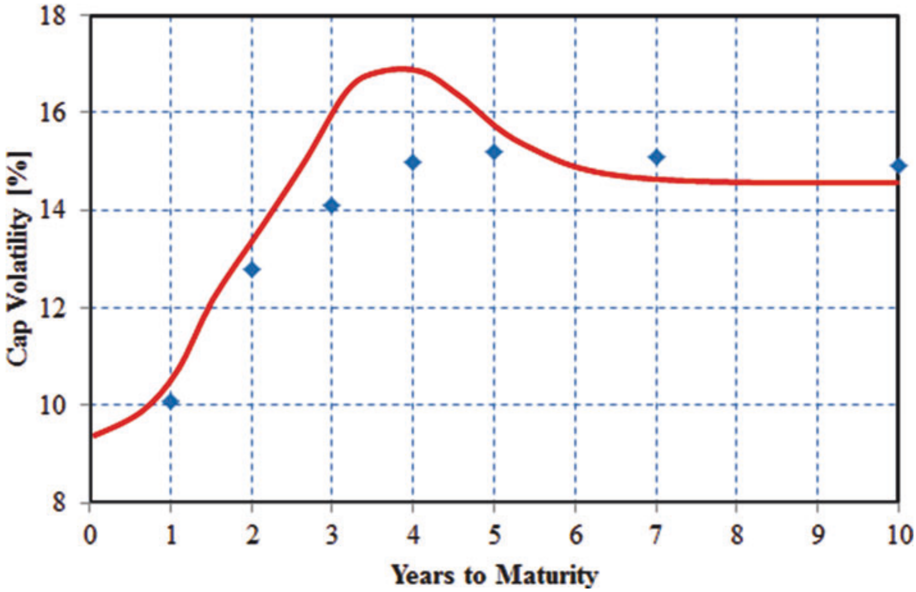

Fig. 19.2 The optimized bootstrapped caplet volatility

Remark that the volatilities cannot be linearly interpolated with respect to time. Instead, you need to interpolate the squared volatility multiplied with time. This gives us the following interpolation formula

$$\sigma^2(t) \cdot t = \sigma^2(T_1) \cdot T_1 + \frac{\sigma^2(T_2) \cdot T_2 - \sigma^2(T_1) \cdot T_1}{T_2 - T_1}(t - T_1)$$

giving

$$\sigma(t) = \sqrt{\sigma^2(T_1) \cdot \frac{T_1}{t} + \frac{\sigma^2(T_2) \cdot T_2 - \sigma^2(T_1) \cdot T_1}{T_2 - T_1}\left(1 - \frac{T_1}{t}\right)}$$

These curves are often combined with statistical confidence bands. In practice it is found that volatilities do revert to a long-run level (as suggested by the ARCH model), which means that the confidence bands are wider at the short end than at the longer end. The bands are often called "volatility cones" due to their shape, and are used by traders to imply the likely movement of volatility through time.

We have just derived forward volatilities from a single ATM par volatility curve. It is however, common practice to use volatility surfaces, that is, a matrix of strike vs. forward start date, when pricing and valuing caps and floors. This allows the smile effect to be incorporated. IR options on 3-month Libor are the most common, probably reflecting the fact that one can get exchange-traded options on 3-month deposit futures for hedging. Therefore, the most liquid volatility surface would also be on 3-month Libor, and volatility surfaces for other tenors represented by an offset surface from the 3-month one. A more complete approach therefore would be to model the entire two-dimensional surface. This surface is likely to contain gaps due to missing maturities and also missing volatilities for particular strikes.

19.1.2 Caps and Floors

As we have seen, an interest-rate cap consists of a series of individual European call options, called caplets. Each caplet can be priced by using a modified version of the Black-76 formula. This is accomplished by using the implied forward rate, F, at each caplet maturity as the underlying asset. The price of the cap is the sum of the price of the caplets that make up the cap. Similarly, the value of a floor is the sum of the sequence of individual put options, called floorlets that make up the floor.

As we know, the Black formula is

$$P_{call} = e^{-rT}(F \cdot N(d_1) - K \cdot N(d_2))$$
$$P_{put} = e^{-rT}(K \cdot N(-d_2) - F \cdot N(-d_1))$$

where F is the forward price and

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Consider the pricing model of a caplet whose ceiling rate is L_c. The holder of the cap receives at time t_i an amount equal to $\alpha_i \max\{L_{i-1}(T_{i-1}) - L_c, 0\}$. The present value of this payment at L_{i-1} is

$$\frac{\alpha_i}{1 + \alpha_i L_{i-1}(T_{i-1})} \max\{L_{i-1}(T_{i-1}) - L_c, 0\} = \max\left\{1 - \frac{1 + \alpha_i L_c}{1 + \alpha_i L_{i-1}(T_{i-1})}, 0\right\}$$

Remember the value of a pure discount bond

$$p(T_{i-1}, T_i) = \frac{1}{1 + \alpha_i L_{i-1}(T_{i-1})}$$

Therefore, the quantity

$$\frac{1 + \alpha_i L_c}{1 + \alpha_i L_{i-1}(T_{i-1})}$$

can be considered as the value at time T_{i-1} of a discount bond that pays $1 + \alpha_i L_c$ at time T. Hence, the payoff in the aforementioned max{.} is the same as that from a put option with expiration date T_{i-1} on a bond with maturity time T_i. The par value of the bond is $1 + \alpha_i L_c$ and the strike price of the put option is unity. Therefore, an interest rate cap can be considered as a portfolio of European put options on discount bonds.

The time-t value of the caplet can then be expressed as:

$$C_i(t, T_{i-1}, T_i) = p(t, T_i) E^t_{Q_{T(i-1)}} [\alpha_i \max\{L_{i-1}(T_{i-1}) - L_c, 0\}]$$

Since $L_{i-1}(T_{i-1})$ is $\mathcal{F}_{T(i-1)}$-measurable, we may write

$$\begin{aligned} C_i(t, T_{i-1}, T_i) &= p(t, T_{i-1}) E^t_{Q_{T(i-1)}} \left[p(T_{i-1}, T_i)\alpha_i \max\{L_{i-1}(T_{i-1}) - L_c, 0\}\right] \\ &= p(t, T_{i-1}) E^t_{Q_{T(i-1)}} \left[\max\{1 - (1 + \alpha_i L_c)\, p(T_{i-1}, T_i), 0\}\right] \end{aligned}$$

If we finally assume that the bond prices $p(t, T)$, under the risk-neutral measure Q, follow a general Gaussian process (as given next), the time-t value of the caplet is given by

$$C_i(t, T_{i-1}, T_i) = p(t, T_{i-1})N(-d_2^{(i)}) - (1 + \alpha_i L_c)\, p(t, T_{i-1})N(-d_1^{(i)}),\ t < T_{i-1}$$

where

$$d_2^{(i)} = \frac{\ln\left\{\frac{(1+\alpha_i L_c)p(t,T_i)}{p(t,T_{i-1})}\right\} - \frac{1}{2}\Sigma_i^2(t)(T_{i-1} - t)}{\Sigma_i(t)\sqrt{T_{i-1} - t}}$$

and

$$d_1 = d_2 + \Sigma_i(t)\sqrt{T_{i-1} - t}$$

and

$$\Sigma_i^2(t) = \frac{1}{T_{i-1} - t} \int_t^{T_{i-1}} \int_{T_{i-1}}^{T_i} \sum_{j=1}^{m} \sigma_F^2(s, u)\,du\,ds$$

The previous analytical expression is complicated. This is because we have based the model on unobservable instantaneous forward rates. This makes the model difficult to implement and calibrate the volatility to market cap data. This motivates to use a market model.

We therefore assume that the underlying forward Libor process is lognormal distributed with zero drift under some "market probability" Q_m. In its simplest form we let the volatility denote the constant Black volatility of the forward Libor process

$$dL_{i-1}(t) = L_{i-1}(t)\sigma_{i-1}^L dW_t^m$$

where W_t^m is a Brownian process under Q_m. The Black formula for the time-t value of the caplet that pays $\alpha_i \max\{L_{i-1}(T_{i-1}) - L_c, 0\}$ at time T_i is given by

$$\begin{aligned} C_i^{Black}(t, T_{i-1}, T_i) &= \alpha_i p(t, T_i) E_{Q_m}^t \left[\max\{L_{i-1}(T_{i-1}) - L_c, 0\}\right] \\ &= \alpha_i p(t, T_i)\left[L_{i-1}(t)N(d_1^{i-1}) - L_c N(d_2^{i-1})\right] \end{aligned}$$

where

$$d_1^{i-1} = \frac{\ln\left\{\frac{L_{i-1}(t)}{L_c}\right\} - \frac{1}{2}\left(\sigma_{i-1}^L\right)^2 (T_{i-1} - t)}{\sigma_{i-1}^L\sqrt{T_{i-1} - t}}$$

and

$$d_2 = d_1 - \sigma_{i-1}^L \sqrt{T_{i-1} - t}$$

We can simplify this as

$$C(t) = \frac{N \cdot \tau}{1 + F \cdot \tau} e^{-r(T-t)} \left[F \cdot N(d_1) - K \cdot N(d_2) \right]$$

where τ is the tenor, N the face value and F the implied forward rate between time t and at the caplets maturity, T. Similarly, for a floorlet, we have

$$F(t) = \frac{N \cdot \tau}{1 + F \cdot \tau} e^{-r(T-t)} \left[K \cdot N(-d_2) - F \cdot N(-d_1) \right]$$

Example 19.1.1

We will illustrate the cap value in a simple example. Suppose we have a caplet, with 6 months to expiry on a 182-day forward rate and a face value of 100 million. The 6-month forward rate is 8% (with act/360 as day-count), the strike is 8%, the risk-free interest rate 7%, and the volatility of the forward rate 28% per annum.

$$F = 0.08, K = 0.08, T = 0.5, r = 0.07, \sigma = 0.28.$$

$$d_1 = \frac{\ln(0.08/0.08) + (0.28^2/2)0.5}{0.28\sqrt{0.5}} = 0.0990, d_2 = d_1 - 0.28\sqrt{0.5} = -0.990$$

$$N(d_1) = 0.5394, \quad N(d_2) = 0.4606$$

$$C(t) = \frac{10^9 \cdot \frac{182}{360}}{1 + 0.08 \cdot \frac{182}{360}} e^{-0.07 \cdot 0.5} \left[0.08 \cdot N(d_1) - 0.08 \cdot N(d_2) \right] = 295.995$$

19.1.3 Swaps and Swaptions

It is usual to distinguish between the two different types of swaptions:

- Payer swaptions. The right but not the obligation to pay fixed rate and receive floating rate in the underlying swap.
- Receiver swaptions. The right but not the obligation to receive fixed rate and pay floating rate in the underlying swap.

Most swaptions (about 90 %) is of European types and are normally priced by using the forward swap rate as input in the Black-76 option-pricing model. The Black-76 value is multiplied by a factor adjusting for the tenor of the swaption, as shown by Smith (1991). This is the practitioner's benchmark swaption model. As illustrated by Jamshidan (1996), the model is arbitrage-free under the assumption of a lognormal swap rate.

To derive a formula for a swaption we will start by studying a forward-starting swap. That is a swap that starts at a future time where we exchange floating against fixed cash flows. A $T_n \times (T_N - T_n)$ swap means a swap that starts at time T_n and have maturity at time T_N.

Denote the reset days for any swap as: T_0, T_1, T_N and define α_i as $T_i - T_{i-1}$. The holder of a forward-starting $T_n \times (T_N - T_n)$ payer swap with tenor $T_N - T_n$ receives fixed payments at times $T_{n+1}, T_{n+2}, \ldots, T_N$ and pays at the same times floating payments.

For each period $[T_i, T_{i+1}]$ the Libor rate $L_{i+1}(T_i)$ is set at time T_i and the floating leg $\alpha_{i+1}L_{i+1}(T_i)$ is received at T_{i+1}. For the same period the fixed leg $\alpha_{i+1}F$ is paid at T_{i+1} where F is the (fixed) swap rate.

The arbitrage-free value at $t < T_n$ of the floating payment made at T_i is given by $p(t, T_i) - p(t, T_{i+1})$. The total value of the floating legs at time t for $t \leqslant T_n$ equals

$$\begin{aligned}\sum_{i=n+1}^{N} \alpha_i \cdot f(t, T_i) \cdot p(t, T_{i-1}) &= \sum_{i=n}^{N-1} \alpha_{i+1} \cdot \frac{1}{\alpha_{i+1}} \frac{p(t, T_i) - p(t, T_{i+1})}{p(t, T_i)} \cdot p(t, T_i) \\ &= \sum_{i=n}^{N-1} \left[p(t, T_i) - p(t, T_{i+1}) \right] = p(t, T_n) - p(t, T_N) \\ &= p_n(t) - p_N(t)\end{aligned}$$

where we have used that the forward rate is given by

$$p(0, t_i) = p(0, t_{i-1}) \cdot \frac{1}{1 + \alpha_i f(t_{i-1}, t_i)} \Rightarrow f(t_{i-1}, t_i) = \frac{1}{a_i} \frac{p(0, t_i) - p(0, t_{i-1})}{p(0, t_i)}$$

If we go back to the FRN, we remember that the value at the starting day is the same as the face value = 1. In a swap, we do not have any final payment of the face value. This gives the swap value at the starting day $t = 0$, as $1 - p(0, T)$. Between to resets we therefore must have the swap value as: $p(t, t_0) - p(t, T)$ where t_0 is the time for the next reset day. This explains the previous formula.

The total value at time t for the fixed side equals

$$\sum_{i=n}^{N-1} F \cdot p(t, T_{i+1}) \alpha_{i+1} = F \sum_{i=n+1}^{N} \alpha_i p_i(t)$$

where F is called the swap rate. This is a **par rate** since it makes the price of the swap to be equal zero when entering the swap contract. Therefore, the total value of the payer swap is given by

$$PS_n^N(t,F) = p_n(t) - p_N(T) - F \sum_{i=n+1}^{N} \alpha_i p_i(t)$$

We therefore define the **forward swap rate** (at par) $R_n^N(t)$ of the $T_n \times (T_N - T_n)$ swap as the value of F for which the total value earlier is zero. That is,

$$R_n^N(t) = F = \frac{p_n(t) - p_N(t)}{\sum\limits_{i=n+1}^{N} \alpha_i p_i(t)}$$

Therefore we also define for each pair n, k with $n < k$, the process

$$S_n^k(t) = \sum_{i=n+1}^{N} \alpha_i p_i(t)$$

as the **accrual factor** or **the value of a basis point** (also called the *level, DV01* Dollar Value change in a shift, *PV0l* Present Value change in a shift, *annuity* or *numerical duration* of the swap).

We then express the swap value as

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{S_n^N(t)}$$

In the market there are no quoted prices for different swaps. Instead there are market quotes for the par swap rates. We see that we can easily compute the arbitrage-free price for a payer swap with the strike rate K as

$$PS_n^N(t, R_n^N(t), K) = \left(R_n^N(t) - K\right) S_n^N(t)$$

A **payer swaption** is then a contract given by;

$$P_n^N(t, R_n^N(T_n), K) = \max\left(R_n^N(T_n) - K, 0\right) S_n^N(T_n)$$

This contract gives the holder the right to enter a swap contract at time T_n with swaption strike (fixed rate) K.

Under the numeraire process S_n^N a payer swaption is then a call option on R_n^N with strike price K. The value of this contract is given by the Black-76 formula:

$$P_n^N(t) = S_n^N(t)\left\{R_n^N(t)N(d_1) - KN(d_2)\right\}$$

where

$$d_1 = \frac{\ln\left\{\frac{R_n^N(t)}{K}\right\} + \frac{1}{2}\sigma_{n,N}^2\,(T_n - t)}{\sigma_{n,N}\sqrt{T_n - t}},$$

$$d_2 = d_1 - \sigma_{n,N}\sqrt{T_n - 1}$$

The constant $\sigma_{n,N}$ is known as the **Black volatility**. Given a market price for a swaption, the Black volatility implied by the Black formula is referred as the **implied Black Volatility**.

We can also write the Black formula as

$$\begin{aligned} P_n^N(t) &= S_n^N(t)\left\{R_n^N(t)N(d_1) - K \cdot N(d_2)\right\} \\ &\equiv \sum_{i=n+1}^{N} \alpha_i p_i(t) \cdot \left\{R_n^N(t) \cdot N(d_1) - K \cdot N(d_2)\right\} \\ &= \varphi(t) \cdot \{F \cdot N(d_1) - K \cdot N(d_2)\} \end{aligned}$$

or

$$P_n^N(t) = \frac{p_n(t) - p_N(t)}{R_n^N(t)}\left\{R_n^N(t) \cdot N(d_1) - K \cdot N(d_2)\right\}$$

Here the function $\varphi(t)$ is a discount function. If we denote the forward swap rate between t_n and t_N as F, we have at t_n

$$\begin{aligned} p_n(t) - p_N(t) &\equiv p(t, t_n) - p(t, t_N) = p(t, t_n)\{1 - p(t_n, t_N)\} \\ &= p(t, t_n)\left\{1 - \frac{1}{(1 + f(t_n, t_N))^{t_N - t_n}}\right\} \equiv p(t, t_n)\left\{1 - \frac{1}{(1 + F)^{t_N - t_n}}\right\} \end{aligned}$$

If we now let $T = t_n$ be the maturity of the swaption, F the forward swap rate (aforementioned $R_n^N(t)$) and introducing m reset days per year (the frequency), we finally have, where PS denote a payer swaption, and RS is a receiver swaption

$$PS = \frac{1 - \frac{1}{(1+F/m)^{\tau \cdot m}}}{F} e^{-rT} \left[F \cdot N(d_1) - K \cdot N(d_2) \right]$$

$$RS = \frac{1 - \frac{1}{(1+F/m)^{\tau \cdot m}}}{F} e^{-rT} \left[K \cdot N(-d_2) - F \cdot N(-d_1) \right]$$

$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2 \cdot T}{\sigma \sqrt{T}}, d_2 = d_1 \sigma \sqrt{T}$$

where

τ = Tenor of swap in years (time between swaption maturity and swap maturity).

F = Forward rate of the underlying swap.

K = Strike rate of the swaption.

r = Risk-free interest rate.

T = Time to swaption expiration in years.

σ = Volatility of the forward-starting swap rate.

m = Compounding's per year in swap rate.

We also used continuous compounding, that is, $p(t, T) = e^{-r(T-t)}$

Example 19.1.2

Consider a 2-year payer swaption on a 4-year swap with semi-annual compounding. The forward swap rate of 7% starts two years from now and ends 6 years from now. The strike is 7.5%, the risk-free interest rate is 6%, and the volatility of the forward-starting swap rate is 20% per annum.

$$\tau = 4.0, m = 2, F = 0.07, K = 0.075, T = 2, r = 0.06, \sigma = 0.20.$$

$$d_1 = \frac{\ln(0.07/0.075) + (0.20^2/2) \cdot 2}{0.20\sqrt{2}} = -0.1025, d_2 = d_1 - 0.20\sqrt{2} = -0.3853$$

$$N(d_1) = 0.4592, \quad N(d_2) = 0.3500$$

$$c = e^{-0.06 \cdot 2} \left[0.07 \cdot N(d_1) - 0.075 \cdot N(d_2) \right] = 0.5227\%$$

With a semi-annual forward swap rate, the upfront value of the payer swaption in per cent of the notional is

$$c \cdot \left[\frac{1 - \frac{1}{(1+0.07/2)^{4 \cdot 2}}}{0.07} \right] = 1.7964\%$$

19.1.3.1 The Greeks

If we return to the Black formulas we can first see that the Greeks are more complicated to calculate than the Greeks in the Black-Scholes formula. Delta for instance can be defined as the derivative of the *Premium* (swaption value) with respect to the forward rate F. Other definitions of delta is as the derivative with respect to the present value of the fixed leg of the swap or as the derivative with respect to the annuity. We can also calculate a delta by shifting the yield curve.

Let's try to explain the general difficulties when defining a delta; options (including swaptions) pricing is based on *models*. Those models have *parameters*. The market, on the other hand, takes this in the opposite direction. There are market prices and the model parameters are selected (calibrated) to achieve the market prices. In this context all the models provide the same (market) prices.

Regarding *delta,* the situation is different. The figures are not calibrated, they are the consequences of the price calibration and intrinsic model dynamic for the rates. You can therefore refer to two different types of deltas. The *theoretical* delta is a ratio. It is often referred to (in particular by Rebonato) as *in-the-model* delta (or hedging). The DV01 is obtained by shifting one rate (or the entire curve) by one basis point. This is (often) incompatible with the model used for the pricing. For that reason it is known as *out-of-the-model* delta (hedging).

In the formula

$$P_n^N(t) = \varphi(t) \cdot \{F \cdot N(d_1) - K \cdot N(d_2)\}$$

we have seen that there is a "hidden" F. Some books and articles give delta as:

$$\Delta = \varphi(t) \cdot N(d_1)$$

alternatively,

$$\Delta = N(d_1)$$

However, by making the approximation:

$$\frac{\Delta P_n^N(t)}{\Delta F} = \frac{P_n^N(t, F + dF) - P_n^N(t, F - dF)}{2 \cdot dF}$$

it is easy to prove that none of the previous deltas gives the correct value. The calculation of delta is quite messy. Remember the calculation of delta for a stock option. We know that d_1 and d_2 also are functions of F. For the stock option d_1 and d_2 also includes the risk-free rate.

To calculation of delta as the derivative with respect to F, we use:

$$PS = \frac{1 - \frac{1}{(1+F/m)^{\tau \cdot m}}}{F} e^{-rT} \left[F \cdot N(d_1) - K \cdot N(d_2)\right]$$

and

$$d_1 = \frac{\ln(F/K) + \frac{1}{2}\sigma^2 \cdot T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

First, we have

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F} = \frac{1}{F \cdot \sigma \cdot \sqrt{T}}$$

then

$$\Delta = \frac{\partial PS}{\partial F} = \frac{\partial}{\partial F}\left\{\frac{1 - \frac{1}{(1+F/m)^{\tau \cdot m}}}{F} e^{-rT}\left[F \cdot N(d_1) - K \cdot N(d_2)\right]\right\}$$

$$= e^{-rT} \frac{\partial}{\partial F}\left\{f(F) \cdot \left[N(d_1) - \frac{K}{F} \cdot N(d_2)\right]\right\}$$

$$\text{where} \quad \left\{f(F) = 1 - \frac{1}{(1+F/m)^{\tau \cdot m}}\right\}$$

$$= e^{-rT} \cdot \left\{\left[N(d_1) - \frac{K}{F} \cdot N(d_2)\right]\frac{\partial f(F)}{\partial F} + f(F) \cdot \frac{\partial}{\partial F}\left[N(d_1) - \frac{K}{F} \cdot N(d_2)\right]\right\}$$

We now have two derivatives, and they are calculated as

$$\frac{\partial}{\partial F}\left[N(d_1) - \frac{K}{F} \cdot N(d_2)\right] = \frac{\partial N(d_1)}{\partial F} + \frac{K}{F^2}N(d_2) - \frac{K}{F}\frac{F}{K}\frac{\partial N(d_1)}{\partial F} = \frac{K}{F^2}N(d_2)$$

and

$$\frac{\partial f(F)}{\partial F} = \frac{\partial}{\partial F}\left\{-(1+F/m)^{-\tau \cdot m}\right\} = \frac{\tau}{(1+F/m)^{\tau \cdot m}}$$

So

$$\Delta = e^{-rT}\left\{\left(1 - \frac{1}{(1+F/m)^{\tau \cdot m}}\right)\frac{K}{F^2}N(d_2) + \frac{\tau}{(1+F/m)^{\tau \cdot m+1}}\left[N(d_1) - \frac{K}{F} \cdot N(d_2)\right]\right\}$$

where we used

$$\frac{\partial N(d(F))}{\partial F} = \frac{\partial}{\partial F}\int_{-\infty}^{d(F)} \phi(x)dx = \frac{\partial}{\partial F}\left[\Phi(x)\right]_{-\infty}^{d(F)} = \frac{\partial d(F)}{\partial F}\left[\phi(d(F)) - \phi(-\infty)\right]$$
$$= \frac{\partial d(F)}{\partial F}\phi(d(F)) = \frac{\partial d(F)}{\partial F}N'(d(F)) = \frac{1}{F\sigma\sqrt{T}}N'(d(F)))$$

and

$$N'(d_2) = \frac{\partial}{\partial F}N(d_1 - \sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}}\frac{\partial d_1}{\partial F}e^{-\left(d_1-\sigma\sqrt{T}\right)^2/2}$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{F\sigma\sqrt{T}}e^{-(d_1)^2/2}e^{d_1\sigma\sqrt{T}}e^{-\sigma^2T/2} = e^{d_1\sigma\sqrt{T}}e^{-\sigma^2T/2}N'(d_1)$$
$$= e^{\ln(F/K)+\sigma^2T/2}e^{-\sigma^2T/2}N'(d_1) = \frac{F}{K}N'(d_1)$$

Greeks in the Black Model

If we ignore the annuity the Greeks in the Black model is given by:

$$\Delta_{call} = \frac{\partial C}{\partial F} = e^{-r(T-t)}N(d_1)$$
$$\Delta_{put} = \frac{\partial P}{\partial F} = e^{-r(T-t)}\left(N(d_1) - 1\right)$$
$$\Gamma = \frac{\partial^2 C}{\partial F^2} = \frac{\partial^2 P}{\partial F^2} = \frac{e^{-r(T-t)}}{F\sigma\sqrt{T-t}} \cdot \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$$
$$\nu = \frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = F \cdot e^{-r(T-t)}\frac{\sqrt{T-t}}{\sqrt{2\pi}} \cdot e^{-d_1^2/2}$$
$$\Theta_{call} = \frac{\partial C}{\partial t} = e^{-r(T-t)}\left(r \cdot F \cdot N(d_1) - r \cdot K \cdot N(d_2) - \frac{F \cdot N'(d_1) \cdot \sigma}{2\sqrt{T-t}}\right)$$
$$\Theta_{put} = \frac{\partial P}{\partial t} = -e^{-r(T-t)}\left(r \cdot F \cdot N(-d_1) - r \cdot K \cdot N(-d_2) + \frac{F \cdot N'(d_1) \cdot \sigma}{2\sqrt{T-t}}\right)$$
$$\rho_{call} = \frac{\partial C}{\partial r} = t \cdot K \cdot e^{-r(T-t)}N(d_2)$$
$$\rho_{put} = \frac{\partial P}{\partial r} = -t \cdot K \cdot e^{-r(T-t)}N(-d_2)$$

19.1.4 Swaps in the Multiple Curve Framework

We saw earlier how to derive the swap rate for a forward-starting swap. If we generalize this for an ordinary swap, under the multiple curve framework we generate the cash flows with one curve (on tenor) and discount with another. Here we will study the difference when using one or two curves.

Denote the reset days for any swap as T_0, T_1, T_N and define α_i as the time interval $T_i - T_{i-1}$. The holder payer swap with tenor $T_N - T_0$ receives fixed payments at times $T_1, T_2, \ldots, T_N$ and pay at the same times floating payments.

For each period $[T_i, T_{i+1}]$ the Libor rate $L_{i+1}(T_i)$ is set at time T_i and the floating leg $\alpha_{i+1} L_{i+1}(T_i)$ is received at T_{i+1}. For the same period the fixed leg $\alpha_{i+1} F$ is paid at T_{i+1} where F is the (fixed) swap rate.

The arbitrage-free value at $0 = t < T_n$ of the floating payment made at T_i is given by $p(T_i) - p(T_{i+1})$. The total value of the floating legs at time t for $t \leqslant T_n$ equals

$$\begin{aligned}
\sum_{i=1}^{N} \alpha_i \cdot f(T_{i-1}, T_i) \cdot p(T_i) &= \sum_{i=1}^{N} \alpha_i \cdot \frac{1}{\alpha_i} \frac{p(T_{i-1}) - p(T_i)}{p(T_i)} \cdot p(T_i) \\
&= \sum_{i=1}^{N} \left[p(T_{i-1}) - p(T_i) \right] = p(0) - p(T_N) \\
&= 1 - p(T)
\end{aligned}$$

where we have used that the forward rate is given by

$$\begin{aligned}
p(t_i) = p(t_{i-1}) \cdot p(t_{i-1}, t_i) &\Rightarrow p(t_{i-1}) \cdot \frac{1}{1 + \alpha_i f(t_{i-1}, t_i)} \\
\Rightarrow f(t_{i-1}, t_i) &= \frac{1}{\alpha_i} \frac{p(t_i) - p(t_{i-1})}{p(t_i)}
\end{aligned}$$

In the previously mentioned analysis the forward rate $f(t, T_i)$ and the discount factor, $p(t, T_i)$ is given by the same curve/tenor. In a multi-curve framework we might generate the cash flows with one curve (as follows, a 3-month tenor curve) and discount with another (in subsequent section, a 6-month tenor curve). Then, we have to modify the calculation as follows

$$\sum_{i=1}^{N} \alpha_i \cdot f_{3M}(T_{i-1}, T_i) \cdot p_{6M}(T_i) = \sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} \cdot p_{6M}(T_i)$$

We see that we cannot simplify this as we did when using the same tenors on both curves. The total value at time t for the fixed side, using a 6-month tenor for discounting equals

$$\sum_{i=1}^{N} F \cdot \alpha_i \cdot p_{6M}(T_i) = F \cdot \sum_{i=1}^{N} \alpha_i \cdot p_{6M}(T_i)$$

where F is the swap rate. This is a **par rate** since it makes the price of the swap to be equal zero when entering the swap contract. So the total value of the payer swap is given by

$$\begin{aligned} PS(F) &= \sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} \cdot p_{6M}(T_i) - F \cdot \sum_{i=1}^{N} \alpha_i \cdot p_{6M}(T_i) \\ &= \sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} - F \cdot \alpha_i \right) \cdot p_{6M}(T_i) \end{aligned}$$

With the old methodology, we should have the result

$$PS(F) = 1 - p(T) - F \sum_{i=1}^{N} \alpha_i \cdot p(T_i)$$

If we use the same tenors (a before the credit crises) for the cash-flow generation as for the discounting we derive the following swap rate:

$$F = \frac{1 - p(T)}{\sum_{i=1}^{N} \alpha_i \cdot p(T_i)}$$

With different tenors we get

$$F = \frac{\sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} \cdot p_{6M}(T_i)}{\sum_{i=1}^{N} \alpha_i \cdot p_{6M}(T_i)}$$

So

$$\sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} - F \cdot \alpha_i \right) = 0$$

$$\Rightarrow$$

$$F \cdot T = \sum_{i=1}^{N} \frac{p_{3M}(T_{i-1}) - p_{3M}(T_i)}{p_{3M}(T_i)} = \sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1})}{p_{3M}(T_i)} - 1 \right)$$

and

$$F = \frac{1}{T} \sum_{i=1}^{N} \left(\frac{p_{3M}(T_{i-1})}{p_{3M}(T_i)} - 1 \right)$$

19.1.5 Swaptions with Forward Premium

In the European, the swaption markets have changed to be traded with a forward premium in contrast to spot premium. In such a way we can minimize the counterparty risk. Therefore we can also discount with the new risk-free interest rate, the EONIA overnight index-swap (OIS) rate.

Say that we want to buy a payer swaption at $t = 0$ with maturity at $t = T$. The swap maturity here is denoted by $t = S$. The present value of the swaption at $t = 0$ is in general given by

$$P(0, T, S) = S_T^S(0) \left\{ F(0, T, S) \cdot N\left(d_1(0, T, S)\right) - K \cdot N\left(d_2(0, T, S)\right) \right\}$$

where $F(0, T, S)$ is the forward swap rate between $t = T$ and $t = S$ contracted at $t = 0$ and

$$d_1(0, T, S) = \frac{\ln\left\{\frac{F(0,T,S)}{K}\right\} + \frac{1}{2}\sigma^2(0, T, S) \cdot T}{\sigma(0, T, S) \cdot \sqrt{T}},$$
$$d_2(0, T, S) = d_1 - \sigma(0, T, S) \cdot \sqrt{T}$$

and

$$S_T^S(0) = \sum_{i=T+\Delta t}^{S} \alpha_i p(t_i, S) = \frac{p(0, T) - p(0, S)}{F(0, T, S)}$$

Here $p(t, S)$ is the forward discount factor (zero coupon) between time t and S. This means that the premium at $t = 0$ is $P(0, T, S)$. If this is a forward premium, we shall at $t = T$ pay

$$premium = \frac{P(0, T, S)}{p(0, T)}$$

We therefore construct a portfolio consisting of the swaption and the premium, so that the total value at $t = 0$ is zero. At any arbitrary time t, the value of our portfolio is

$$
\begin{aligned}
V(t,T,S) &= S_T^S(t)\,\{F(t,T,S)\cdot N\,(d_1(t,T,S)) - K\cdot N\,(d_2(t,T,S))\} \\
&\quad - p(t,T)\cdot premium \\
&= \frac{p(t,T)-p(t,S)}{F(t,T,S)}\,\{F(t,T,S)\cdot N\,(d_1(t,T,S)) - K\cdot N\,(d_2(t,T,S))\} \\
&\quad - p(t,T)\cdot premium
\end{aligned}
$$

where $F(t,T,S)$ is the forward swap rate between $t = T$ and $t = S$ at time t and

$$
d_1(t,T,S) = \frac{\ln\left\{\frac{F(t,T,S)}{K}\right\} + \frac{1}{2}\sigma^2(t,T,S)\cdot(T-t)}{\sigma(t,T,S)\cdot\sqrt{T-t}},
$$

$$
d_2(t,T,S) = d_1 - \sigma(t,T,S)\cdot\sqrt{T-t}
$$

19.1.6 The Normal Black Model

Usually the underlying security is assumed to follow a lognormal process (or Geometric Brownian Motion). However, there are some traders who believe that the normal process describes the real market more closely than that of lognormal counterpart. This model is also known as the Bachelier's model.

Let us assume that the current future price, strike price, risk-free interest rate, volatility and time to maturity as denoted as f, K, r, σ and $T-t$ respectively. Let us also assume that the current future price follows the following normal process:

$$
df = \mu dt + \sigma dW_t
$$

where μ is a constant drift. For instruments like swaptions, f represents the forward rate.

Let us start by studying the behaviour of the delta-hedged portfolio which consists of long delta shares of future contract and short one derivative in question. Say, call it Π. Let us also denote the value of derivative by g. Then, the value of the delta-hedged portfolio is given by:

$$
\Pi = g - \frac{\partial g}{\partial f} f
$$

So applying Ito's lemma using the earlier mentioned SDE into the changes of the portfolio value, one gets

$$
\begin{aligned}
d\Pi &= dg - \frac{\partial g}{\partial f} df \\
&= \left(\frac{\partial g}{\partial t} + \mu \frac{\partial g}{\partial f} + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial f^2} \right) dt + \sigma \frac{\partial g}{\partial f} dW - \frac{\partial g}{\partial f} \left(\mu dt + \sigma dW \right) \\
&= \left(\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial f^2} \right) dt
\end{aligned}
$$

We want the aforementioned quantity to be a Q-martingale under the discounted expectation with risk-free rate. This is the same as stating that the previous quantity equals the gain from the risk-free interest rate for the portfolio value. So, we have:

$$
d\Pi = r\Pi dt
$$

Since it costs nothing to enter into a futures contract, one has $\Pi = g$. Thus, we obtain the following PDE

$$
\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial f^2} = rg
$$

In a risk-neutral world the process is giving as

$$
df = \sigma dV_t
$$

with the trivial solution, from integration over the interval $[t, T]$:

$$
f(T) = f(t) + \sigma \left(V_T - V_t \right)
$$

We see that f is a Gaussian process; $N[f_t, \sigma^2(T-t)]$, that is, with mean $f(t)$ and variance $\sigma^2(T-t)$. By the application of Feynman-Kač, we obtain the following solution

$$
\begin{aligned}
g(t, f_T) &= e^{-r(T-t)} E^Q \left[\Phi(T) \right] = \frac{e^{-r(T-t)}}{\sigma \sqrt{2\pi (T-t)}} \int_{-\infty}^{\infty} \Phi(T) \cdot e^{-\frac{(f_T - f_t)^2}{2\sigma^2 (T-t)}} df_T \\
&= \left\{ f_T = f_t + \sigma \sqrt{T-t} \cdot z \right\} \frac{e^{-r(T-t)}}{\sigma \sqrt{2\pi (T-t)}} \\
&\quad \times \int_{-\infty}^{\infty} \Phi(T) \cdot e^{-\frac{\left(\sigma \sqrt{T-t} \cdot z\right)^2}{2\sigma^2 (T-t)}} \sigma \sqrt{T-t} dz \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(T) \cdot e^{-\frac{z^2}{2}} dz
\end{aligned}
$$

Here

$$\Phi(T) = \begin{cases} (f_T - K)^+ & \text{for a Call} \\ (K - f_T)^+ & \text{for a Put} \end{cases}$$

For the Call we have

$$\Pi_C(t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_T - K)^+ \cdot e^{-\frac{z^2}{2}} dz$$

$$= e\frac{-r(T-t)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_t + \sigma\sqrt{T-t} \cdot z - K)^+ \cdot e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} (f_t + \sigma\sqrt{T-t} \cdot z - K) \cdot e^{-\frac{z^2}{2}} dz = A - B$$

Set $f_t = F$ and with $z_0 = \frac{(F-K)}{\sigma\sqrt{T-t}}$ we get

$$A = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} (F - K) \int_{z_0}^{\infty} e^{-\frac{z^2}{2}} dz = e^{-r(T-t)} (F - K) \cdot N[z_0]$$

$$B = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{z_0}^{\infty} \sigma\sqrt{T-t} \cdot z \cdot e^{-\frac{z^2}{2}} dz = e^{-r(T-t)} \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-\frac{z_0^2}{2}}$$

Then, the fair values of call C (payer swaption, PS) and put P (receiver swaption, RS) are expressed as

$$C = e^{-r(T-t)} \left[(F - K) \cdot N(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2} \right]$$

$$P = e^{-r(T-t)} \left[(K - F) \cdot N(-d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2} \right]$$

where

$$d = \frac{F - K}{\sigma\sqrt{T-t}}$$

and

τ = Tenor of swap in years (time between swaption maturity and swap maturity).

F = Forward rate of the underlying swap.

K = Strike rate of the swaption.
r = Risk-free interest rate.
T = Time to swaption expiration in years.
σ = Volatility of the forward-starting swap rate.
m = Compounding's per year in swap rate.

To apply this on swaptions we need, as before, to multiply C and P with the annuity

$$\frac{1 - \frac{1}{(1+F/m)^{\tau \cdot m}}}{F}$$

The Greeks can easily be calculated by simple differentiations

$$\Delta_C = \frac{\partial C}{\partial F} = e^{-r(T-t)} \cdot N(d)$$

$$\Delta_P = \frac{\partial P}{\partial F} = -e^{-r(T-t)} \cdot N(-d)$$

$$\Gamma = \frac{\partial^2 C}{\partial F^2} = \frac{\partial^2 P}{\partial F^2} = \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-d^2/2}$$

$$\nu = \frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = e^{-r(T-t)} \frac{\sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2}$$

$$\Theta_C = \frac{\partial C}{\partial t} = e^{-r(T-t)} \times \left(-r \cdot (F-K) \cdot N(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2 \cdot \pi}} e^{-d^2/2} - \frac{\sigma}{2\sqrt{2 \cdot \pi \cdot t}} e^{-d^2/2} \right)$$

$$\Theta_P = \frac{\partial P}{\partial t} = e^{-r(T-t)} \times \left(r \cdot (F-K) \cdot N(-d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2 \cdot \pi}} e^{-d^2/2} - \frac{\sigma}{2\sqrt{2 \cdot \pi \cdot t}} e^{-d^2/2} \right)$$

19.1.6.1 Convexity Adjustments

A standard bond or interest rate swap has a convex price-yield relationship. To price options with the Black-76 model when the underlying asset is a derivative security with a payoff function linear in the bond or swap yield, the yield should be adjusted for lack of convexity value.

Examples of derivatives where the payoff is a linear function of the bond or swap yield are constant maturity swaps (CMS) and constant maturity treasury swaps (CMT). The closed-form formula published by Brotherton-Ratcliffe and Iben (1993) assumes that the forward yield is lognormal distributed.

$$adj. = -\frac{1}{2}\frac{\frac{\partial^2 P}{\partial y_F^2}}{\frac{\partial P}{\partial y_F}} \cdot y_F^2 \left(e^{\sigma^2 T} - 1\right)$$

where

P = Bond or fixed side swap value.

y_F = Forward yield.

T = Time to payment date in years.

σ = Volatility of the forward yield.

Example 19.1.3

Consider a derivative instrument with a single payment 5 years from now that is based on the notional principal times the yield of a standard 4-year swap with annual payments. The forward yield of the 4-year swap starting 5 years in the future and ending 9 years in the future is 7%. The volatility of the forward swap yield is 18%. Calculate the convexity adjustment of the swap yield.

The value of the fixed side of the swap with annual yield is equal to the value of a bond where the coupon is equal to the forward swap rate/yield y_F.

$$P = \frac{c}{1+y_F} + \frac{c}{(1+y_F)^2} + \frac{c}{(1+y_F)^3} + \frac{1+c}{(1+y_F)^4}$$

The partial derivative of the swap with respect to the yield is

$$\frac{\partial P}{\partial y_F} = -\frac{c}{(1+y_F)^2} - \frac{2c}{(1+y_F)^3} - \frac{3c}{(1+y_F)^4} - \frac{4(1+c)}{(1+y_F)^5}$$
$$= \{c = y_F = 0.07\} = -3.3872$$

and the second partial derivative with respect to the forward swap rate is

$$\frac{\partial^2 P}{\partial y_F^2} = \frac{2c}{(1+y_F)^3} + \frac{6c}{(1+y_F)^4} + \frac{12c}{(1+y_F)^5} + \frac{20(1+c)}{(1+y_F)^6} = 15.2933$$

The convexity adjustment can now be found as:

$$adj. = -\frac{1}{2}\frac{15.2933}{-3.3872} \cdot 0.07^2 \left(e^{0.18^5} - 1\right) = 0.0019$$

The convexity-adjusted rate is then equal to 7.19% (0.07+0.0019).

19.1.6.2 Vega of the Convexity Adjustment

By taking the derivative of the convexity adjustment, we get the convexity adjustment's sensitivity to a small change in volatility

$$\nu = -\frac{\frac{\partial^2 P}{\partial y_F^2}}{\frac{\partial P}{\partial y_F}} \cdot y_F^2 \sigma T e^{\sigma^2 T}$$

19.1.6.3 Implied Volatility From the Convexity Value in a Bond

If the convexity adjustment is known, it is possible to calculate the implied volatility by simply rearranging the convexity adjustment formula

$$\sigma = \sqrt{\ln\left(\frac{adj.}{-\frac{1}{2}\left(\frac{\partial^2 P}{\partial y_F^2} \Big/ \frac{\partial P}{\partial y_F}\right) \cdot y_F^2} + 1\right)\frac{1}{T}}$$

19.1.7 European Short-Term Bond Options

European bond options can be priced in the Black-76 model by using the forward price of the bond at expiration as the underlying asset.

$$c = e^{-rT}(F \cdot N(d_1) - K \cdot N(d_2))$$
$$p = e^{-rT}(K \cdot N(-d_2) - F \cdot N(-d_1))$$

where F is the forward price of the bond at the expiration of the option, and

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

This model does not take into consideration the pull to par effect of the bond. At maturity, the bond price must be equal to principal plus the coupon. For this reason, the uncertainty of a bond will first increase and then decrease.

The Black-76 model assumes that the uncertainty (variance) of the underlying asset increases linearly with time to maturity. Pricing of

European bond options using this approach should thus be limited to options with short time to maturity relative to the time to maturity of the bond. A rule of thumb used by some traders is that the time to maturity of the option should be no longer than one-fifth of the time to maturity on the underlying bond

Example 19.1.4

Consider a European put option with 6 months to expiry and strike price 122 on a bond with forward price at option expiration equal to 122.5. The volatility of the forward price is 4%, and the risk-free discount rate is 5%. Calculate the option's value.

$$F = 122.5, K = 122, T = 0.5, r = 0.05, \sigma = 0.04.$$

$$d_1 = \frac{\ln(122.5/122) - (0.04^2/2) \cdot 0.5}{0.04\sqrt{0.5}} = 0.1587, d_2 = d_1 - 0.04\sqrt{0.5} = 0.1305$$

$$N(-d_1) = 0.4369, \quad N(-d_2) = 0.4481$$

$$p = e^{-0.05 \cdot 0.5}\,[122N(-d_2) - 122.5N(-d_1)] = 1.1155$$

19.1.8 The Schaefer and Schwartz Model

Schaefer and Schwartz (1987) modified the Black-Scholes model for pricing bond options to take into consideration that the price volatility of a bond increases with duration

$$c = S \cdot N(d_1) - Ke^{-rT}N(d_2)$$
$$p = Ke^{-rT}N(-d_2) - S \cdot N(-d_1)$$

where $\sigma = \left(\lambda S^{\alpha-1}\right) D$, and

$$d_2 = \frac{\ln(S/K) - (r + \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Here D is the duration of the bond after the option expires. λ is estimated from the observed price volatility σ_0 of the bond. α is a constant that Schaefer and Schwartz suggest should be set to 0.5.

$$\lambda = \frac{\sigma_0}{S^{\alpha-1} \cdot D^*}$$

where D^* is the duration of the bond today.

Example 19.1.5

Assume that the duration of the bond is 8 years and that the observed price volatility of the bond is 12%. This gives: $\lambda = 0.15$.

In Table 19.1 we use this value and compares the option prices from the Schaefer and Schwartz formula with option prices from the Black-76 formula.

Table 19.1 Option prices from Schaefer and Schwartz and Black-76

Bond Duration	Base Volatility (%)	Adjusted Volatility (%)	Black-76 Value	Modified Black-76 value
1	12.0	1.5	5.5364	0.6929
2	12.0	3.0	5.5364	1.3857
3	12.0	4.5	5.5364	2.0783
4	12.0	6.0	5.5364	2.7707
5	12.0	7.5	5.5364	3.4628
6	12.0	9.0	5.5364	4.1545
7	12.0	10.5	5.5364	4.8457
8	12.0	12.0	5.5364	5.5364

20

Convertibles

20.1 Convertible Bonds

A convertible bond is a security issued by a company that may be converted from debt to equity (and vice versa) at various prices and stages in the life cycle of the contract (e.g. the time to maturity). There are many types of convertible bonds with various conversion properties and complex structures. Common examples of convertible bonds are **Convertible Preferred Stock bonds**, **Zero-Coupon convertibles**, **Mandatory convertibles**, to name but a few. The traditional (simplest) convertible bond is one that is a fixed coupon paying bond when the stock price S is below some predetermined conversion price K (i.e. $S < K$), and may be converted to a predetermined number of stocks (the conversion ratio) when above (i.e. $S > K$). It may immediately become apparent that the traditional convertible bond is effectively a bond with an imbedded call warrant (option) on the stock of the issuing company with strike price K.

Since a convertible bond is a liability of the issuer, if the company goes into liquidation, the convertible bondholder has priority over most other parties except pure bondholders.

Returning to our traditional convertible bond example, for the moment we ignore credit rating issues, the convertible price B_{con} is close to its fixed income value (bond) B of an equivalent pure bond from the same issuer when deep out-of-the-money. Hence,

$$B_{con} \sim B \quad \text{for } S \ll K.$$

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When the stock price is very high and exceeds the conversion price K, the convertible bond becomes stock-like when deep in-the-money. Hence,

$$B_{con} \sim S \quad \text{for } S \gg K.$$

Moreover, given that the convertible is effectively a bond plus a warrant on the stock, the price of a convertible must be comparable to the sum of the two individual components. From arbitrage considerations for options, we know that the price of a call warrant W must satisfy

$$W > \max[S - K, 0]$$

prior to expiry. Ignoring issues related to credit risk, when the warrant element is out-of-the-money the convertible is worth the fixed income value B. Therefore, the convertible has to satisfy the relation

$$B_{con} > \max[S - K, 0] + B$$

given the hybrid nature of the convertible bond. Here the price of the convertible bond has been approximated as

$$B_{con} = B + W$$

In Fig. 20.1 we show the price track of a traditional convertible bond. The convertible price (solid red line) is always above the stock price track as already discussed. The red dotted line demonstrates the price track of a bond plus warrant in the absence of credit risk.

The true price track of the convertible is found to fall below the fixed income value ($B_{con} < B$) when the stock price falls to low levels due to the widening of credit risk spread when the company's stock price falls. As the stock price falls to low levels there is an increasing correlation between the price of the convertible and the stock; however, as the price rises to very high values, the correlation becomes insignificant (as the credit rating improves). A fall below the fixed income value is also seen in the pure bonds due to poorer credit rating.

The following terms are used to describe various sections of the price track of the convertible:

1. Distressed debt –In this region the convertible is on close to a default event. If a default event occurs, a sum proportional to the

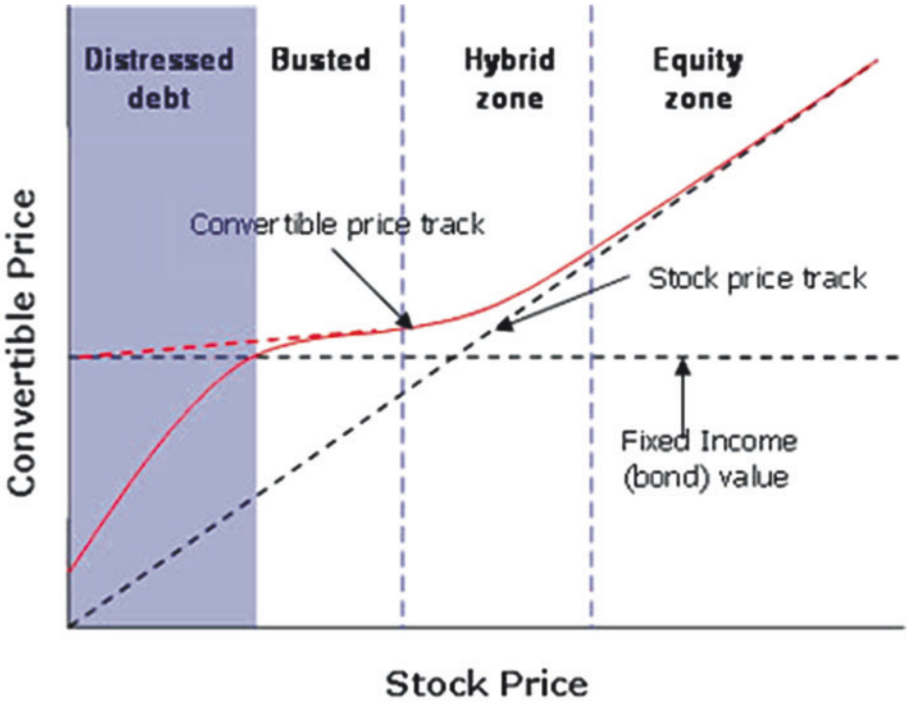

Fig. 20.1 The price track of a convertible bond

recovery rate R is paid out to the holder of a convertible. The value of the convertible is highly sensitive to the credit risk spread (a parameter often referred to as omicron) in this region.

2. Busted convertible–A term often used to describe a convertible that is out-of-the money but above the distressed zone.
3. Hybrid zone–The convertible shows behaviour between the stock and a pure bond.
4. Equity zone–The convertible price is more equity-like than debt. Credit risk factors become insignificant since the company's credit rating is high due to the high stock value.

The hybrid nature of convertibles is often exploited by arbitrageurs in the rather popular practice of convertible arbitrage. Convertible arbitrage is especially successful at times of high volatility in stock price, producing high returns with relatively low risk. Delta, gamma and other more sophisticated hedging strategies are used to capture the low-risk profits (though not completely risk-free). Instances can occur when the equity options imbedded are priced different to those of the equivalent pure options that may exist in the market for various reasons. These situations, when a relative price difference is observed, are exploited by arbitrageurs in a long-short trade.

20.1.1 A Model for Convertibles

The stock is modelled as a lognormal Brownian process

$$dS = \mu S dt + \sigma S dZ_1$$

and the interest rate as

$$dr = u(r,t)dt + w(r,t)dZ_2$$

where Z_1 and Z_2 are two independent Wiener processes with a correlation ρ. The drift $u(r,t)$ and volatility $w(r,t)$ is dependent on the interest rate model. We make the choice

$$dr = (a_1 - b_1 r)dt + w dZ_2$$

where all parameters are time dependent. The value of the convertible V depends on the stock price, the interest rate and of time, $V(S,r,t)$, and given by the following PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \rho\sigma S w \frac{\partial^2 V}{\partial S \partial r} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + rS\frac{\partial V}{\partial S} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0$$

This is found by hedging the two processes against each other and with the introduction of the market price of risk. The value of the convertible must be: $V(S,r,t) \geq nS$ where n is the number of stocks on exercise. We get

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \rho\sigma S w \frac{\partial^2 V}{\partial S \partial r} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + rS\frac{\partial V}{\partial S} + (a_1 - b_1 r)\frac{\partial V}{\partial r} - rV = 0$$

This equation can be solved with a finite difference method.

21

A New Framework

21.1 Pricing Before and After the Crisis

Ten years ago if you had suggested that a sophisticated investment bank did not know how to value a plain **vanilla** interest rate swap, people would have laughed at you. But that isn't too far from the case today.

We will now give an introduction to yield curve constructions and how this has been changed since after the financial crisis.

21.1.1 Introduction

Pricing complex interest rate derivatives requires modelling the future dynamics of the yield curve term structure. Most of the literature considers the existence of the *current* zero-coupon yield curve as a given, and its construction is often neglected, or even obscured, as it is considered to be more of an art than science. Actually, any yield curve term structure modelling approach will fail to produce good/reasonable prices if the current term structure is not correct.

Financial institutions, software houses and practitioners have developed their own proprietary methodologies in order to extract the current zero-coupon yield curve term structure from quoted prices on subsets of liquid market instruments. These can be divided into two groups: "best fit" and "exact fit" algorithms. "Best-fit" algorithms start by assuming a functional form for the term structure and calibrate its parameters such as to minimize the re-pricing error of the chosen

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set of calibration instruments. An example of this is the Smith-Wilson approach to discounting used by regulators for insurance companies.

In banks, "exact-fit" algorithms are often preferred in practice. Such algorithms fix the zero-coupon yield curve on a time grid of N points in order to *exactly* re-price N pre-selected market instruments, often referred to as benchmarks. The implementation of these algorithms is often incremental, extending the yield curve step by step in increasing order of maturity for the selected instruments, in a "bootstrap" approach.

Intermediate yield curve values are then obtained by interpolation on the bootstrapping grid. Here different interpolation algorithms are available but little attention has been devoted in the literature to the fact that interpolation is often already used during bootstrapping, not just after that, and that the interaction between bootstrapping and interpolation can be subtle if not nasty (see e.g. Patrick Hagan, Interpolation method for curve constructions).

While naive algorithms may fail to deal with market subtleties such as date conventions, the intra-day fixing of the first floating payment of a swap, the turn-of-year effect, the futures convexity adjustment, etc., even very sophisticated algorithms used in a naive way may fail to provide relevant estimates of forward Euribor rates in difficult market conditions, such as those observed since the summer of 2007 and the so-called *subprime credit crunch crisis*. Today using just one single curve is not enough to account for forward rates of different tenors, such as 1, 3, 6, 12 months, because of the large basis swap spreads presently quoted in the market.

Prior to the credit crisis, (zero-coupon) yield curve modelling was reasonably well understood. The underlying fundamental principles had existed for over 15 years with steady evolutions in areas that were most relevant to options and complex products. Credit and liquidity issues were ignored as their effects were minimal. Pricing a single-currency interest rate swap was relatively straightforward. A single "default free" zero-coupon yield curve was calibrated to liquid market products, and future cash flows of other instruments were discounted and evaluated using this single curve. There was little variation between implementations, and results across the market were consistent.

Following the credit crisis, yield curve modelling has undergone nothing short of a revolution. During the credit crisis, credit and liquidity problems appeared in several markets. This drove apart previously closely-related interest rates. For example, Euribor basis swap spreads dramatically increased and the spreads between Euribor and Eonia overnight indexed swaps (OIS) diverged. In addition, the impact of counterparty credit on valuation and risk management dramatically increased.

Existing modelling and infrastructure no longer seemed to work and a re-assessment of the basic principles has taken place. Currently a new interest rate modelling framework is evolving which is based on OIS discounting and the integration of credit value adjustments (CVAs). Pricing a single-currency interest rate swap (IRS) now takes into account the difference between projected rates such as Euribor that include credit risk and the rates appropriate for discounting cash flows that are risk free or based on funding cost. CVAs take into account the likelihood that the counterparty will default, along with the expected exposure given default, the volatility of these expected exposures, and wrong way risk.[1]

One of the many consequences of the liquidity crisis that started in the second half of 2007 has been a strong increase in quoted basis spreads in the market between single-currency interest rate instruments, in particular for swaps. This is characterized by different underlying rate tenors (e.g. Euribor3M, Euribor6M, etc.), reflecting the increased liquidity risk, and an increased preference of financial institutions for receiving payments with higher frequency (quarterly instead of semi-annually, for instance). Such asymmetry has induced the "segmentation" of interest rate markets into various sub-areas, with 1M, 3M, 6M and 12M underlying rate tenors. Each area is characterized, in principle, by its own internal dynamic, reflecting the different views and interests of the market participants.

In order to price derivatives we must now know

1. the underlying interest rate (Libor/Eonia, etc.);
2. the matching market prices for plain vanilla derivatives, which are used to construct zero-coupon yield curves and the term structure of volatilities, as well as for calibration and hedging;
3. the transaction mechanics: collateral and liquidity/funding issues;
4. the counterparties: credit and default issues.

[1] Wrong-way risk is defined by the International swaps and Derivatives Association (ISDA) as the risk that occurs when "exposure to a counterparty is adversely correlated with the credit quality of that counterparty".

21.1.2 After the Crises – How the Market Has Changed

In Fig. 21.1, we show the market quotations on 31 September 2008, for six basis swap curves corresponding to the four Euribor tenors 1, 3, 6 and 12 months.

As one can see, the basis spreads are monotonically decreasing from over 100 to around four basis points. There is neither way nor any good reason to ignore such quotations in a market pricing framework of interest rate derivatives. Before the credit crunch of August 2007, the basis spreads were just a few basis points.

The consequences of the credit crunch crisis that started in August 2007 can also be seen in the historical series of the Euribor 6-month (6M) rate vs. the Eonia OIS rate as in Fig. 21.2. The change in the spread is essentially a consequence of the different credit and liquidity risk reflected by Euribor and Eonia rates. This divergence is not a consequence of any difference in counterparty risk between these contracts. Euribor and OIS rates are exchanged in the interbank market by risky counterparties, but they also depend on different fixing frequences of the underlying rates.

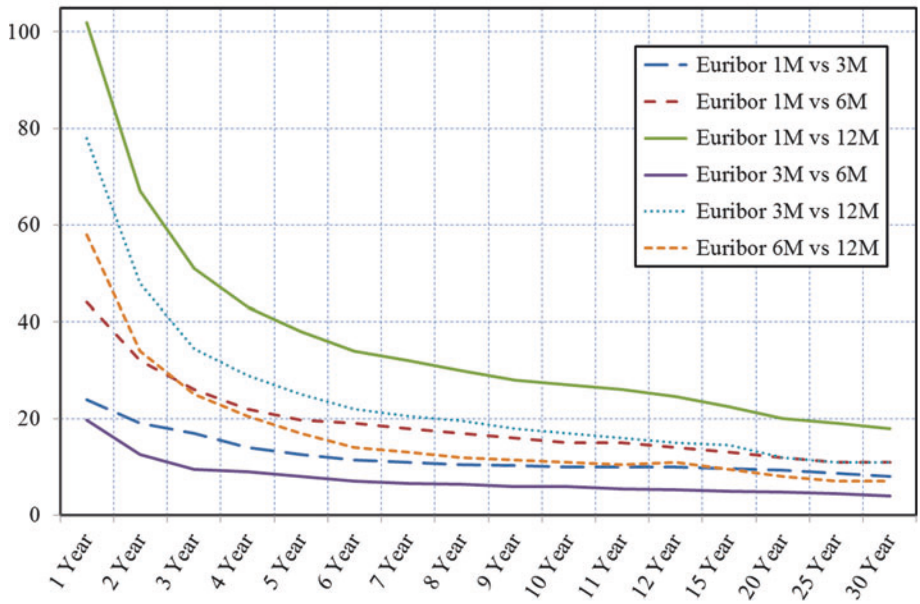

Fig. 21.1 A typical overnight index swap

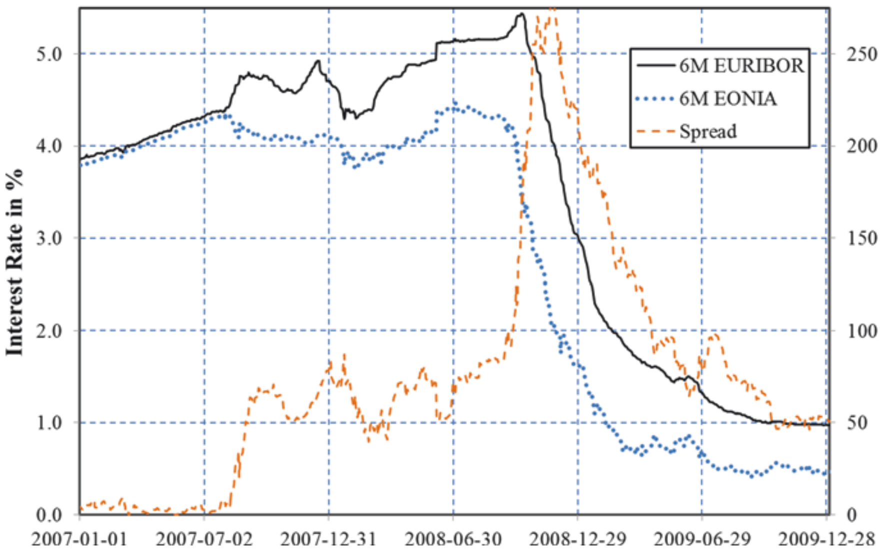

Fig. 21.2 The 6-month Euribor vs. Eonia overnight indexed swap rate

In Fig. 21.3, we show the basis spread between some of the term structures of interest rates in June 2011

The different influence of credit risk on Libor and overnight rates is shown in Fig. 21.4, where we compare the historical series for the Euribor-OIS spread above credit default swaps (CDS) spreads for some main banks in the Euribor Contribution Panel (Commerzbank, Deutsche Bank, Barclays, Santander, Royal Bank of Scotland and Credit Suisse). We observe that the Euribor-OIS basis explosion in August 2007 exactly matches the CDS explosion, corresponding to the generalized increase of the default risk seen in the interbank market.

An effect of the credit crunch has been the great increase of collateral agreements (Credit Support Annex (CSA) agreements) in an attempt to reduce the counterparty risk of over-the-counter (OTC) derivatives. Nowadays most of the counterparties in the interbank market have mutual collateral agreements. In 2010, more than 70% of all OTC derivatives transactions were collateralized.[2]

The main feature of the CSA is a margination mechanism similar to those adopted by central clearing houses for standard instruments exchange (e.g. futures). Shortly, at every margination date, the two

[2] International swaps and Derivatives Association, ISDA (2010).

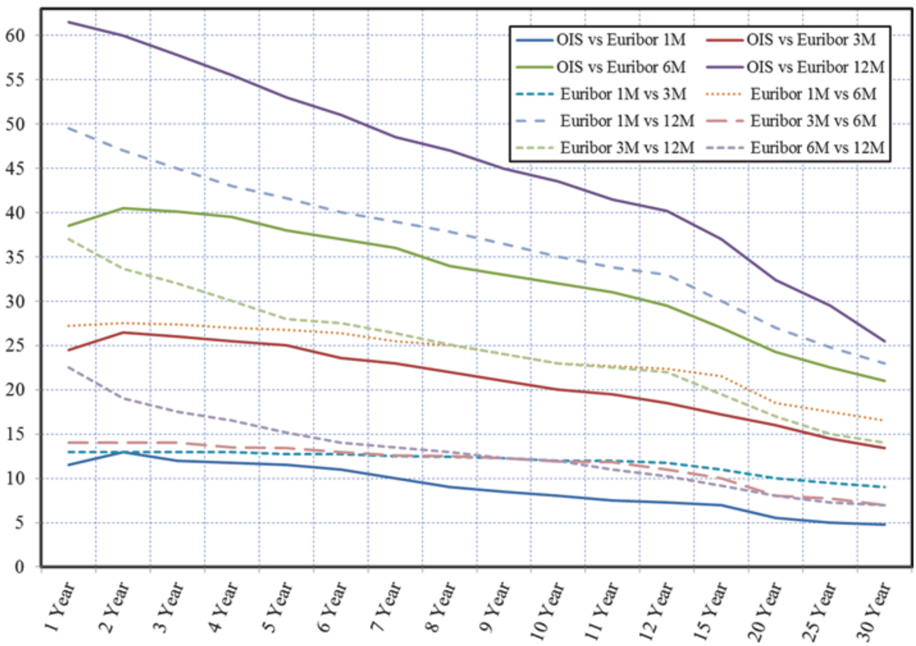

Fig. 21.3 The EUR basis preads for market data in June 2011

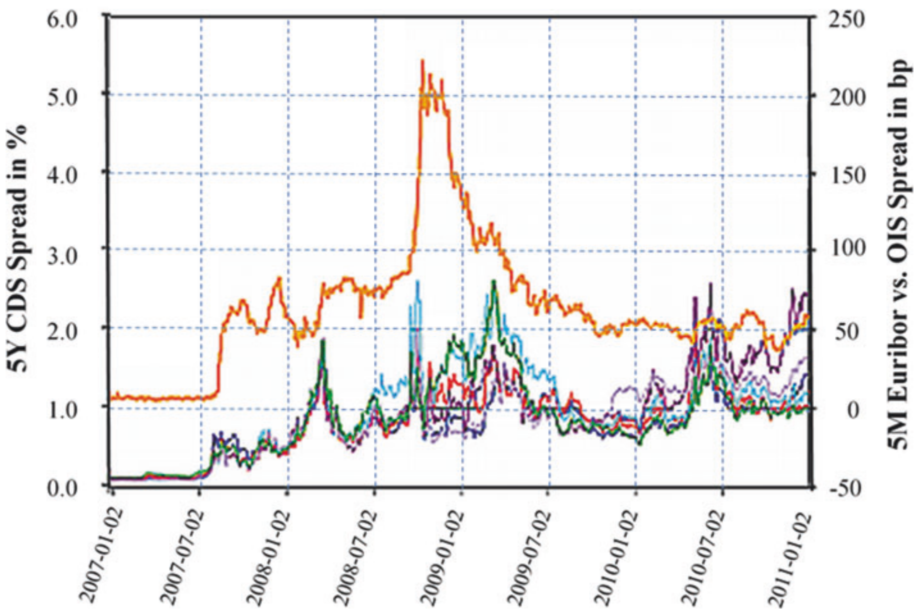

Fig. 21.4 The 5 months Euribir-OIS spread and credit default spread for some main banks in Euorope during the financial crisis

counterparties check the value of the portfolio of mutual OTC transactions and regulate the margin, adding to or subtracting from the collateral account the corresponding mark to market variation with respect to the preceding margination date.

We can also look at CSA as a hedging mechanism, where the collateral amount hedges the creditor against the event of default of the debtor. The most diffused CSA provides a daily margination mechanism and an overnight collateral rate.

An important consequence of the diffusion of collateral agreements among the interbank counterparties is that we can consider the prices of derivatives quoted in the interbank market as counterparty risk-free OTC transactions. A second important consequence is that, by no arbitrage, the CSA margination rate and the discounting rate of future cash flows must match, hence the name "CSA discounting". Since the OIS curve is what is usually used, the alternative name is "OIS discounting". Such a discounting curve is also the best available proxy for a risk-free yield curve.

21.1.2.1 Discount Rate – The Risk-Free Interest Rate, Then and Now

Before the crisis, the risk-free rate was defined in either two ways:

1. The rate at which the government borrows. This assumes that governments do not default and that their bond yields are not contaminated by liquidity or tax premiums.
2. The (Libor) rate at which the big international (Libor rated) banks borrow at short maturities (e.g. 3 months). This assumes that the credit spread over a short horizon of a highly rated bank is practically zero.

Now, many actors in the markets use the OIS/Eonia as the risk-free interest rate. This is obtained by compounding the overnight interest rate at which Libor rated banks borrow. One way to justify this new definition is to argue that zero credit risk at short horizons is valid for 1 day but not for longer periods. Another justification is that OIS/Eonia is the rate that is paid on cash collateral and is therefore the correct discounting rate for collateralized forward contracts and swaps. The latter argument does not require that Libor rated banks have zero credit risk at even overnight horizons.

21.1.2.2 Reasons to Change the Idea of the Risk-Free Rates

The argument that highly rated entities have zero credit risk over short time horizons requires perfect observability of the balance sheet (asset values and debt). However, the crisis in 2008 showed that bank balance sheets are terribly opaque. Banks valued illiquid assets not by marking to market but by marking to model (see model risk) or sometimes marking to their own belief. Similarly, the true liabilities of the bank were hidden using off balance sheets. This meant that the true distance to default could be much lower than the estimated distance to default based on observable parameters.

The unreliability of accounting information creates a "jump to default" risk and creates non-negligible credit spreads over short time horizons. So, even 1-month Libor can no longer be regarded as risk free. Whether overnight Libor can be regarded as risk free is an open question.

We also have to remember, even if a bank is Libor rated today (i.e. it can borrow at Libor today), this does not guarantee that it can borrow at Libor in the next quarter because it may no longer be Libor rated then. The Libor rate for the next quarter will be the rate at which a bank can borrow if it is Libor rated on that date. This makes a big difference between 6-months Libor and the 3-months Libor rate.

This implies that an interest rate swap whose floating leg is 3-months Libor is not the same as a swap whose floating leg is 6-months Libor. The floating leg payments on the first swap are expected to be lower and therefore the fixed leg should also be lower. A tenor swap in which both legs are floating – say 3-months Libor on one leg and 6-months Libor on the other leg – should include a tenor spread[3] on the 3-months leg to ensure that the swap is fair at inception. We are assuming here that the interest rate swaps are free of credit (counterparty) risk as discussed later when we turn to collateralization.

Similar explanations can be given for the cross-currency swap (CCS) spread. In a basis CCS, both legs are floating but in different currencies. For example, US dollar Libor on a dollar notional amount may be exchanged for Japanese yen Libor on an equivalent yen notional with an exchange of principals at the end. If Libor is regarded as risk free, then

[3] This tenor spread is usually called basis spread, but I think tenor spread is a better name. In the following I will call this spread basis spread

the two legs are floating rate bonds that must be worth par and the swap should trade flat. In reality however, for several years, there has been a premium on the yen leg of this swap. Credit quality differences could be one explanation for this, though liquidity issues might also play an important role.

21.1.3 A Multi-Curve Framework

To set up this framework, we denote with M_x, $x = \{d, f_1, \ldots, f_n\}$ a multiple distinct interest rate sub-market, characterized by the same currency and by distinct money market accounts B_x, such as

$$B_x(t) = \exp\left\{\int_0^t r_x(t')dt'\right\},$$

where r_x(t) are the associated short rates. We also have multiple yield curves C_{x}^{p} in the form of a continuous term structure of discount factors,

$$C_{\mathrm{x}}^{p} = \{T \to p_x(t_0,\ T),\ T \geqslant t_0\},$$

where t_0 is the reference date of the curves (e.g. settlement date, or today) and $p_x(t,\ T)$ the price at time $t \geqslant t_0$ of the M_x-zero-coupon bond for maturity T, such that $p_x(T, T) = 1$. In each sub-market M_x we postulate the usual no-arbitrage relation,

$$p_x(t,\ T_2) = p_x(t,\ T_1)p_x(t,\ T_1,\ T_2),\ t \leqslant T_1 < T_2,$$

where $p_x(t,\ T_1,\ T_2)$ denotes the M_x forward discount factor from time T_1 to time T_2, prevailing at time t. The financial meaning is that, in each market M_x, given a cash flow of one unit of currency, at time T_2, its corresponding value at time $t < T_2$ must be unique. This must be true, both if we discount in one single step from T_2 to t, using the discount factor $p_x(t,\ T_2)$, and if we discount in two steps, first from T_2 to T_1, using the forward discount $p_x(t,\ T_1,\ T_2)$ and then from T_1 to t, using $p_x(t, T_1)$.

We also define continuously compounded zero-coupon rates $z_x(t_0, T)$ and simply compounded instantaneous forward rates $f_x(t_0, T)$

such that

$$p_x(t_0, T) = \exp\{-z_x(t_0, T) \cdot \tau_c(t_0, T)\} = \exp\left\{-\int_0^t f_x(t_0, u)du\right\},$$

$$\ln p_x(t_0, T) = -z_x(t_0, T) \cdot \tau_c(t_0, T) = -\int_0^T f_x(t_0, u)du,$$

where $\tau_c(T_1, T_2) = \tau(T_1,\ T_2,\ dc_c)$ and dc_c, the day count convention for the zero rate. From the relationships mentioned earlier it is immediate to observe that

- $z_x(t_0,\ T)$ is the average of $f_x(t_0,\ u)$ over $[t_0,\ T]$;
- if rates are non-negative, $p(t_0,\ T)$ is a monotone non-increasing function of T such that $0 < p(t_0,\ T) \leqslant 1 \forall T > t_0$; the instantaneous forward curve C_x^f is the most severe indicator of yield curve smoothness, since anything else is obtained through its integration, therefore being smoother by construction.

Using the rate formulas we define two other curves associated with C_x, a zero curve and an instantaneous forward rate curve

$$\begin{aligned} C_x^z &= \{x(t_0, T), \quad T \geqslant t_0\} \\ C_x^f &= \{x(t_0, T), \quad T \geqslant t_0\} \end{aligned},$$

where z_x and f_x are given from the previous equation as

$$\begin{aligned} z_x(t_0, T) &= -\frac{1}{\tau_c(t_0, T)} \cdot \ln p_x(t_0, T) \\ f_x(t_0, u) &= -\frac{\partial}{\partial t} \ln p_x(t_0, t)\big|_{t=T} = z_x(t_0, T) + \tau_c(t_0, T)\frac{\partial}{\partial t} z_x(t_0, t)\big|_{t=T}. \end{aligned}$$

In the following we will denote with C_x the generic curve and we will specify the particular typology (discount, zero or forward curve) if necessary.

Denoting with $F_x(t; T_1,\ T_2)$ the discretely compounded forward rate corresponding to the M_x forward discount factor $p_x(t,\ T_1,\ T_2)$, resetting at time T_1 and covering the time interval $[T_1,\ T_2]$, we have

$$p_x(t, T_1, T_2) = \frac{p_x(t, T_2)}{p_x(t, T_1)} = \frac{1}{1 + F_x(t; T_1, T_2) \cdot \tau_x(T_1, T_2)},$$

where $\tau_x(T_1,\ T_2)$ is the year fraction between times T_1 and T_2 with day-count dc_x. We then obtain the familiar no-arbitrage expression

$$F_x(t;T_1T_2) = \frac{1}{\tau_x(T_1,T_2)}\left[\frac{1}{p_x(t,T_1,T_2)} - 1\right] = \frac{p_x(t,T_1) - p_x(t,T_2)}{\tau_x(T_1,T_2)\cdot p_x(t,T_2)}.$$

This can be also derived as the fair value condition at time t of the forward rate agreement (FRA) contract with pay-off at maturity T_2 given by

$$FRA_x(T_2;T_1,T_2,K,N) = N\cdot\tau_x(T_1,T_2)\left[L_x(T_1,T_2) - K\right]$$
$$L_x(T_1,T_2) = \frac{1 - p_x(T_1,T_2)}{\tau_x(T_1,T_2)\cdot p_x(T_1,T_2)},$$

where N is the nominal amount, $L_x(T_1,\ T_2)$ the T_1-spot Libor rate for maturity T_2 and K the (simple compounded) strike rate (sharing the same day-count convention for simplicity). Introducing expectations we have $\forall t \leqslant T_1 < T_2$:

$$\begin{aligned} FRA_x(T_2;T_1,T_2,K,N) &= p_x(t,T_2)\cdot E_{t,x}^{Q(T_2)}\left[FRA_x(T_2;T_1,T_2,K,N)\right],\\ &= N\cdot p_x(t,T_2)\cdot\tau_x(T_1,T_2)\left\{E_{t,x}^{Q(T_2)}\left[L_x(T_1,T_2)\right] - K\right\},\\ &= N\cdot p_x(t,T_2)\cdot\tau_x(T_1,T_2)\left\{F_x(t;T_1,T_2) - K\right\}, \end{aligned}$$

where the expectation is taken in the $M_x - T_2$-forward measure corresponding to the numeraire $p_x(t,T_2)$, at time t with respect to measure Q and filtration $\mathcal{F}_t$, encoding the market information available up to time t, and we have assumed the standard martingale property of the forward rates

$$F_x(t;T_1,T_2) = E_{t,x}^{Q(T_2)}\left[F_x(T_1;T_1,T_2)\right] = E_{t,x}^{Q(T_2)}\left[L_x(T_1,T_2)\right]$$

to hold in each interest rate market M_x. The previous assumptions imply that each sub-market M_x is internally consistent and has the same properties as the "classical" interest rate market had before the crisis.

The value of a FRA at time t can therefore be written as

$$\begin{aligned} FRA_x(t;T_1,T_2,K,N) &= N\cdot\left[\frac{p_x(t,T_1) - p_x(t,T_2)}{p_x(t,T_2)} - \tau_x(T_1,T_2)\cdot K\right]\cdot p_x(t,T_2),\\ &= N\cdot\left[p_x(t,T_1) - (1+\tau_x(T_1,T_2))\cdot K\cdot p_x(t,T_2)\right]. \end{aligned}$$

Regarding swap rates, given two increasing dates vectors $\mathbf{T} = \{T_0, \ldots, T_n\}$, $\mathbf{S} = \{S_0, \ldots, S_m\}$, $T_n = S_m > T_0 = S_0 \geqslant t_0$, and an interest rate swap with a floating leg paying at times S_j, $j = 1, \ldots, m$, the Euribor rate with tenor $[S_{j-1}; S_j]$ fixed at time S_{j-1}, plus a fixed leg paying a fixed rate at times T_i, $i = 1, \ldots, n$, the corresponding simple compounded fair swap rate on curve C_x with day count convention dc_S is given by

$$S_x(t, \mathbf{T},\mathbf{S}) = \frac{\sum_{j=1}^{m} p_x(t, S_j)\tau_F(S_{j-1}, S_j)F_x(t, S_{j-1}, S_j)}{A_x(t, \mathbf{T})} \frac{p_x(t, T_0) - p_x(t, T_n)}{A_x(t, \mathbf{T})},$$

where

$$A_x(t, \mathbf{T}) = \sum_{i=1}^{n} p_x(t, T_i)\tau_S(T_{i-1}, T_i)$$

is the forward annuity on curve C_x and we have defined $\tau_S(T_{i-1}, T_i) = \tau(T_{i-1}, T_i, dc_S)$. Notice that on the right-hand side of the previous equation for S_x, we have used the definition of the forward rate and the telescopic property of the summation. Actually the telescopic property would hold exactly only if the forward rates end dates equal the next forward rate start dates, with no period's gaps or overlaps.

This is not true in general, because start and end dates are adjusted with their business day convention, and the resulting periods do not concatenate exactly. Typically, such date mismatch does not exceed one business day (which sometimes can be three calendar days). In practice, on the one hand the error is small, of the order of 0.1 basis points; on the other hand nothing prevents us from using the exact dates and accrual periods.

21.1.3.1 The Single-Curve Framework

So how did the market practice for pricing and hedging interest rate derivatives change through the credit crunch crisis? Using the notation described earlier we start by considering a general single-currency interest rate derivative with m future coupons with pay-offs $\pi = \{\pi_1, \ldots, \pi_m\}$, with $\pi_i = \pi_i(F_x)$, generating m expected cash flows $c = \{c_1, \ldots, c_m\}$ on the future dates $T = \{T_1, \ldots, T_m\}$, with $t < T_1 < \ldots < T_m$.

The pre-crisis standard market practice was based on a single-curve procedure, well known to the financial world, that can be summarized as follows:

1. Select *one* finite set of the most convenient (e.g. liquid) vanilla interest rate instruments traded in real time in the market with increasing maturities. For instance, a very common choice in the EUR market was a combination of short-term EUR deposit, medium-term futures on Euribor3M and medium-long-term swaps on Euribor6M.
2. Build *one* yield curve, C_d using the selected instruments plus a set of bootstrapping rules (e.g. pillars, priorities, interpolation method, etc.)
3. Compute the relevant forward rates using the same yield curve C_d as

$$F_d(t; T_{i-1}, T_i) = \frac{p_d(t, T_{i-1}) - p_d(t, T_i)}{\tau_d(T_{i-1}, T_i) \cdot p_d(t, T_i)},$$
$$\text{where } t \leqslant T_{i-1} < T_i,\ i \in \{1, \ldots, m\}$$

4. Compute cash flows c_i as expectations at time t of the corresponding coupon pay-offs $\pi_i(F_d)$ with respect to the T_i-forward measure associated to the numeraire $p_d(t, T_i)$ from the same yield curve C_d,

$$c_i = c(t; T_i, \pi_i) = E_{t,d}^{Q(T_i)}\left[\pi_i(F_d)\right].$$

5. Compute the relevant discount factors $p_d(t,\ T_i)$ from the same yield curve C_d.
6. Compute the derivative's price at time t as the sum of the discounted cash flows,

$$\pi(t; T_i, \pi_i) = \sum_{i=1}^{m} p_d(t, T_i) \cdot c(t; T_i, \pi_i) = \sum_{i=1}^{m} p_d(t, T_i) \cdot E_{t,d}^{Q(T_i)}\left[\pi_i(F_d)\right].$$

7. Compute the delta sensitivity with respect to the yield curve C_d and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the *same* set of vanillas.

21.1.3.2 The Multi-Curve Framework

Unfortunately, the pre-crisis approach outlined earlier is no longer consistent, at least in its simple formulation, with the present market conditions.

- First, it does not take into account the market information carried by basis swap spreads, now much larger than in the past and no longer negligible.
- Second, it does not take into account that the interest rate market is segmented into sub-areas corresponding to instruments with distinct underlying rate tenors, characterized, in principle, by different dynamics (e.g. short rate processes). Thus, pricing and hedging an interest rate derivative on a single yield curve mixing different underlying rate tenors can lead to "dirty" results, incorporating the different dynamics, and eventually the inconsistencies, of distinct market areas, making prices and hedge ratios less stable and more difficult to interpret. On the other side, the more the vanillas and the derivative share the same homogeneous underlying rate, the better should be the relative pricing and the hedging.
- Third, by no arbitrage, discounting must be unique: two identical future cash flows of whatever origin must display the same present value (PV); hence we need a unique discounting curve.

In principle, a consistent credit and liquidity theory would be required to account for the interest rate market segmentation. This would also explain the reason why the asymmetries cited earlier do not necessarily lead to arbitrage opportunities, once counterparty and liquidity risks are taken into account. Unfortunately such a framework is not easy to construct. In practice, interest rate derivatives with a given underlying rate tenor should be priced and hedged using vanilla interest rate market instruments with the same underlying rate tenor.

The post-crisis market practice may be summarized in the following working procedure:

1. Build **one discounting curve** C_d using the preferred selection of vanilla interest rate market instruments and bootstrapping procedure;
2. Build **multiple distinct forwarding curves** $C_{f1}, \ldots, C_{fn}$ using the preferred selections of distinct sets of vanilla interest rate market instruments, each homogeneous in the underlying

Xibor rate tenor (typically with 1M, 3M, 6M and 12M tenors) and bootstrapping procedures.

3. Compute the relevant forward rates with tenor f using the corresponding yield curve as

$$F_f(t;T_{i-1},T_i) = \frac{p_f(t,T_{i-1}) - p_f(t,T_i)}{\tau_f(T_{i-1},T_i)\cdot p_f(t,T_i)},$$
$$\text{where } t \leqslant T_{i-1} < T_i, \quad i \in \{1,\ldots,m\};$$

4. Compute cash flows c_i as expectations at time t of the corresponding coupon pay-offs $\pi_i(F_f)$ with respect to the discounting T_i-forward measure associated to the numeraire $p_d(t,\ T_i)$, as

$$c_i = c(t;T_i,\pi_i) = E_{t,d}^{Q(T_i)}\left[\pi_i(F_f)\right];$$

5. Compute the relevant discount factors $p_d(t,\ T_i)$ from the discounting yield curve $C_{d.}$;
6. Compute the derivative's price at time t as the sum of the discounted cash flows,

$$\pi(t;T_i) = \sum_{i=1}^{m} p_d(t,T_i)\cdot c(t;T_i,\pi_i) = \sum_{i=1}^{m} p_d(t,T_i)\cdot E_{t,d}^{Q(T_i)}\left[\pi_i(F_f)\right];$$

7. Compute the delta sensitivity with respect to the market pillars of each yield curve $C_d,\ C_{f1},\ldots,C_{fn}$ and hedge the resulting delta risk using the suggested amounts of the *corresponding* set of vanillas.

The FRAs are now priced under the T_2 forward measure associated to the numeraire $p_x(t,\ T_2)$ as

$$\begin{aligned}\mathrm{FRA}_x(t;T_1,T_2,K) &= p_x(t,T_2)\cdot\tau_x(T_1,T_2)\left\{E_{t,x}^{Q(T_2)}\left[L_x(T_1,T_2)\right]-K\right\},\\ &= p_x(t,T_2)\cdot\tau_x(T_1,T_2)\left[F_x(t;T_1,T_2)-K\right].\end{aligned}$$

21.1.3.3 Pricing in Single- and Multi-Curve Framework

Now, we will study and compare a 5.5Y maturity EUR floating swap leg on Euribor1M (not directly quoted in the market) in the two frameworks.

In the *single curve framework* this is commonly priced using discount factors and forward rates calculated on the same depo-futures-swap curve cited earlier. The corresponding delta risk is hedged using the suggested amounts (hedge ratios) of 5Y and 6Y Euribor6M swaps. Notice that the expectation in step 3 is taken with respect to the pricing measure $E_{t,d}^{Q(Ti)}$ associated to the numeraire $p_d(t, T_i)$ of the discounting curve. Any other equivalent measure associated to different numeraire may be used as well.

In the *multiple curve framework* the forward rates are calculated on the C_{1M} forwarding curve, bootstrapped using Euribor1M vanillas only, plus discount factors calculated on the discounting curve C_d. The delta sensitivity is computed by shocking one by one the market pillars of both C_{1M} and C_d curves and the resulting delta risk is hedged using the suggested amounts (hedge ratios) of 5Y and 6Y Euribor1M swaps plus the suggested amounts of 5Y and 6Y instruments from the discounting curve C_d.

In the *single curve framework* approach, a unique yield curve is built and used to price and hedge any interest rate derivative on a given currency. This is equivalent to assuming that there exists a unique fundamental underlying short rate process able to model and explain the whole term structure of interest rates of all tenors. It is also a relative pricing approach, because both the price and the hedge of a derivative are calculated relatively to a set of vanillas quoted in the markets. We notice also that it is not strictly guaranteed to be arbitrage-free, because discount factors and forward rates obtained from a given yield curve through interpolation are, in general, not necessarily consistent with those obtained by a no-arbitrage model; in practice bid-ask spreads and transaction costs hide any arbitrage possibilities.

The *multiple-curve framework* is consistent with the present market situation, but it is also more demanding. First, the discounting curve clearly plays a special and fundamental role and must be built with particular care. This "pre-crisis" obvious step has become, in the present market situation, a very subtle and controversial point. In fact, while the forwarding curves construction is driven by the underlying rate homogeneity principle, for which there is (now) a general market consensus, there is no longer, at the moment, any general consensus for the discounting curve construction.

At least two different practices can be encountered in the market:

1. the old "pre-crisis" approach (e.g. the depo, futures/FRA and swap curve cited earlier) that can be justified with the principle of maximum liquidity, and

2. the OIS curve, based on the overnight rate (Eonia for EUR), considered as the best proxy to a risk-free rate available on the market because of its 1-day tenor, justified with collateralized (riskless) counterparties.

Second, building multiple curves requires multiple quotations: many more bootstrapping instruments must be considered (deposits, futures, swaps, basis swaps, FRAs, etc., on different underlying rate tenors), which are available in the market with different degrees of liquidity and can display transitory inconsistencies.

Third, non-trivial interpolation algorithms are crucial for producing smooth forward curves.

Fourth, multiple bootstrapping instruments imply multiple sensitivities, so hedging becomes more complicated.

Last but not least, pricing libraries, platforms, reports, etc., must be extended, configured, tested and released to manage multiple and separated yield curves for forwarding and discounting, not a trivial task for quants, risk managers, developers and IT people.

The static multiple-curve pricing and hedging methodology described earlier can be extended, in principle, by adopting multiple distinct models for the evolution of the underlying interest rates with tenors $f_1, \ldots, f_n$ to calculate the future dynamics of the yield curves $C_{f1}, \ldots, C_{fn}$ and the expected cash flows. The volatility/correlation dependencies carried by such models imply, in principle, bootstrapping multiple distinct variance/covariance matrices and hedging the corresponding sensitivities using volatility- and correlation-dependent vanilla market instruments.

21.1.4 Bootstrapping with Multiple Curves

21.1.4.1 Settings

A yield curve is a complex object that results from many different features that concur to shape the curve. We have different types of yield curves, for example, the **discount factor curve** $C^p{}_x$ the **zero-coupon yield curve** $C^z{}_x$ and the **instantaneous forward rate curve** $C^f{}_x$. Since the discount factor curve is observed to be monotonically decreasing, the zero rates are chosen to be continuous, as in

$$p_x(t_0, T) = \exp\{-z_x(t_0, T) \cdot \tau_c(t_0, T)\} = \exp\left\{-\int_0^t f_x(t_0, u)du\right\}.$$

The associated **year fraction** must be monotonically increasing with increasing time intervals and additive, such that

$$\tau_C(T_1, T_2) + \tau_C(T_2,\ T_3) = \tau_C(T_1,\ T_3).$$

The **day count convention** satisfying the aforementioned conditions that will be used is the common *actual*/365(*fixed*), such that

$$\tau_C(T_1,\ T_2) := \tau[T_1\ T_2;\ actual/365(fixed)] = (T_2 - T_1)/365.$$

The **forward rates** are chosen to be simply compounded as with an associated year fraction (Euribor rates are quoted as *actual/360* so that

$$\tau_F(T_1,\ T_2) := \tau[T_1\ T_2;\ actual/360(fixed)] = (T_2 - T_1)/360.$$

The **reference date**, t_0 can be, today, spot (two business days after today according to the chosen calendar) or, in principle, any business day after today. Once the yield curve at spot date is available, the corresponding yield curve at today can be obtained using the discount between these two dates implied by O/N and T/N deposits.

We choose a **time grid** of the yield curve as a predetermined vector of dates, defined by the set of maturities associated with the selected bootstrapping instruments. The first point in the time grid is the reference date t_0.

Finally we have the **bootstrapping instruments**, quoted in the market, chosen as input for the bootstrapping procedure. We will use an algorithm that ensures exact re-pricing of the choosen input bootstrapping instruments. We also use an **interpolation algorithm** for calculating the yield curve outside the time grid points. Notice that interpolation is also used during the bootstrapping procedure. In principle, we can interpolate on discounts, zero rates or log discounts. We build curves for all needed **currencies** for their specific **calendars** used to determine holidays and business days. In some cases we also specify the bid, mid or ask price chosen for the market instruments, if quoted.

21.1.4.2 Market Instruments

In the current market situation, similar instruments may display very different price levels, liquidity, and may also give erratic forward rates.

Therefore, the first step for multiple yield curve construction is a very careful selection of the corresponding sets of bootstrapping instruments while they roughly cover different maturities and overlap in significant areas. For this reason we select those with more liquid ones with a tighter bid/ask spread.

We start by examining these instruments in detail. In order to fix the data set once for all, we use the quotes on liquid instruments observed in the market.

We begin with the interest rate **deposits** (depos) which are over-the-counter zero-coupon contracts that start at a reference date t_0 (today or spot), and pay the interest accrued until maturity with a given rate fixed at t_0. Let $R_x^{Depo}(t_0,\ T_i)$ be the quoted rate associated to the i:th deposit with maturity T_i and underlying rate tenor $x = t_0$ months. The implied discount factor at time T_i is given by the following relation:

$$p_x(t_0, T_i) = \frac{1}{1 + R_x^{Depo}(t_0, T_i) \cdot \tau_F(t_0, T_i)}, \quad t_0 < T_i.$$

The previous expression can be used to bootstrap the yield curve C_x at point T_i.

We continue with FRAs which are forward starting deposit contacts. For instance the 3x9 FRA is a 6 months deposit starting 3 months forward. In some markets FRAs are quoted between IMM days. In EUR, FRAs do concatenate exactly; for example, the 6x9 FRA starts when the preceding 3x6 FRA ends. The underlying forward rate fixes two working days before the forward start date.

Market FRAs provide direct empirical evidence that a single curve cannot be used to estimate forward rates with different tenors. This can be seen if we observe the levels in the market. For instance, if the 1x4 FRA3M between March 18 and June 18, $(\tau_{F;1x4} = 0.25556)$, is $F_{1x4}^{mkt} = 1.696\%$ and the 4x7 FRA3M between June 18 and September18, $(\tau_{F;4x7} = 025556)$, is $F_{4x7}^{mkt} = 1.580\%$, we can compound these two rates to obtain the implied 1x7 FRA6M between March 18 to September 18, $(\tau_{F;1x7} = 0.50556)$, as

$$F_{1x7}^{implied} = \frac{\left(1 + F_{1x4}^{mkt}\tau_{F;1x4}\right) \cdot \left(1 + F_{4x7}^{mkt}\tau_{F;4x7}\right) - 1.0}{} = 1.641\%$$

while the market quote for the $1x7$ FRA6M might be $F_{1x7}^{mkt} = 1.831\%$, that is, 19 basis point higher. The difference is a liquidity risk premium in the market.

Market FRAs can be used together with deposits to construction of the short-term structure of the yield curve C_x. If $F_x(t;\ T_{i-1},\ T_i)$ is the i:th Euribor forward rate resetting at time T_{i-1} with tenor $x = T_i - T_{i-1}$ months associated to the i:th FRA with maturity T_i, the implied discount factor at time T_i is obtained by

$$p_x(t_0, T_i) = \frac{p_x(t_0, T_{i-1})}{1 + F_x(t_0, T_{i-1}, t_i) \cdot \tau_F(T_{i-1}, T_i)}, \quad t_0 < T_{i-1} < T_i.$$

Note that the closer the forward starting date of each FRA T_{i-1} gets to the current date t_0 the closer each FRA rate gets to the matching spot deposit rate

$$\lim_{T_{i-1} \to t_0} F_x(t_0, T_{i-1}, T_i) = R_x^{depo}(t_0, T_i).$$

Interest rate futures are the exchange-traded counterparts to the over-the-counter FRAs. While FRAs have the advantage of being more customizable, futures are highly standardized contracts which can be bought and sold at some exchanges. The most common contracts are traded at the Intenational Money Market in Chicago (so-called *IMMFutures*). They refer to Euribor3M or USD Libor3M. Some contracts are also traded in London at LIFFE. These standardized contracts expire the third Wednesday, every March, June, September and December (the IMM dates). The closing rates for these contracts will be fixed by the exchange on the third Wednesday of the maturity month, the last trading day being the preceding Monday (because of the 2 days of settlement).

Notice that such a date grid is not always regular: if S_i is the maturity date of the ***i'th*** futures, then S_i and T_i, such that $\tau_F(S_i, T_i) = 3M$, are the underlying FRA3M start and end dates, respectively, and, in general, $T_i \neq S_{i+1}$. On some exchanges, futures are quoted in terms of prices instead of rates, for example, eurodollar futures. Then, the relation between rate and price is

$$P_x^{Fut}(t_0, S_i, T_i) = 100 - R_x^{Fut}(t_0, S_i, T).$$

Since exchange-traded futures have a daily marking-to-market mechanism they do not have exactly the same pay-off as FRAs. An investor who is long a futures contract will have a loss when the futures price increases (and the futures rate decreases) but he will finance such loss at a lower rate. On the other hand, when the futures price decreases the daily profits will be reinvested at a higher rate. This means that the volatility of the forward rates and their correlation to the spot rates have to be accounted for; hence a *convexity adjustment* is needed to convert the rate R_x^{Fut} implied in the futures price to its corresponding forward rate *Fx*,

$$F_x(t_0, S_i, T_i) = R_x^{Fut}(t_0, S_i, T) - C_x(t_0, S_i, T_i).$$

In other words, the trivial unit discount factor implied by daily margination according to the rules of the exchange introduces a pricing measure mismatch with respect to the corresponding FRA case that generates a volatility-correlation-dependent convexity adjustment. The calculation of the convexity adjustment thus requires a model for the evolution of the underlying interest rates. While advanced approaches are available in literature, a standard practitioner's recipe is based on a simple short rate 1-factor Hull & White model.

EUR futures on Euribor3M uses in their names, the letters "H", "M", "U" and "Z", stand for March, June, September and December expiries, respectively. A future FUT3MH6 means a March 2016 expiry.

Futures on an *x*-tenor Euribor can be used as bootstrapping instruments for the construction of short-medium term structure section of the yield curve. Be aware of the fact that futures contracts have expiration dates gradually shrinking to zero and generate rolling pillars that periodically jump and overlap the depo and FRA pillars. Hence some *priority* rule must be used in order to decide which instruments should be excluded from the bootstrapping procedure.

Given the i'th futures market quote $P_x^{Fut}(t_0,\ S_i,\ T_i)$ with underlying FRA maturity T_i, the implied discount factor at T_i is given by

$$p_x(t_0, T_i) = \frac{p_x(t_0, T_{i-1})}{1 + \left\{R_x^{Fut}(t_0, S_i, T_i) - C_x(t_0, S_i, T_i)\right\} \cdot \tau_F(S_i, T_i)}.$$

The aforementioned expression can be used to bootstrap the yield curve C_x at point T_i once point S_i is known.

Interest rate swaps are OTC contracts in which two counterparties agree to exchange fixed against floating rate cash flows. The EUR market quotes standard plain vanilla swaps starting at spot date with annual fixed leg vs. floating leg indexed to *x*-months Euribor rate paid with *x*-months frequency. Such swaps can be regarded as portfolios of FRA contracts (the first one being actually a deposit). The day count convention for the quoted (fair) swap rates is 30/360 (*bond basis*).

Swaps can be selected as bootstrapping instruments for the construction of the medium-long term structure section of the yield curve.

By setting $T_0 = S_0 = t = t_0$ and $T_n = S_m = T_i = S_j$ in

$$S_x(t, T, S) = \frac{\sum_{j-1}^{m} p_x(t, S_j)\tau_F(S_{j-1}, S_j)F_x(t, S_{j-1}, S_j)}{A_x(t, T)} = \frac{p_x(t, T_0) - p_x(t, T_n)}{A_x(t, T)},$$

we obtain

$$\begin{aligned} S_x(t, T_i) &= \frac{\sum_{k=1}^{j} p_x(t_0, S_k)\tau_F(S_{k-1}, S_k)F_x(t_0, S_{k-1}, S_k)}{A_x(t_0, T_i)}, \\ &= \left[\sum_{k=1}^{j-1} p_x(t_0, S_k)\tau_F(S_{k-1}, S_k)F_x(t_0, S_{k-1}, S_k) + p_x(t_0, S_{k-1}) - p_x(t_0, T_i)\right] \\ &\quad \times \frac{1}{A_x(t_0, T_{i-1}) + p_x(t_0, T_i)\tau_S(T_{i-1}, T_i)}, \end{aligned}$$

where the last discount factor $p_x(t_0, T_i)$ has been separated in the second line and $\tau_S(T_1,\ T_2) = \tau[T_1,\ T_2,\ 30/360(\textit{bond basis})]$. $A_x(t, T)$ is the *annuity* factor. This can be inverted to find $p_x(t_0,\ T_i)$ as

$$\begin{aligned} p_x(t_0, T_i) = \Bigg[&\sum_{k=1}^{j-1} p_x(t_0, S_k)\tau_F(S_{k-1}, S_k)F_x(t_0, S_{k-1}, S_k) \\ &+ p_x(t_0, S_{k-1}) - S_x(t_0, T_{i-1})A_x(t_0, T_{i-1})\Bigg] \times \frac{1}{1 + S_x(t_0, T_i)\tau_S(T_{i-1}, T_i)}. \end{aligned}$$

This last formula can be used to bootstrap the yield curve at point $T_i = S_j$ once the curve points at $\{T_1, \ldots, T_{i-1}\}$ and $\{S_1, \ldots, S_{j-1}\}$ are known. Since the fixed leg frequency is annual and the floating leg frequency is given by the underlying Euribor rate tenor, we have that $\{T_1, \ldots, T_i\} \subseteq \{S_1, \ldots, S_j = T_i\}$ for any given fixed leg date T_i. Hence some points between $p_x(t_0,\ T_{i-1})$ and $p_x(t_0,\ T_i)$ may be unknown and one must resort to interpolation and, in general, to a numerical solution.

Interest rate (single-currency) **basis swaps** are floating vs. floating swaps admitting underlying rates with different tenors. The EUR market quotes standard plain vanilla basis swaps as portfolios of two swaps with the same fixed legs and floating legs paying Euribor xM and yM, for example, 3M vs. 6M, 1M vs. 6M, 6M vs. 12M, etc. Basis swaps are a fundamental element for long-term multi-curve bootstrapping, because they allow one to imply levels for non-quoted swaps on Euribor 1M, 3M and 12M, which can be selected as bootstrapping instruments for the corresponding yield curves construction. If $\Delta_{x6M}(t_0,\ T_i)$ is the quoted basis spread for a basis swap receiving Euribor xM and paying Euribor6M plus spread for maturity T_i, we simply have

$$S_x(t_0,\ T_i) = S_{6M}(t_0,\ T_i) + \Delta_{x6M}(t_0, T_i).$$

Note that the bootstrapping of yield curves requires extrapolation of basis swap quotations.

21.1.4.3 Interpolation

The choosen interpolation method determines how reasonable the yield curve will be. For instance, linear interpolation of discount factors is an obvious but extremely poor choice. Linear interpolation of zero rates or log discounts are popular choices leading to stable and fast bootstrapping procedures, but unfortunately they produce horrible forward curves, with a sag saw or piecewise-constant shape.

In Fig. 21.5, we show examples of *bad* (but very popular!) interpolation scheme, linear interpolation. The lower curve is the Swedish zero swap curve in April 2016 based on deposits (O/N, T/N, 1W, 1M, 2M and 3M), IMM FRAs (with maturities between June 2016 and June 2018) and swaps (3Y–10Y, 12Y, 15Y, 20Y and 30Y). The upper curve shows the forward curve.

While the zero curve displays a smooth behaviour, the forward curve is non-smooth with oscillations that can exceed several basis points. Such discontinuities in the forward curves correspond to angle points (knees) in the zero curves, generated by linear interpolation that forces them to suddenly turn around a market point.

It is recommended to only use the most liquid swaps, with maturities 3–10, 12, 15, 20, 25 and 30 years in the bootstrapping. The remaining less liquid quotations for 11, 13, 14, 16–19, 21–24,

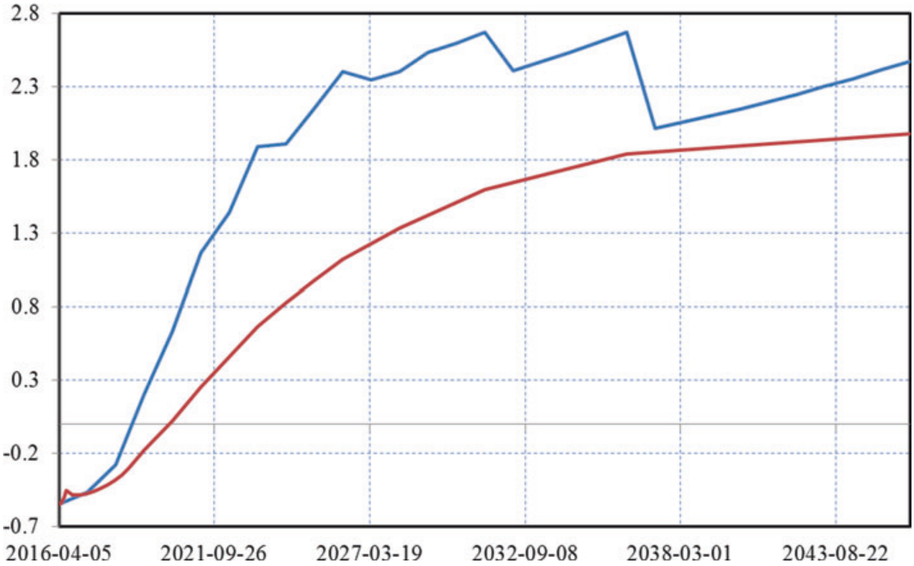

Fig. 21.5 A bootstrap of SEK swap curve with linear interpolation. This shows the very bad shape of the forward curve

26–29 years maturity should only be included in the linear interpolation schemes in order to reduce the amplitude of the forward curve oscillations.

The choice of cubic interpolations is a very delicate issue. Simple splines suffer of well-documented problems such as spurious inflection points, excessive convexity and lack of locality after input price perturbations. Some researchers found the classic Hyman monotonic cubic filter[4] applied to spline interpolation of log discounts to be the easiest and perhaps the best approach. Its monotonicity ensures non-negative forward curves and actually removes most of the unpleasant waviness. Notice that the Hyman filter can be applied to any cubic interpolants. This helps to address the non-locality of spline using alternative more local cubic interpolations.

21.1.4.4 No-Arbitrage and Forward Basis

Now, we wish to understand the consequences of the assumptions given earlier in terms of no arbitrage. First, we notice that in the

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[4] James M. Hyman. Accurate monotonicity preserving cubic interpolation. *SIAM Journal on Scientific and Statistical Computing*, 4(4):645–654, 1983.

multiple-curve framework the classic single-curve no-arbitrage relations are broken. For instance,

$$p_d(t,T_2) = p_d(t,T_1)\cdot p_f(t,T_1,T_2) \qquad t`T_1 \le T_2$$
$$p_f(t,T_1,T_2) = \frac{1}{1+F_f(t;T_1,T_2)\cdot \tau_f(T_1,T_2)}.$$

No arbitrage between distinct yield curves C_d and C_f can be immediately recovered by taking into account the forward basis, the forward counterparty of the quoted market basis, as

$$p_f(t,T_1,T_2) = \frac{1}{1+F_d(t;T_1,T_2)\cdot BA_{fd}(t;T_1,T_2)\cdot \tau_d(T_1,T_2)}$$

or through the equivalent simple transformation rule for forward rates

$$F_f(t;\ T_1,\ T_2)\ \tau_f(T_1,\ T_2) = F_d(t;\ T_1,\ T_2)\ \tau_d(T_1,T_2)BA_{fd}(t;\ T_1,\ T_2).$$

From this equation we can express the **forward basis** as a ratio between forward rates or, equivalently, in terms of discount factors from C_d and C_f curves as

$$BA_{fd}(t;T_1,T_2) = \frac{F_f(t;T_1,T_2)\cdot \tau_f(T_1,T_2)}{F_d(t;T_1,T_2)\cdot \tau_d(T_1,T_2)} = \frac{p_d(t,T_2)\cdot p_f(t,T_1) - p_f(t,T_2)}{p_f(t,T_2)\cdot p_d(t,T_1) - p_d(t,T_2)}.$$

Obviously the following alternative additive definition is completely equivalent

$$p_f(t,T_1,T_2) = \frac{1}{1+\left[F_d(t;T_1,T_2) + BA'_{fd}(t;T_1,T_2)\right]\cdot \tau_d(T_1,T_2)}$$
$$\begin{aligned} BA'_{fd}(t;T_1,T_2) &= \frac{F_f(t;T_1,T_2)\cdot \tau_f(T_1,T_2) - F_d(t;T_1,T_2)\cdot \tau_d(T_1,T_2)}{\tau_d(T_1,T_2)} \\ &= \frac{1}{\tau_d(T_1,T_2)}\left[\frac{p_f(t,T_1)}{p_f(t,T_2)} - \frac{p_d(t,T_1)}{p_d(t,T_2)}\right] \\ &= F_d(t;T_1,T_2)\left[BA_{fd}(t;T_1,T_2) - 1\right] \end{aligned}$$

which is more useful for comparisons with the market basis spreads. Notice that if $C_d = C_f$ we recover the single-curve case $BA_{fd}(t,\ T_1,\ T_2) = 1,\ BA'_{fd}(t,\ T_1,T_2) = 0$.

The forward basis in the previous equations is a straightforward consequence of the aforementioned assumptions, essentially the existence of two yield curves and no arbitrage. Its advantage is that it allows for a direct computation of the forward basis between forward rates for any time interval $[T_1,\ T_2]$, which is the relevant quantity for pricing and hedging interest rate derivatives. In practice its value depends on the market basis spread between the quotations of the two sets of vanilla instruments used in the bootstrapping of the two curves C_d and C_f.

The approach can be inverted to bootstrap a new yield curve from a given yield curve plus a given forward basis, using the following recursive relations:

$$p_{d,i} = \frac{p_{f,i} \cdot BA_{fd,i}}{p_{f,i-1} - p_{f,i} + p_{f,i} \cdot BA_{fd,i}} p_{d,i-1} = \frac{p_{f,i}}{p_{f,i-1} - p_{f,i} \cdot BA'_{fd,i} \cdot \tau_{d,i}} p_{d,i-1}$$

$$p_{f,i} = \frac{p_{d,i}}{p_{d,i} + \left(p_{d,i-1} - p_{d,i}\right) \cdot BA_{fd,i}} p_{f,i-1} = \frac{p_{d,i}}{p_{d,i-1} + p_{d,i-1} \cdot BA'_{fd,i} \cdot \tau_{d,i}} p_{f,i-1},$$

where $\tau_x(T_{i-1}, T_i) = \tau_{x,i}$, $p_x(t,\ T_i) = p_{x,i}$ and $BA_{fd}(t,\ T_{i-1},\ T_i) = BA_{fd,i}$. Given the yield curve C_x up to step $p_{x,i-1}$ plus the forward basis for the step $i - 1 \to i$, the earlier equations can be used to obtain the next step $p_{x,i}$.

We now discuss a numerical example of the forward basis in a realistic market situation where we consider the four underlying interest rates $I = \{I_{1M}, I_{3M}, I_{6M}, I_{12M}\}$, where I = Euribor index, and we bootstrap from market data five distinct yield curves $C = \{C_d,\ C_{1M},\ C_{3M},\ C_{6M},\ C_{12M}\}$, using the first one for discounting and the others for forwarding.

We build the discounting curve C_d following a "pre-crisis" traditional recipe from the most liquid deposit, IMM futures/FRA on Euribor3M and swaps on Euribor6M. The other four forwarding curves are built from convenient selections of deposits, FRAs, futures, swaps and basis swaps with homogeneous underlying rate tenors; a smooth and robust algorithm (monotonic cubic spline on log discounts) is used for interpolations. Different choices (e.g. an Eonia discounting curve) as well as other technicalities of the bootstrapping are described by Ferdinando M. Ametrano and Marco Bianchett (see references).

The corresponding multiplicative forward basis curves can be calculated as

$$BA_{fd}(t;T_1,T_2) = \frac{F_f(t;T_1,T_2)\cdot\tau_f(T_1,T_2)}{F_d(t;T_1,T_2)\cdot\tau_d(T_1,T_2)} = \frac{p_d(t,T_2)\cdot p_f(t,T_1)-p_f(t,T_2)}{p_f(t,T_2)\cdot p_d(t,T_1)-p_d(t,T_2)}$$

and the additive forward basis are given by

$$\begin{aligned} BA'_{fd}(t;T_1,T_2) &= \frac{F_f(t;T_1,T_2)\cdot\tau_f(T_1,T_2)-F_d(t;T_1,T_2)\cdot\tau_d(T_1,T_2)}{\tau_d(T_1,T_2)}, \\ &= \frac{1}{\tau_d(T_1,T_2)}\left[\frac{p_f(t,T_1)}{p_f(t,T_2)}-\frac{p_d(t,T_1)}{p_d(t,T_2)}\right] \\ &= F_d(t;T_1,T_2)\left[BA_{fd}(t;T_1,T_2)-1\right]. \end{aligned}$$

21.1.5 Modern Pricing

According to Bianchetti and Morini (2010), the new market situation has induced a sort of "segmentation" of the interest rate market into sub-areas, mainly corresponding to instruments with 1, 3, 6 and 12 months underlying rate tenors. These are characterized, in principle, by different internal dynamics, liquidity and credit risk premia, reflecting the different forwarding horizon views and interests of the market participants ("on average" i.e. views are weighted by how much Capital each investor is willing and able to invest according to his/her opinion).

In response to the crisis, the classical pricing framework, based on a single yield curve used to calculate forward rates and discount factors, has been abandoned, and a new modern pricing approach has been created by practitioners. This new methodology takes into account the market segmentation as an empirical fact and incorporates the new interest rate dynamics into a multiple curve framework as follows.

- **Discounting curves**: these are the yield curves used to discount futures cash flows. These curves must be constructed and selected so that they reflect the funding cost of the bank in combination with the actual nature of the specific contract that generates the cash flows. In particular:

  - an OIS-based curve is used to discount cash flows generated by a contract under CSA with daily margination and overnight collateral rate;
  - a funding curve is used in case of contracts without CSA;
  - in case of non-standard CSA (e.g. different margination frequency, rate, threshold, etc.), an appropriate funding curve should in principle be selected, but we will not discuss this topic here.
- We stress that the funding curve for non-CSA contracts is specific to each counterparty that will have its specific funding curve. This modern discounting methodology is called CSA-discounting.
- **Forwarding curves**: these are the yield curves used to compute forward rates. As discussed before, the curve must be constructed and selected according to the tenor and typology of the rate underlying the actual contract to be priced. For instance, a swap floating leg indexed to Euribor6M requires a Euribor6M forwarding curve constructed from quoted instruments with Euribor6M as the underlying rate.

During the pricing process it is important to generate all cash flows from the curve based on instruments with the same tenor and then discount with the matching discount curve. It is also important to use an exact fit bootstrap method and a good interpolation method.

21.1.6 Pricing Under Collateralization

The financial crisis in 2008 was a catalyst for significant changes in the market. Financial practitioners witnessed a tremendous increase in basis swap spreads, implying a divergence from implied rates and traded rates in interest rate markets. When collateral agreements increased in use and when counterparties began discounting at the overnight rates dictated by the credit support annex, the world had forever changed.

As a result, many financial institutions are currently in the process of migrating to new market standards. But questions remain as to the potential impact on existing portfolios and how to effectively manage instruments with longer-dated maturities when spreads in Libor vs. OIS rates diverge.

Collateral discounting and the impact of standardization in the market are adding a whole new level of complexity to derivative pricing and risk management. Market participants are seeking a deeper understanding when it comes to the potential consequences of moving to collateral discounting.

With such collateral agreements, also called CSA agreements, we have to make changes in the valuation formulas. The swap market has already moved to the dual-curve approach, with the London Clearing House (LCH) using OIS discounting for clearing swaps, and the International swaps and Derivatives Association (ISDA) standardization on the imminent horizon. Other markets, including swaptions, caps/floors, exotics and equities, are still evolving. Now, receivers of fixed rates gain under OIS discounting, while payers lose when compared to the old ways. Taking into account recent studies of the swap market moving towards the multi-curve approach, we come to the conclusion that swaps can be significantly mispriced under the single curve framework.

In the market, one can observe that spreads and OIS rate risks are close for par swaps, but differ significantly for out-of-the-money (OTM) swaps (vs. ATM swaps). Under the single curve approach, one neglects the risk that actually exists, while under the dual-curve approach one can estimate these risks.

When we price under a collateral agreement we make two simplifying assumptions:

1. We consider **full collateralization** (with zero thresholds) by cash.
2. The collateral is adjusted continuously with zero minimum transfer amount (MTA).

Most CSA contracts include a threshold and an MTA to avoid posting small amounts of cash each day. The CSA contracts can be one sided or bilateral. About 16% of the agreements are unilateral (one sided) while 84% bilateral.

21.1.6.1 Problems in the Old-Style Implementation

We start by considering a plain vanilla tenor basis swap,[5] typically 3-months floating against 6-months floating rate as illustrated in Fig. 21.6.

[5] It is also common that payment of short-tenor leg is compounded and paid at the same time with the other leg.

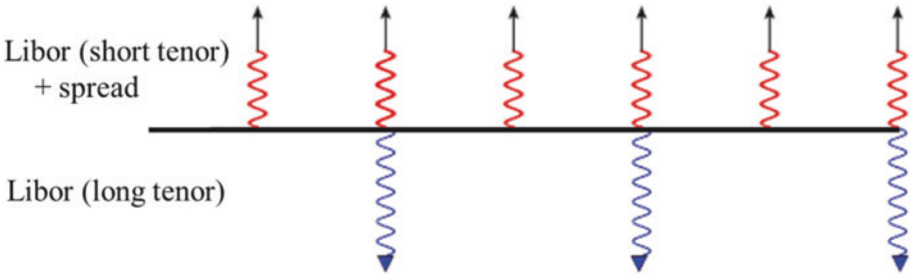


Fig. 21.6 A 3-month floating rate (the upper cash flows) against a 6-month floating rate (the lower cash flows). The arrow above the upper "wave" represents the spread over the floating rate

In the old-style implementation, the spread between the rates in Fig. 21.6 was typically zero (or very near zero). However, since 2008 the spread is quite significant and also volatile as seen in Fig. 21.7, where tenor swap spreads for maturities 1, 3, 5, 7, 10 and 20 years are shown. This spread represents the difference in risk; the long tenor has a higher risk than the short tenor.

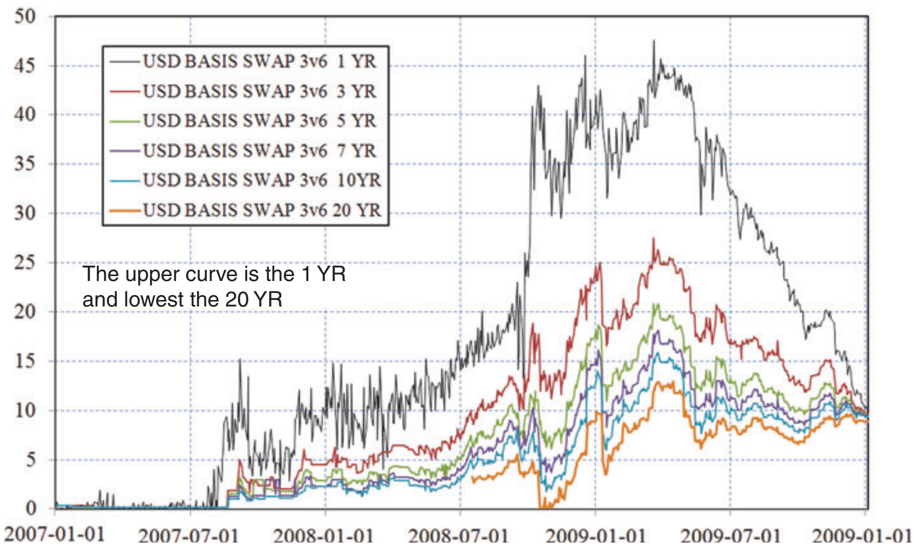


Fig. 21.7 Historical data for USD 3-month vs. 6-month TS spread. The curves are given in the same order as the legends[6]

[6] Source Bloomberg.

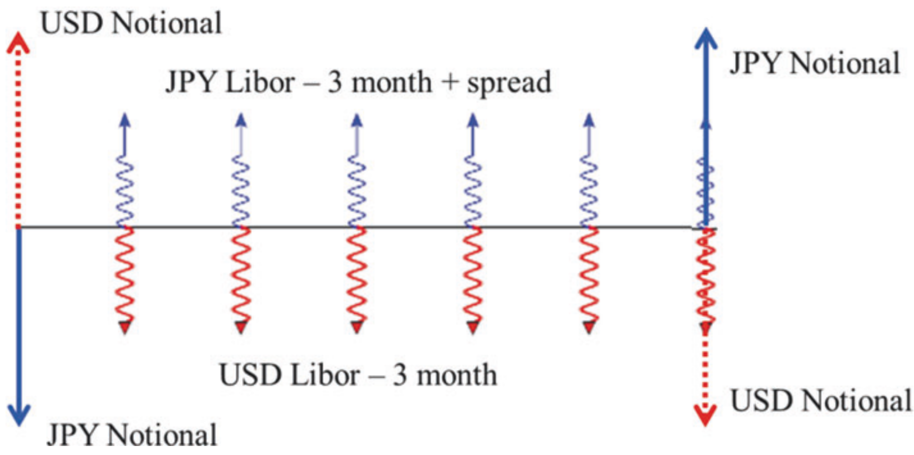


Fig. 21.8 A 3-month floating rate in JPY (with a constant spread) against a 3-month floating rate in USD. The arrow above the upper "wave" represents the JPY spread over USD

Similar problems are found if we consider cross-currency swaps (CCS or CIRS) as shown in Fig. 21.8.

In the old-style implementation the spread in a CCS was zero. However, since 2007 the spread is quite significant and volatile. Here we have seen a drastic change in recent years. This is shown in Fig. 21.9 where we show the cross-currency USD/JPY spreads for maturities 1, 3, 5, 7 and 10 years.

Before 2008, most banks did their funding on the interbank markets; that is, the trades were made without collateral. Then we had the situation as in Fig. 21.10.

According to the old view, the purchase of an OTC derivative was funded with an unsecured external loan.

Here counterparty *R* (*Red*) buys a derivative of *B* (*Blue*) where the value of the derivative is given by some optional payment. To fund the trade, *R* takes a loan from another party or internally in his own bank.[7] The interest A pays on this loan is (supposed to be) given by Libor. We then have unsecured funding. A Libor rate is §, an unsecured offer rate in the interbank (deposit) market and Libor discounting is appropriate for such unsecured trades between financial firms with Libor credit

[7] If the trader is a person who funds the trade with his/her savings from his/her own money-market account, he/she will typically earn less interest on this amount than the running rate on his/her own money-market account.

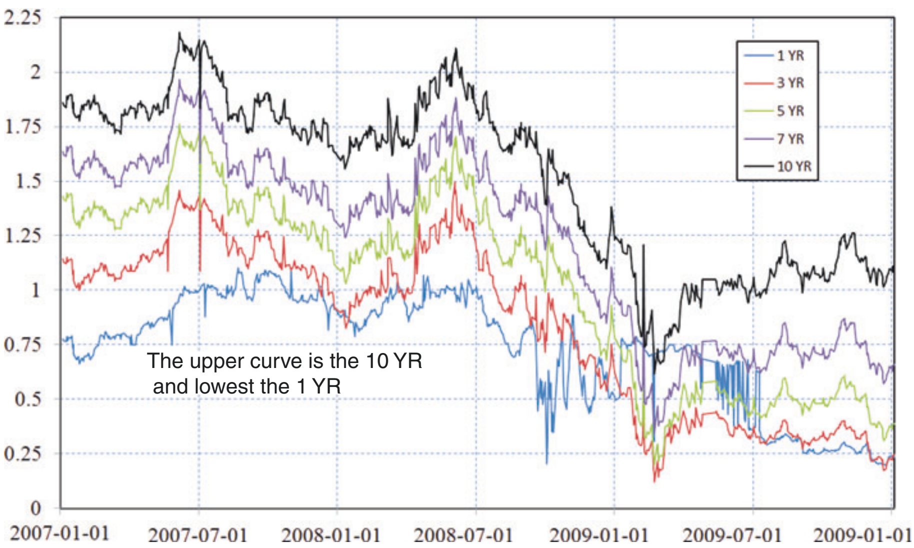


Fig. 21.9 Historical data for USD/JPY cross-currency spread. The curves are given in the reverse order as the legends[8]

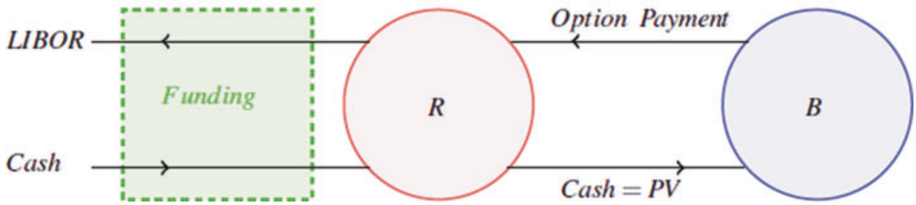


Fig. 21.10 Funding via the nterbank market

quality. In addition, Libor discounting makes the present value of the loan equal to zero.

After the credit crisis in 2008, banks started requesting collateral. This modifies the pricing of OTC derivatives. The current view is to require collateral, so the funding is secured, and the picture looks like Fig. 21.11.

In the new view, we finance the trade by "a loan" from the counterparty via collateral from counterparty *B*.

Here, the outright cash flow, that is, the collateral is equal to the PV of the trade so no external funding is needed. And when the PV of the trade changes, *R* pays back or receives more collateral from *B*. This collateral is (ideally) paid/received once every day. Also, the party who

[8] Source Bloomberg.

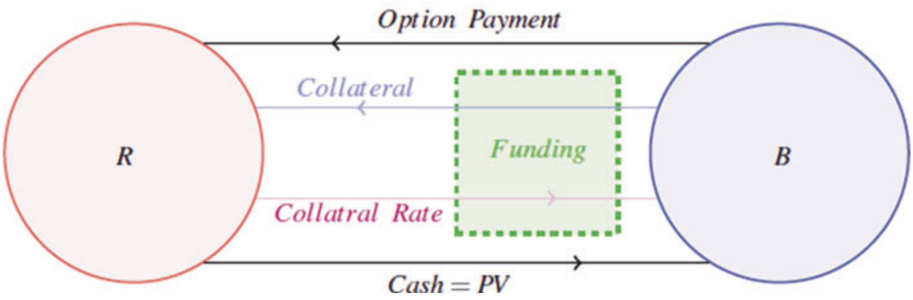


Fig. 21.11 Funding via collateral

holds the collateral must pay the other party interest. The size of this interest is specified in the collateral agreement. Since the collateral is posted on a daily basis, the relevant default risk is an overnight risk. Therefore, the collateral rate is usually the over-night (O/N) rate, so we use OIS to bootstrap the discount curve. When funding is provided by the collateral agreement Libor discounting is inappropriate.

21.1.6.2 Collateral Agreements

In Table 21.1 we provide some details for a given collateral agreement.

In many CSA agreements you also have the opportunity to post (government) bonds. When posting bonds there normally is a haircut. This haircut is a percentage that is subtracted from the market value of the asset that is being used as collateral. The size of the haircut reflects the perceived risk associated with holding the asset. The higher the haircut, the safer the loan as regarded by the lender. For example,

Table 21.1 A simplified collateral agreement

Base currency	USD
Eligible currency	USD, EUR, GBP
Independent amount	5M
Haircuts	[Schedule]
Threshold	50M
Minimum transfer amount	500,000
Rounding	Nearest 100,000 USD
Valuation agent	Party A
Valuation date	Daily, NewYorkBusinessDay
Notification time	2PM, NewYorkBusinessDay
Interest rate	OIS, EONIA, SONIA
Daycount	Act/360

United States Treasury bills, which are seen as fairly safe, might have a haircut of only 10%. For most other bonds the collateral is higher.

In most CSA agreements you also have to consider the MTA and the **threshold amount**. The MTA is the smallest amount of currency that is allowed for transfer as collateral. For large banks, the MTA is usually in the USD 100,000 range, but can be lower. The threshold amount is the amount of unsecured credit risk that two counterparties are willing to accept before any demand for additional collateral demand will be made. The counterparties typically agree to a threshold amount prior to committing to the deal, but this is a source of on-going friction between OTC counterparties and their brokers.

All these choices in the CSA agreements complicate the application of the theoretical priniciples. However, we will omit this here since it is a rather complex subject. But we need to remember this when dealing with the full valuation in a practical situation. For simplicity, we will pursue the analysis of CSAs with MTA and threshold amount set to zero and in addition assume that the posting frequency is instantaneous. Instantaneous posting enables us to approximate the collateral with an integral instead of a sum. Furthermore, we only use cash as collateral.

Also note that, except for the MTA and thresholds, CSA typically does not involve hedging 100% of the credit exposure. As illustrated in Fig. 21.12, a 6-year swap (paying fixed at 2.5% on $1,000,000 notional)

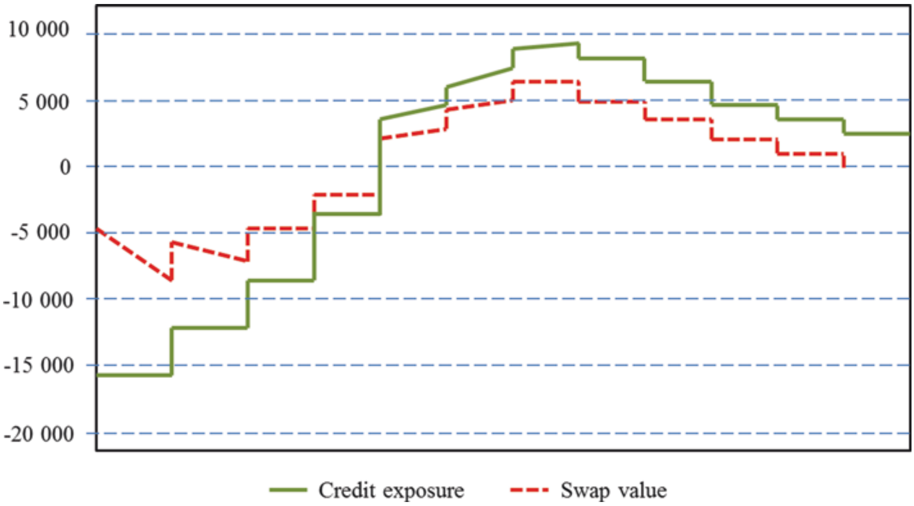


Fig. 21.12 A 6-year swap paying fixed rate at 2.5% on 1,000,000 notional. The collateral amount is the difference between the credit exposure and the swap value

cannot be 100% funded with collateral. The reason is that the value of the trade changes faster than the collateral frequency, and MTA and threshold considerations must also be taken into account, in addition to cash flow events. We must also consider the movements in the value of the collateral itself.

Fig. 21.12 shows the total exposure of a swap over the lifetime of the trade, and a hypothetical collateral amount. Since imperfections exist, the total exposure of the trade is not perfectly hedged. The mismatch between the current exposure and the amount in the collateral account means funding is not purely OIS. This is the reason for funding value adjustments (FVA), which we will discuss later.

21.1.6.3 Overnight Index Swaps

For an OIS the floating rate is a daily compounded O/N rate and the market quotes; the fixed rates are called OIS rates[9] (see Fig. 21.13).

Figure 21.13 shows the Libor vs. OIS spread in USD and JPY over a few years. According to the old view, an inappropriate kind of discounting was used for secured trades which mispriced future cash flows. This has as a significant impact on multi-currency trades where the change is given by

$$\textit{Change} \approx \textit{Notional} \times \textit{Duration} \times (\textit{Difference in discount rate})$$

This is also inconsistent with CCS. The implied Foreign Exchange (FX) forwards are off market, leading to mispricing of foreign Libors. Long-dated FX products are most affected. We also get wrong forward Libors and overestimation of forward Libors with short tenors.

Fig. 21.14 shows the 3-month Libor vs. OIS spread in USD and JPY over a few years.

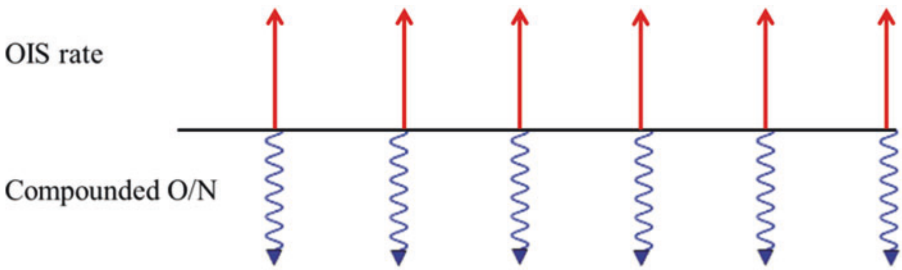


Fig. 21.13 A typical overnight index swap

[9] Usually, there is only one payment for tenors (maturities) shorter than a year.

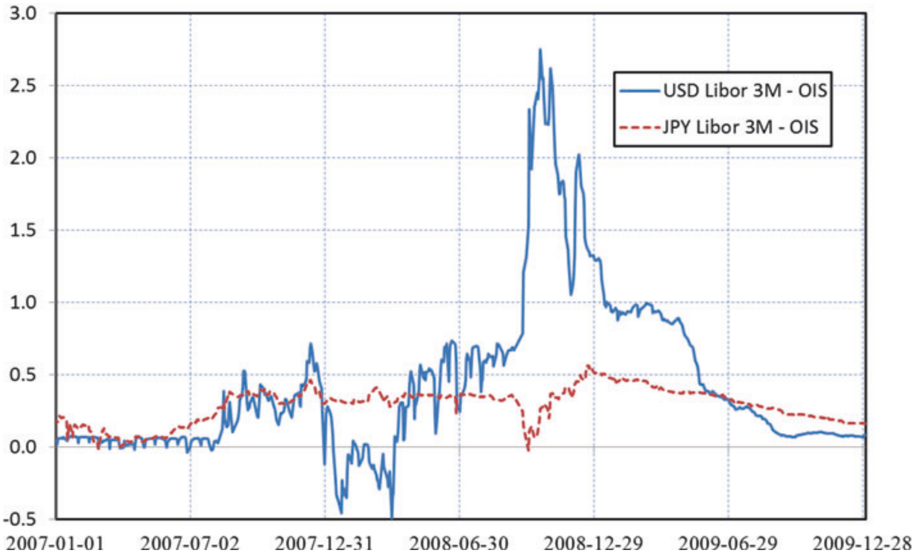


Fig. 21.14 A 3-month Libor rate vs. the overnight index swap spread in USD and JPY

The bootstrapping of OIS curves is made by the same formula that we use when we bootstrap interest rate swaps

$$D(t, T_N) = \frac{D(t, T_0) - R_N^{OIS} \sum_{n=1}^{N-1} \Delta_n D(t, T_n)}{1 + R_N^{OIS} \cdot \Delta_n}.$$

Here we get the discount factors from the quoted OIS rates R_N^{OIS} with maturity T_N and day fraction Δ_n. The continuously compounded zero-coupon rate is then calculated by

$$Z_N^{OIS} = -100 \cdot \frac{\ln\left(D(t, T_N)\right)}{\frac{d_N}{365}}$$

21.1.7 Pricing with Collateral Agreements

With no collateral (or collateral in domestic rate r^d), the price of a contingent claim paying $\Pi(T)$ at maturity T is given by the Feynmann-Kaè representation

$$\Pi^d(t) = E^{Q_d}\left[\exp\left\{\int_t^T r^d(u)du\right\} \cdot \Pi^d(T)|\mathcal{F}_t\right] = E^{Q_d}\left[\frac{B^d(t)}{B^d(T)} \cdot \Pi^d(T)|\mathcal{F}_t\right].$$

Here the expectation is taken with respect to the domestic martingale (risk-neutral) probability measure Q^d and the domestic money-market account

$$\begin{cases} dB^d(t) = r^d(t)B^d(t)dt \\ B^d(0) = 1 \end{cases} \quad \Rightarrow \quad B^d(t) = \exp\left\{\int_0^t r^d(u)du\right\}$$

so

$$\exp\left\{-\int_t^T r^d(u)du\right\} = \frac{B^d(t)}{B^d(T)}.$$

If there is a collateral agreement where we are supposed to get or pay cash collateral in a foreign currency f, then we have to add an extra term in the pricing formula for the aforementioned contingent claim. Say that we are in the money and get collateral from our counterparty. In this case we are free to invest this collateral in the market. However, we also have to pay interest to our counterparty since he/she still owns the collateral. This collateral rate is specified in the CSA agreement between the counterparties. If we denote the instantaneous return (or cost when it is negative) by holding the collateral in the foreign currency f at time t by $y^f(t)$, we have

$$y^f(t) = r^f(t) - c^f(t).$$

Here $r^f(t)$ denotes the "risk-free" interest rate, assumed to be equal to the funding or repo rate for uncollateralized assets and $c^f(t)$ the collateral rate in the foreign currency f. In the case of cash collateral, the collateral rate $c(t)$ is usually given by the overnight rate, that is, the OIS rate of the corresponding currency. When using treasury bonds as collateral, the collateral rate is usually the repo rate. For corporate bonds, we can use Libor plus a spread.

The collateral rate is pre-specified in the collateral agreement. Therefore, we let $r(t)$ accumulate in a money-market (bank) account and $c(t)$ in a collateral account. When we get collateral, we invest this money in the market.

The spot measure is taken to be that measure under which uncollateralized assets grow at $r(t)$, and collateralized assets grow at $c(t)$. Both uncollateralized $p(t, T)$ and collateralized $D(t, T)$ zero-coupon bonds pay 1 unit in the domestic currency at time T.

If we denote the PV of the derivative at time t by $\Pi^d(t)$ (in terms of the domestic currency d), the collateral amount posted from the counterparty in the foreign currency f is given by $\Pi^f(t) \equiv \Pi^d(t)/fx^{d,f}(t)$, where $fx^{d,f}(t)$ is the foreign exchange rate at time t,representing the price of the unit amount of currency f in terms of currency d. These considerations lead to the following calculations for the collateralized derivative price:

$$\Pi^d(t) = E^{Q_d}\left[\frac{B_t^f}{B_T^f}\cdot\Pi_T^d|\mathcal{F}_t\right] + fx_t^{d,f}\cdot E^{Q_f}\left[\int_0^T \frac{B_t^f}{B_s^f}y_s^f\left(\frac{\Pi_s^d}{fx_s^{d,f}}\right)ds|\mathcal{F}_t\right].$$

We start by calculating the expectation

$$E^{Q_f}\left[\int_0^T \frac{B_t^f}{B_s^f}y_s^f\left(\frac{\Pi_s^d}{fx_s^{d,f}}\right)ds|\mathcal{F}_t\right]$$
$$= E_t^{Q_d}\left[\int_0^T \frac{B_t^f}{B_s^f}\cdot\frac{B_s^f}{B_s^d/fx_s^{d,f}}\cdot\frac{B_t^d/fx_t^{d,f}}{B_t^f}\cdot y_s^f\left(\frac{\Pi_s^d}{fx_s^{d,f}}\right)ds|\mathcal{F}_t\right],$$

where we have used $B^f(s) = B^d(s)/f_x^{(d,f)}(s)\forall_s$ in order to change the measure from Q_f to Q_d. This can be simplified into

$$E^{Q_f}\left[\int_0^T \frac{B_t^f}{B_s^f}\cdot y_s^f\left(\frac{\Pi_s^d}{fx_s^{d,f}}\right)ds\right] = E_t^{Q_d}\left[\int_0^T \frac{B_t^d}{B_s^d}\cdot\frac{fx_s^{d,f}}{fx_t^{d,f}}\cdot y_s^f\left(\frac{\Pi_s^d}{fx_s^{d,f}}\right)ds\right],$$
$$= \frac{1}{fx_t^{d,f}}E_r^{Q_d}\left[\int_0^T \frac{B_t^d}{B_s^d}\cdot y_s^f\cdot\Pi_s^d ds\right],$$
$$= \frac{1}{fx_t^{d,f}}E_t^{Q_d}\left[\int_0^T \exp\left\{-\int_t^s r_u^d du\right\}\cdot y_s^f\cdot\Pi_s^d ds\right].$$

Using this formula we have

$$\Pi_t^d = E_t^{Q_d}\left[\exp\left\{-\int_t^T r_u^d du\right\}\cdot\Pi_T^d + \int_t^T \exp\left\{-\int_t^s r_u^d du\right\}\cdot y_s^f\cdot\Pi_s^d ds\right]$$

which is equivalent to

$$\Pi_t^d = \exp\left\{\int_0^t r_u^d du\right\}$$
$$\times\ E_t^{Q_d}\left[\exp\left\{-\int_0^T r_u^d du\right\}\cdot \Pi_T^d + \int_t^T \exp\left\{-\int_t^s r_u^d du\right\}\cdot y_s^f \cdot \Pi_s^d ds\right],$$
$$= B_t^d \cdot E_t^{Q_d}\left[\frac{\Pi_T^d}{B_T^d}\cdot + \int_t^T \frac{\Pi_s^d}{B_s^d}\cdot y_s^f ds\right],$$
$$= B_t^d \cdot E_t^{Q_d}\left[\frac{\Pi_T^d}{B_T^d}\cdot + \int_0^T \frac{\Pi_s^d}{B_s^d}\cdot y_s^f ds - \int_0^t \frac{\Pi_s^d}{B_s^d}\cdot y_s^f ds\right],$$
$$= B_t^d \cdot \left(E_t^{Q_d}\left[\frac{\Pi_T^d}{B_T^d} + \int_0^T \frac{\Pi_s^d}{B_s^d}\cdot y_s^f ds\right] - \int_0^t \frac{\Pi_s^d}{B_s^d}\cdot y_s^f ds\right).$$

Now we see that

$$E_t^{Q_d}\left[\frac{\Pi_T^d}{B_T^d} + \int_0^T \frac{\Pi_s^d}{B_s^d}\cdot y_s^f ds\right] = \frac{\Pi_t^d}{B_t^d} + \int_0^t \frac{\Pi_s^d}{B_s^d}\cdot y_s^f ds = M(t)$$

since the expectation is a martingale under the domestic money-market account. Here we have also defined $M(t)$. Remark that $E_d^{Q_d}[M(T)] = M(t)$. Next, use Ito's lemma on

$$M(t) = \frac{1}{B_t^d}\cdot \Pi_t^d + \int_0^t \frac{1}{B_s^d}\cdot y_s^f \cdot \Pi_s^d ds.$$

The result is

$$dM_t = \frac{\partial M_t}{\partial t}dt + \frac{\partial M_t}{\partial B_t^d}dB_t^d + \frac{\partial M_t}{\partial \Pi^d(t)}d\Pi^d(t)$$
$$= \frac{\Pi_t^d}{B_t^d}\cdot y_t^f dt - \frac{\Pi_t^d}{\left(B_t^d\right)^2} r_t^d \cdot B_t^d dt + \frac{1}{B_t^d}d\Pi_t^d = \left(\frac{\Pi_t^d}{B_t^d}\cdot y_t^f - \frac{\Pi_t^d}{B_t^d}\cdot r_t^d\right)dt + \frac{1}{B_t^d}d\Pi_t^d$$
$$= \exp\left\{-\int_0^t r_u^d du\right\}\cdot\left(y_t^f - r_t^d\right)\cdot \Pi_t^d dt + \exp\left\{-\int_0^t r_u^d du\right\}\cdot d\Pi_t^d$$

or

$$\exp\left\{\int_0^t r_u^d du\right\} dM_t = \left(y_t^f - r_t^d\right) \cdot \Pi_t^d dt + d\Pi_t^d$$

that is,

$$d\Pi_t^d = \left(y_t^f - r_t^d\right) \cdot \Pi_t^d dt + \exp\left\{\int_0^t r_u^d du\right\} dM_t.$$

We can now solve the previously mentioned Stochastic Differential Equation (SDE):

$$\begin{cases} d\Pi = \alpha \cdot \Pi dt + dB \\ \Pi(t) = \pi \end{cases}$$

where we use

$$\alpha = \left(y^f(t) - r^d(t)\right)$$

and

$$dB = \exp\left\{\int_0^t r^d(u) du\right\} dM(t).$$

Set $X = e^{-\alpha t}\Pi$ and use Ito

$$dX = \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial \Pi} d\Pi = -\alpha \cdot X \cdot dt + \alpha \cdot X \cdot dt + e^{-\alpha t} dB = e^{-\alpha t} dB.$$

If we integrate this we get

$$X(T) - X(t) = \int_t^T e^{-\alpha u} dB \Rightarrow e^{-\alpha T}\Pi(T) - e^{-\alpha t}\Pi(t) = \int_t^T e^{-\alpha u} dB$$

and by taking expectation value we get

$$\Pi(t) = e^{\alpha(T-t)} E_t^{Q_d}\left[\Pi(T)\right].$$

Finally we have the following theorem.

Theorem 21.1. *Suppose that $\Pi^d(T)$ is a derivative's pay-off at time T in terms of the domestic currency d and that the foreign currency f is used as the collateral for the contract. Then, the value of the derivative at time t, $\Pi^d(t)$ is given by*

$$\Pi^d(t) = E_t^{Q_d}\left[\exp\left\{-\int_t^T r^d(s)ds\right\}\cdot\exp\left\{\int_t^T y^f(s)ds\right\}\Pi^d(T)\right],$$

$$= D^d(t,T)E_t^{T_d}\left[\exp\left\{\int_t^T\left(y^d(s)-y^f(s)\right)ds\right\}\Pi^d(T)\right].$$

Here we also have defined the collateralized zero-coupon bond of currency d as

$$D^d(t,T) = E_t^{Q_d}\left[\exp\left\{-\int_t^T c^d(s)ds\right\}\right]$$

or equivalently

$$c^d(t,T) = -\frac{\partial}{\partial T}\ln D^d(t,T)$$

and the collateralized forward measure T_d, where the collateralized zero-coupon bond is used as the numeraire.

In the case where the deal and collateral currencies are different, (d) and (f) respectively, we define the foreign collateralized zero-coupon bond $D(d,f)$ by

$$D^{d,f}(t,T) = E_t^{Q_d}\left[\exp\left\{-\int_t^T c^d(s)ds\right\}\cdot\exp\left\{-\int_t^T y^{d,f}(s)ds\right\}\right].$$

In particular, if c^d and $y^{d,f}(t) = (r^d(t) - c^d(t)) - (r^f(t) - c^f(t))$ are independent, we have

$$D^{d,f}(t,T) = D^d(t,T)\cdot\exp\left\{-\int_t^T y^{d,f}(s)ds\right\},$$

where

$$y^{d,f}(t,s) = -\frac{\partial}{\partial s} \ln E_t^{Q_d}\left[\exp\left\{-\int_t^s y^{d,f}(s)ds\right\}\right].$$

We also have some corollaries to the Theorem 21.1.

Corollary 21.1 *When the foreign "risk-free" rate is the foreign collateral rate, that is, when $r^f(t) = c^f(t)$ and the collateral is posted in foreign currency, the PV of the derivative at time t becomes*

$$\Pi(t) = E_t^Q\left[\exp\left\{-\int_t^T r^d(s)ds\right\}\cdot \Pi(T)\right] = p(t,T)E_t^{T_d}\left[\Pi(T)\right],$$

where $p(t, T)$ is an uncollateralized zero-coupon bond and $E_t^{T_d}$ the expectation under the domestic forward measure with information given up to time t.

Corollary 21.2 *When the domestic "risk-free" rate is the domestic collateral rate, that is, when $r^d(t) = c^d(t)$ and the collateral is posted in domestic currency, again we have*

$$\Pi(t) = E_t^Q\left[\exp\left\{-\int_t^T r^d(s)ds\right\}\cdot \Pi(T)\right] = p(t,T)E_t^{T_d}\left[\Pi(T)\right].$$

Corollary 21.3 *If the foreign and the domestic economies are interchanged so that the domestic risk-free rate is equal to the domestic collateral rate, that is, $r^d(t) = c^d(t)$, and the collateral is posted in domestic currency, then the previous equation is transformed into*

$$\Pi^f(t) = p^f(t,T)E_t^{T_f}\left[\Pi^f(T)\right].$$

This result clearly shows the fact that the effective funding cost is given by the collateral rate, regardless of the risk-free rate of the corresponding currency.

It is also possible to construct a maturity T dependent measure associated with the collateralized zero-coupon bonds

$$D^d(t,T) = E_t^{Q_d}\left[\exp\left\{-\int_t^T c^d(s)ds\right\}\right]$$

that is similar in some ways to forward measures. Define $\bar{Q}^T$ (with expectation $\bar{E}^T$) by $\bar{Q}^T = Z(T) \cdot Q$, where

$$Z(T) = \frac{\exp\left\{-\int_0^T c^d(s)ds\right\}}{D(0,T)}.$$

Then

$$\bar{E}_t^T[X(T)] = \frac{E_t^{Q_d}\left[\exp\left\{-\int_0^T c^d(s)ds\right\} \cdot X(T)\right]}{E_t^{Q_d}\left[\exp\left\{-\int_0^T c^d(s)ds\right\}\right]}$$

$$= \frac{E_t^{Q_d}\left[\exp\left\{-\int_t^T c^d(s)ds\right\} \cdot X(T)\right]}{D(t,T)}.$$

If the spread $y^d(t)$ is deterministic, then $\bar{Q}^T$becomes the standard T-forward measure Q^T because

$$Z(T) = \frac{\exp\left\{-\int_0^T \left[r^d(s) - y^d(s)\right] ds\right\}}{E_t^{Q_d}\left[\exp\left\{-\int_0^T \left[r^d(s) - y^d(s)\right] ds\right\}\right]} = \frac{1}{B^d(T) \cdot p^d(0,T)},$$

which is the Radon-Nikodym derivative for Q^T. In particular, $\bar{Q}^T = Q^T$ when $y^d(t) = 0$ and the collateral rate is equal to the overnight rate, that is, $c^d(t) = r^d(t)$.

From the definition of $D(t, T)$, for any $0 \leqslant s \leqslant t \leqslant T$

$$E_s^{Q_d}\left[\exp\left\{-\int_0^t c^d(s)ds\right\}D^d(t,T)\right] = E_s^{Q_d}\left[E_t^{Q_d}\left[\exp\left\{-\int_0^T c^d(s)ds\right\}\right]\right]$$

$$= E_s^{Q_d}\left[\exp\left\{-\int_0^T c^d(s)ds\right\}\right] = \exp\left\{-\int_0^s c^d(s)ds\right\}D(s,T)$$

giving a result that will prove to be useful, for example, in modelling $c^d(t)$ and $D(t, T)$ as SDEs.

Corollary 21.4 *The value of the collateralized zero-coupon bond discounted by the collateral rate*

$$\exp\left\{-\int_0^t c^d(s)ds\right\}D(t,T)$$

is a Q^d-martingale, confirming thet the drift under Q^d of $D(t, T)$ is $c^d(t)$.

In the literature $\bar{Q}^T$ is often referred to as the T-forward measure induced by $D(t, T)$ as numeraire because it makes collateralized trades $\bar{Q}^T$-martingales; that is, changing measures from Q^d to $\bar{Q}^T$ allows Theorem 21.1 to be restated as the followng corollary.

Corollary 21.5 *When payment and pricing currencies are different*

$$E_t^{\bar{Q}^T}\left[\exp\left\{-\int_0^T \left[y^d(s)-y^f(s)\right]ds\right\}\cdot\Pi(T)\right]$$

$$= \frac{\Pi(t)\cdot\exp\left\{-\int_0^t \left[y^d(s)-y^f(s)\right]ds\right\}}{D(t,T)}$$

and when payment and pricing currencies are the same

$$E_t^{\bar{Q}^T}\left[\Pi(T)\right] = \frac{\Pi(t)}{D(t,T)}.$$

21.1.7.1 Pricing with Multiple Currency Collateralization

With full collateralization in multiple currencies, the funding spread between currencies i and k is given by $y^{(i,k)} = y^i - y^k = \left(r^i - c^i\right) - \left(r^k - c^k\right)$. The pricing formula can then be written as (remember: $y^j(t) = r^j(t) - c^j(t)$)

$$\Pi^d(t) = E_t^{Q_d}\left[\exp\left\{-\int_t^T \max_{j\in C}\left(r^d(u) - y^j(u)\right)du\right\}\cdot \Pi^d(T)\right],$$
$$= D^d(t,T)\cdot \exp\left\{-\int_0^T \max_{j\in C}\left(y^{(d,j)}(u)\right)du\right\}\cdot E_t^{T_d}\left[\cdot \Pi^d(T)\right].$$

Here we use a blended cheapest-to-deliver (CTD) curve through the life of the trade constructed for each unique CSA agreement. If we have an agreement with six possible collateral currencies (USD, EUR, GBP, JPY, CHF and CAD), we need to bootstrap 29 different interest rate curves, as shown in Fig. 21.15.

Also, note that we have the following spread relations:

$$\begin{cases} y^{(i,j)} = -y^{(j,i)} & \forall t > 0 \\ y^{(i,j)} = y^{(i,k)} + y^{(k,j)} & \forall t > 0 \end{cases}.$$

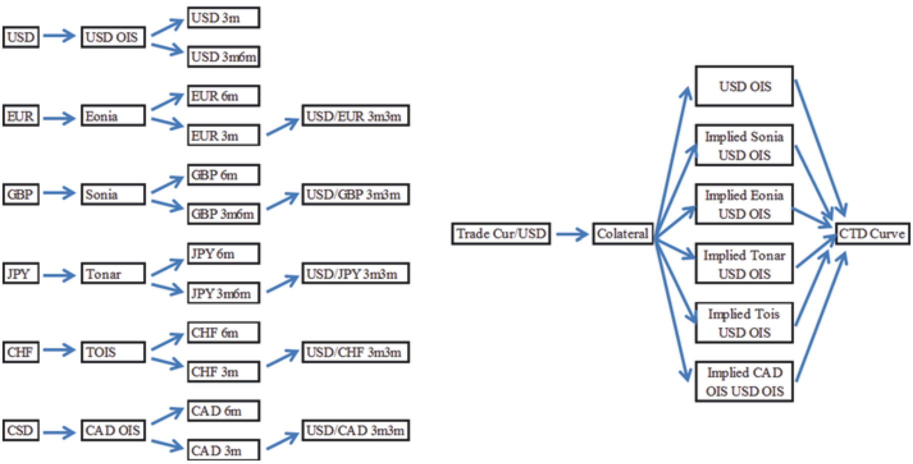


Fig. 21.15 The complicated bootstrap process if 6 collateral currencies are used

21.1.7.2 What Currency Gives the CTD?

The CTD currency is given by the one with the highest interest rate. The reason is that the standard contracts in the market always stipulate that it is the party who posts the collateral that can decide on the currency. By choosing the highest paying currency for the collateral he/she will get the highest rate of return. This contractual feature is closely related to the fact that the party who posts the collateral can at any time decide to get the collateral back and then post another collateral in another currency. The receiver of the collateral, on the other hand, has to pay interest in the currency chosen by his/her counterparty.

In some agreements, the counterparties can also post bonds. When bonds are posted, a haircut is made depending on the credit quality of the bonds.

The party holding the collateral can do what he/she likes with the cash or the bonds, like selling it in the market. But note that he/she must return exactly the same collateral when the other party wants it back and it can be difficult to buy back the same bond when market liquidity is low.

When you have to post collateral yourself, you might not be able to deliver the CTD currency. Then you might have to deliver the next CTD currency or even another one.

If you could decide to return the interest rate in any other currency than the one your counterparty chooses, then for you the currency with the lowest rate would be CTD.

In Fig. 21.16, we illustrate the CTD curve for two different currencies, GBP (SONIA) and EUR (EONIA).

21.1.7.3 Bootstrap Technique

Before we end this section, we will describe how to bootstrap in some different currencies. The exact bootstrap procedure depends on what liquid instruments are available in each market.

In general, curve stripping is based on market instruments such as

- OIS
- swaps (floating Xibor vs. fixed)
- Basis swaps also known as tenor basis swaps (e.g. Xibor 3M vs. Xibor 6M)

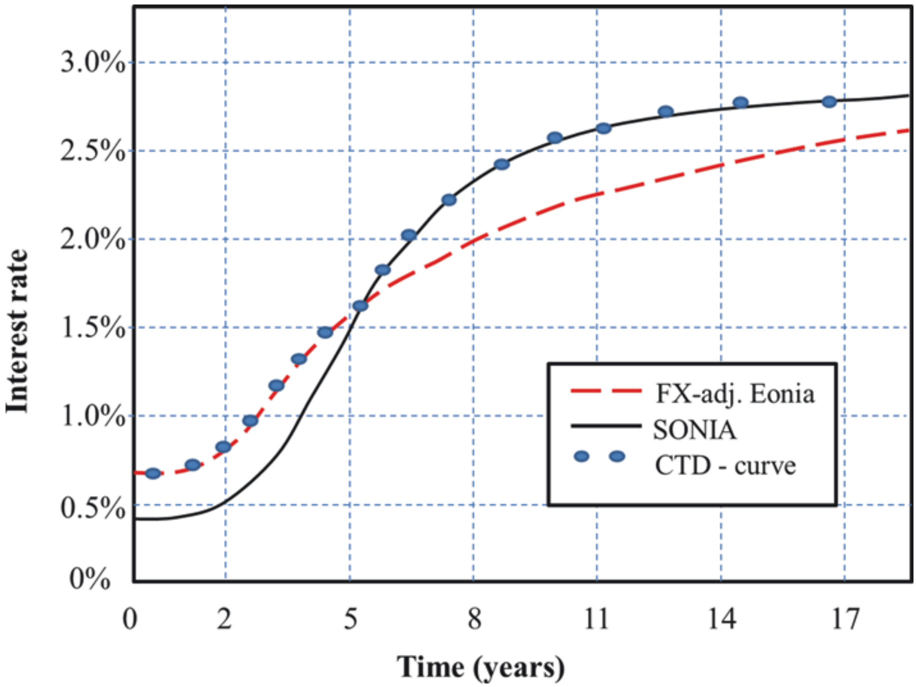


Fig. 21.16 A CTD curve for two currencies, GBP (SONIA) and EUR (EONIA)

- Cross-currency basis swaps (CCBS) (e.g. USD Libor 3M vs. GBP Libor 3M)

The building blocks of these instruments are

- Fixed cash flows: $Fixed * YF * Notional$
- Libor payments: $Libor * YF * Notional$

To price these we only need the elementary bits

- Discount factors: $DF = PV(1 \text{ unit of currency})$
- Forwards: $FWD = PV(Libor)/DF$

Then, the PVs are given by

- $PV(\text{Fixed cash flow}) = Fixed * YF * Notional * DF$
- $PV(\text{Floating cash flow}) = YF * Notional * FWD * DF$,

where YF is the year fraction.

Once we know the discount factors for all maturities and the forward rates for all maturities and tenors that is, the **dscount curve**

$t \rightarrow DF(t)$ and the **forward curves** $t \rightarrow FWD^{\delta}(t)$ for all tenors δ we are able to price all linear instruments.

In practice, the discount factors and the forward rates are stripped from market instruments for a set of maturities and tenors. For other maturities and tenors the values are obtained by interpolation. Here we have some practical issues:

How is this interpolation performed?

How are the curves represented? In terms of discount factors, forwards directly, etc.?

Liquid OIS markets exist for the G5 currencies (USD, EUR, GBP, JPY and CHF) and AUD and CAD among others. If there is no OIS market or the market is not liquid enough we cannot find an OIS curve. Therefore we cannot strip the projection curves in the usual manner. One possible idea is then to turn to a cross-currency market which is liquid enough and try to simultaneously strip the projection curve and the implied discounting curve from local swaps and CCBS.

In the single-currency case, we first do the stripping of the OIS curve, typically done using OIS. OIS is a fixed/float interest rate swap with the floating leg based on the published overnight rate index.

$$\textit{O/N rate} + \textit{OIS swaps} \rightarrow \textit{OIS curve used for discounting}$$

Next, we do the stripping of the projection curves (e.g. 1M, 3M, 6M, 1Y curves) given the OIS curve. Here we use instruments indexed on Xibor:

$$\textit{Cash deposits} + \textit{FRA/futures} + \textit{swaps} \diamond \textit{projection curve}$$

In the cross-currency case we first assume that the domestic and foreign curves for all needed tenors have been already stripped. Then we strip the implied foreign basis curve

$$\textit{FX forwards} + \textit{CCBS} \diamond \textit{Foreign discount curve}$$

For the 3-month tenor, we now have Table 21.2.

In the most liquid markets we have the following available curves (Table 21.3).

For different currencies, the trading conventions and instrument dependencies are slightly different.

Table 21.2 The 3-month tenor in the bootstrap process

Domestic floating leg		Foreign floating leg	
3MDOM index		3MFOR index + spread	
Projection Curve	DOM3M swap Curve	Projection Curve	FOR3M swap curve
Discount curve	DOM OIS curve	Discount curve	Implied DOM3M/FOR3M basis curve

Table 21.3 Available curves for the most liquid markets

Currency	Overnight rate	Standard curve	Forward curves	Basis curves
USD	Fed funds effective rate	3M USD Libor	Muni swaps	1M vs. 3M 3M vs. 6M 3M vs. 12M 3M Prime/Libor Basis swap Muni vs 3M Libor T-bill vs 3M Libor
EUR	EONIA	6M Euribor	1M Euribor 3M Euribor 12M Euribor	3M vs. 6M 6M vs. 12M
JPY	MUTAN	6M JPY Libor	1M JPY Libour 3M JPY Libour	1M vs. 3M 3M vs. 6M
GBP	SONIA	6M GBP Libor	1M GBP Libour 3M GBP Libour 12M Euribor	3M vs. 1M 12M vs. 6M 12M GBP Libor
CHF	TOIS	6M CHF Libor	3M CHF Libour 1M CHF Libour	12M vs. 6M
CAD	Bnak of Canada Overnight Repo Rate (CORRA)	6M CAD-BA		6M vs. 3M&3M vs. 1M

21.1.8 Market Instruments

We now start to present some formulas that are used for the curve generation. By a time schedule, we mean times $t < T_0 < T_1 < \ldots < T_N$ and the corresponding year fractions $\{\Delta_n\}_{n=1}^{N}$ where Δ_n is the year fraction from T_{n-1} to T_n. We also denote by $L_x(T_{n-1})$ the Libor rate with tenor x that starts at T_{n-1} (the rate is usually fixed 2 days before T_{n-1}).

21.1.8.1 Overnight Index Swap

If the collateral and the pay currency are the same, we have the following definition of the *forward overnight rate*: For a given currency

i, collateral rate c_i and times T_1 and T_2, the time t forward overnight rate is defined by

$$O(t,T_1,T_2) = \frac{1}{\Delta(T_1,T_2)}\left\{\frac{D(t,T_1)}{D(t,T_2)} - 1\right\}.$$

An OIS in a currency i with collateral rate c_i is a contract where you exchange a daily compounded floating overnight rate for a fixed rate at pre-specified points in time. The fixed rate OIS_N is chosen so that the PV of the swap, according to market discounting conventions, becomes zero today

$$OIS_N \sum_{n=1}^{N} \Delta_n D(t,T_n) = \sum_{n=1}^{M} E_t^{Q_i}\left[\exp\left\{-\int_0^{S_m} c^i(s)ds\right\}\right.$$
$$\left.\cdot\left(\prod_{j=1}^{J_m}\left(1 + O(U^*_{j-1,m}, U_{j-1,m}, U_{j,m})\right)\cdot\delta_{j,m} - 1\right)\right],$$

where $\{T_n\}$ and $\{S_m\}$ are the time schedules for the fixed leg and the floating leg respectively, $\{U_{j,m}\}$ is the time schedule for the daily compounded overnight rate, $U*_{j-1,m}$ is the fixing date for the rate in the period $(U_{j-1,m}, U_{j,m}]$ and Δ_n, $\delta_{j,m}$ are the year fractions for the periods $(T_{n-1}, T_n]$ and $(U_{j-1,m}, U_{j,m}]$. Here, $U_{0,m} = S_{m-1}$, $U_{Jm,m} = S_m$, $T_0 = S_0$ and $T_N = S_M$. By using the forward overnight rate and the approximation

$$O(U^*_{j-1,m}, U_{j-1,m}, U_{j,m}) \approx O(U_{j-1,m}, U_{j-1,m}, U_{j,m}),$$

we get

$$E_i^{Q_i}\left[\exp\left\{-\int_t^{S_m} c^i(s)ds\right\}\cdot\left(\prod_{j=1}^{J_m}\left(1 + O(U^*_{j-1,m}, U_{j-1,m}, U_{j,m})\cdot\delta_{j,m}\right) - 1\right)\right]$$
$$\approx E_i^{Q_i}\left[\exp\left\{-\int_t^{S_m} c^i(s)ds\right\}\cdot\left(\prod_{j=1}^{J_m}\left(\frac{1}{D(U_{j-1,m},U_{j,m})}\right) - 1\right)\right]$$
$$= E_i^{Q_i}\left[\exp\left\{-\int_t^{S_{m-1}} c^i(s)ds\right\}\right] - E_i^{Q_i}\left[\exp\left\{-\int_t^{S_m} c^i(s)ds\right\}\right].$$

In the last step, we were splitting up the integral and used the tower property of conditional expectation: if $\mathcal{H} \subset \mathcal{G}$ then $\mathrm{E}[\mathrm{E}[X|\mathcal{G}]|\mathcal{H}] = \mathrm{E}[X|\mathcal{H}] = \mathrm{E}[\mathrm{E}[X|\mathcal{H}]|\mathrm{G}]$.

Finally, we end up with the following approximation (where we ignore that the fixing is made in two trading days before the start of the time interval).

$$\text{(OIS)} \qquad OIS_N \sum_{n=1}^{N} \Delta_n D(0, T_n) = D(0, T_0) - D(0, T_N)$$

We can also get the previous equation from a simplification of a collateralized OIS

$$OIS_N \sum_{n=1}^{N} \Delta_n E^Q \left[\exp\left\{ -\int_0^{T_n} c(s)ds \right\} \right]$$
$$= \sum_{n=1}^{N} E^Q \left[\exp\left\{ -\int_0^{T_n} c(s)ds \right\} \left(\exp\left\{ -\int_{T_{n-1}}^{T_n} c(s)ds \right\} - 1 \right) \right]$$

by using the collateralized zero-coupon bond. The collateralized zero-coupon bond price can then be bootstrapped as

$$D(t, T_N) = \frac{D(t, T_0) - OIS_N \sum_{n=1}^{N} \Delta_n D(t, T_n)}{1 + OIS_N \cdot \Delta_N}.$$

If the collateral and the pay currency are *not* the same, we use that fair value of a payer OIS swap in currency i with collateral in currency j, collateral rate c_j and quote OIS

$$PV = \sum_{m=1}^{M} \delta_m \cdot D(t, S_m) \cdot \exp\left\{ -\int_t^{S_m} y^{i,j}(s)ds \right\} \cdot O(t, S_{m-1}, S_m)$$
$$- OIS_N \sum_{n=1}^{N} \Delta_n \cdot D(t, T_n) \cdot \exp\left\{ -\int_t^{T_n} y^{i,j}(s)ds \right\}.$$

21.1.8.2 Forward Rate Agreements (FRA)

Given times T_1 and T_2 the $T_1 \times T_2$ FRA traded in the market is a contract paying out to the buyer at time T_1 the amount

$$\frac{\Delta(T_1, T_2) \cdot (L_x(T_1) - K)}{1 - \Delta(T_1, T_2) \cdot L_x(T_1)},$$

where K is the pre-specified fixed rate of the FRA, x is the tenor starting at T_1 and ending in T_2, and $\Delta(T_1, T_2)$ is the year fraction for the tenor interval $(T_1, T_2]$. Under certain model assumptions and reasonable market conditions it can be shown that

$$\text{FRA)} \qquad K \approx E_t^{T_2^c}\left[L_x(T_1)\right],$$

where the expectation is taken with respect to the forward measure at T_2 discounted with the current (time t) price of a zero-coupon bond with expiry T_2 using the collateral rate in currency i (domestic).

21.1.8.3 Interest Rate Swaps

An IRS in currency i with collateral in currency j with collateral rate c_j and quote IRS is given by

$$\text{(IRS)} \qquad \begin{aligned} & IRS_N \sum_{n=1}^{N} \Delta_n D(t, T_n) \cdot \exp\left\{-\int_t^{T_n} y^{i,j}(s)ds\right\} \\ &= \sum_{m=1}^{M} \delta_m D(t, T_m) \cdot E_t^{S_m^c}\left[L_x(S_{m-1})\right] \exp\left\{-\int_t^{S_m} y^{i,j}(s)dx\right\} \end{aligned}$$

and a collateralized tenor swap (TS) by

$$\text{(TS)} \qquad \begin{aligned} & \sum_{n=1}^{N} \Delta_n D(t, T_n) \left\{E^{T_n^c}\left[L(T_{n-1}, T_n, \tau_S)\right] + TS_N\right\} \cdot \exp\left\{-\int_t^{T_n} y^{i,j}(s)ds\right\} \\ &= \sum_{m=1}^{M} \delta_m D(t, S_m) \cdot E^{S_m^c}\left[L_x(S_{m-1}, S_m, \tau_L)\right] \cdot \exp\left\{-\int_t^{S_m} y^{i,j}(s)dx\right\} \end{aligned}.$$

Market quotes of collateralized OIS, IRS and TS and proper spline method allow us to determine

$$\{D(0,T)\}, \qquad \left\{E^{T_m}\left[L(T_{m.-1},T_m,\tau)\right]+TS_N\right\}$$

for all relevant $T,\ T_m$ and tenor τ.

21.1.8.4 Cross-Currency Swaps

When dealing with multiple currencies, we also need to consider CCS and FX forwards. The cross-currency formula for currencies $i,\ j$, with collateral in currency k, and collateral rate c_k and quote s is given by

$$\begin{aligned}
&\sum_{n=1}^{N}\delta_n D(c_i,t,T_n)\cdot\exp\left\{-\int_t^{T_N}y^{i,k}(s)ds\right\}\left\{E^{T_n^{c,i}}\left[L\left(T_{n-1},T_n\right)\right]+s\right\}\\
&\quad -D(c_i,t,T_0)\cdot\exp\left\{-\int_t^{T_0}y^{i,k}(s)ds\right\}\\
&\quad +D(c_i,t,T_N)\cdot\exp\left\{-\int_t^{T_N}y^{i,k}(s)ds\right\}\\
\text{(CCS)}\quad &=\sum_{n=1}^{M}\delta_m D(c_j,t,S_m)\cdot\exp\left\{-\int_t^{S_m}y^{j,k}(s)ds\right\}E^{S_M^{c,j}}\left[L\left(S_{m-1},S_m\right)\right]\\
&\quad -D(c_j,t,S_0)\cdot\exp\left\{-\int_t^{S_0}y^{j,k}(s)ds\right\}\\
&\quad +D(c_j,t,S_N)\cdot\exp\left\{-\int_t^{S_N}y^{j,k}(s)ds\right\}
\end{aligned}$$

21.1.8.5 FX Forwards

A collateralized FX forward rate at time t for the currency pair (i,j) with collateral in currency k with collateral rate c_k and maturity T is defined as the solution $f_x^{(i,j)}(t,T)$ of the following equation:

$$0 = f_x^{(i,j)}(t,T)E_t^{Q_i}\left[\exp\left\{-\int_t^T r^i(s)ds\right\}\cdot\exp\left\{\int_t^T y^{i,k}(s)ds\right\}\right]$$
$$-f_x^{(i,j)}(t)E_t^{Q_j}\left[\exp\left\{-\int_t^T r^j(s)ds\right\}\cdot\exp\left\{\int_t^T y^{j,k}(s)ds\right\}\right],$$

where an amount of currency i is exchanged for one unit of currency j, assuming a collateralization in currency k. Then we have

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{p^j(t,T)}{p^i(t,T)}\cdot\frac{E_t^{T_j}\left[\exp\left\{\int_t^T y^{j,k}(s)ds\right\}\right]}{E_t^{T_i}\left[\exp\left\{\int_t^T y^{i,k}(s)ds\right\}\right]}.$$

If the spread y is stochastic, the currency triangle, such as USD/JPY × EUR/USD = EUR/JPY, holds only for the same collateral currency. When y is deterministic, this becomes

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{p^j(t,T)}{p^i(t,T)} = f_x^{(i,j)}(t)\frac{D^j(t,T)}{D^i(t,T)}\cdot\exp\left\{\int_t^T y^{i,j}(t,s)ds\right\}$$

which is independent of the choice of collateral currency. In a bootstrap all the previous except $y^{i,j}$ can be observed in the market. This can only be used for short maturities due to the lack of liquidity in the forward market for longer maturities. In the previous equation

$$y^{i,j}(t,T) = -\frac{\partial}{\partial T}\ln\left(E_t^{T_i}\left[\exp\left\{-\int_t^T y^{i,j}(s)ds\right\}\right]\right).$$

21.1.9 Curve Calibration

In this section, we describe our procedure for generating the curves needed. We fix $t = 0$ as today and we will write $E_0[\cdot] = E[\cdot]$.

Definition 21.1 By a *collateral pair*, we mean $(\mathrm{i}, \mathrm{c_i})$, where i is a currency and $\mathrm{c_i}$ a collateral rate in currency i.

Definition 21.2 Given a currency i and a collateral pair $(\mathrm{j},\ \mathrm{c_j})$, we call the mapping

$$\mathrm{T} \mapsto \mathrm{D_{c^i}(0, T)} \cdot \exp\left\{ \int_0^{\mathrm{T}} \mathrm{y^{i,j}(s)ds} \right\}$$

a *discount curve* and denote it by $\mathrm{D^i_{j,c^j}}(0, \mathrm{T})$.

Definition 21.3 Given a currency i and a set of collateral pairs $\left\{\left(\mathrm{f_j}, \mathrm{c^j}\right)\right\}_{\mathrm{i}=1}^{\mathrm{n}}$, we call the mapping

$$\mathrm{T} \mapsto \mathrm{D_{c^i}(0, T)} \cdot \exp\left\{ \int_0^{\mathrm{T}} \max_{\mathrm{j}} \left(\mathrm{y^{i,f_j}(s)}\right) \mathrm{ds} \right\}$$

a *CTD curve* and denote it by $\mathrm{D^i_{j,c^j}}(0, \mathrm{T})$.

Definition 21.4 Given a tenor x and a currency i we call the mapping $\mathrm{T} \mapsto \mathrm{E^{T(c^i)}}\left[\mathrm{L_x(S;T)}\right]$ a *forward curve*, where S, T are the start and maturity date, respectively.

21.1.9.1 Single-Currency Collateral

Given a currency *i* from the set of available currencies, a set of tenors (the number depending on *i*) and a collateral pair, our task is to generate the corresponding discount and forward curves. Due to liquidity problems, we have to make some more assumptions regarding the quotes that we observe in the markets. Moreover, in some currencies the complete lack of OIS quotes forces us to make some sort of approximations. In this section we describe the currency-independent assumptions and we will come back to certain currency-dependent issues later.

- A CCS quote observed in the market is interpreted to be of constant notional type. Mark-to-market (MtM) CIRS are not considered in the curve generation.
- The CCS quote in Equation (CCS) is independent of the collateral pair (j, c_j).

The last assumption makes it possible for us to compute for example EUR discounting under SEK collateral where the collateral rate is the Swedish repo rate (such a cross-currency quote is definitely not available in the markets).

21.1.9.2 Stripping in EUR

In EUR, the available instruments are OIS, IRS, FRA, CIRS and FX forward and the collateral pair is (EUR, EONIA). The OIS and IRS quotes are assumed to be under (EUR, EONIA).

We start to bootstrap the (EUR, EONIA) discounting curve using Equation (OIS) given earlier and then the forward rate curves for the tenors {1M, 3M, 6M} using Equations (FRA) and (IRS)

These are the consistency conditions needed to get the market quotes of various swaps. We have denoted the market observed OIS rate, IRS rate and TS spread respectively as OIS_N, IRS_M and TS_N, where the subscripts represent the lengths of swaps. $\{T_i\}_{i\geqslant 0}$ are the reset/payment times of each instrument. We distinguish the day-count fraction of the fixed and floating legs by Δ and δ, respectively. These are not necessarily the same for different instruments. $L(T_{m-1};\ T_m,\ \tau)$ is the Libor with tenor τ whose reset and payment times are T_{m-1} and T_m, respectively. In Equation (IRS), we have distinguished between two different tenors, τ_S and $\tau_L(>\tau_S)$, although we have used the same payment frequencies in the fixed and floating legs of the IRS.

R_{OIS} is defined as

$$R_{OIS}(t) = \frac{-\ln\left(D(t,T)\right)}{T-t}.$$

For the forward Libor, the zero-rate curve R_t is determined reclusively through the relation

$$\begin{aligned}
E^{T_m^c}\left[L\left(t,T_{m-1},T_m,\tau\right)\right] &= \frac{1}{\delta_m}\left(\frac{e^{-R_\tau(T_{m-1})\cdot(T_{m-1}-t)}}{e^{-R_\tau(T_m)\cdot(T_m-t)}}-1\right)\\
&= \frac{1}{\delta_m}\left(\frac{D(t,T_{m-1})}{D(t,T_m)}-1\right)\\
&= \frac{1}{\delta_m}\left(\frac{D(t,T_{m-1})-D(t,T_m)}{D(t,T_m)}\right)
\end{aligned}$$

21.1.9.3 Stripping in USD

In the United States, 30-day fed funds futures and 3-months OIS futures are commonly used, although there are only 36 standard 30-day FF futures contracts and eight standard 3M OIS futures contracts, based on www.cmegroup.com. Therefore, one can only bootstrap the OIS curve up to 2 or 3 years.

In order to capture longer-term OIS discounting, one could then bootstrap the short end of OIS curve from fed funds OIS (and/or fed funds futures) and then bootstrap the long end through fed funds basis swap (FFBS) (typically combined with IRS to get Libor information). In the United States, the instruments are typically FFBS ranging from 1y to 30y, which are averaged index basis swaps. This has become the common practice in the USD market. However, the actual implementation methodology could vary in several ways. We describe these in **Method 1, Method 2** and **Method 3** in the following sections.

Method 1: Approximation Approach

By ignoring discrepancy business day count adjustment and compounding crude adjustment (see Bloomberg (2012), Extending USD OIS curves using Fed Funds Basis swap quotes), we obtain

$$OIS(t)_{adj} \approx 4 \cdot \left(\left(1 + \frac{OIS(t)_{approx.}}{360} \right)^{90} - 1 \right),$$

where

$$OIS(t)_{approx.} \approx \left(1 + \frac{r_Q - FFBS(t)}{4} \right)^{4} - 1$$

$$r_Q = 4 \cdot \left(\left(1 + \frac{IRS(t)_{360/365}}{2} \right)^{2/4} - 4 \right),$$

where $IRS(t)$ is the market quote of swap rate.

$FFBS(t)$ is the market spread quote of FFBS.

One can observe from market quotes that the implied OIS rate has approximately deterministic spread to IRS and such an OIS has embedded market information from FFBS. Moreover, the OIS is reasonably close to market OIS for long tenors, so using this adjusted OIS for longer (say, 30y) tenors is justified and computationally efficient.

Thus this approximation method makes the following assumptions that practitioners must recognize.

- Ignore conventions like business day count, calendar, roll convention, basis, spot lag, schedule mismatch, etc.
- The compounding approximation assumes "flat curve with rate equal to the difference between Libor and Fed Funds Basis while ignoring weekends and holidays".
- The error between adjusted OIS and OIS market quotes is a combination of approximation errors and liquidity.
- Thanks to the simplifying assumptions this method is very fast.

Method 2: Brute Force Approach

The brute force approach jointly solves both FFBS and IRS to par. We know analytics for both FFBS and the IRS, as shown in Table 21.4.

Table 21.4 The fed funds basis swap and IRS

Fed funds basis swap	
Pay	Receive
Libor leg	Fed fund leg
–3m Libor	+ weighted average Fed fund + basis spread
IRS	
Receive	Pay
Libor leg	Fixed leg
+3m Libor	– fixed swap rate

Note:

- This is the most accurate approach because it solves for discount factors for both OIS discounting curve and Libor 3m forward curve jointly.
- Schedules among legs between FFBS and IRS may not align, which burdens the solver and is very sensitive to choice of interpolation method.
- Computationally expensive (especially the weighted average on daily forward fed funds effective rate feature).

Method 3: Synthetic Approach

The synthetic approach is to bootstrap the OIS curve by re-pricing FFBS to par given an IRS. The idea is represented as in Table 21.5.

Through the synthetic construction in the previous graph, we can have a single instrument (fixed vs. floating (weighted average FF +

Table 21.5 Re-pricing fed fund basis swaps to par with IRS

Synthetic fed fund swap	
Receive	Pay
Fed fund leg	Fixed leg
+ weighted average fed fund + basis spread	– fixed swap rate

basis spread)) per tenor, which would fit into most standard curve stripping frameworks as a regular bootstrapping procedure.

Note:

- This is faster than Method 2.
- This is more accurate than Method 1.
- Schedule alignment is still an issue.
- Tenors have to be matched so as to create a synthetic FF average swap.

21.1.10 The Bootstrap

We now assume we have OIS, IRS, FRA, CIRS and FX forward and the collateral pair is (USD, FEDL01) where FEDL01 is the USD fed fund rate. The OIS and IRS quotes are assumed to be under (USD, FEDL01).

We start to bootstrap the (USD, FEDL01) discounting curve as shown previously and then the forward rate for tenors {1M; 3M}. The 6M forward rate curve can be built using the 6M-3M tenor basis swaps to create a synthetic 6M swap curve.

Remarks: In the previous calculations for EUR and USD, we have assumed that all the instruments are collateralized by the cash of domestic currency, which is the same as the payment currency. You may worry about the possibility that the market quotes contain significant contributions from market participants who use a foreign currency as collateral. In fact, some of the major financial firms prefer USD cash collateral regardless of the payment currency of the contracts. This gives rise to additional factors in discounting as in

$$\begin{aligned}\Pi^d(t) &= E_t^{Q_d}\left[\exp\left\{-\int_t^T r^d(s)ds\right\}\cdot\exp\left\{\int_t^T y^f(s)ds\right\}\Pi^d(T)\right],\\ &= D^d(t,T)E_t^{T_d}\left[\exp\left\{\int_0^T\left(y^d(s)-y^f(s)\right)ds\right\}\Pi^d(T)\right].\end{aligned}$$

Changes in IRS/TS quotes are very small and impossible to distinguish from the bid/offer spreads in normal circumstances, because the correction appears both in the fixed and floating legs which keeps the market quotes almost unchanged. However, the PVs of off-the-market swaps can be significantly affected when the collateral currency is different.

21.1.10.1 Stripping in SEK

In SEK, the available instruments are also OIS (STINA swaps, which are not real OIS, but T/N), IRS, FRA, CIRS and FX forwards. The collateral pairs are (SEK, STIB1D) and (SEK, SWRRATE). Here STIB1D is the T/N rate (STINA) and SWRRATE the Swedish repo rate. The OIS quotes are assumed to be under (SEK, STIB1D) and the IRS quotes are assumed to be under (EUR, EONIA).

The assumption about the IRS quote makes the bootstrap procedure a little bit different from earlier. We begin to bootstrap (SEK, ST IB1D) as usual for OIS. Then we need to add a cross-currency formula so we simultaneously can bootstrap the discounting curve for SEK with collateral pair (EUR, EONIA) and the SEK 3M forward curve.

The (SEK, SWRRATE) is generated as follows:

1. Bootstrap a curve from available SEK RIBA (Riksbanks futures) instruments and compute the daily instantaneous forward rates up to the maturity of the last RIBA instrument. This will be the short end of the resulting curve.
2. Compute the spread between the last instantaneous forward rate in the first step and the corresponding rate generated from (SEK, STIB1D).
3. Extrapolate the spread and convert the resulting instantaneous forwards to a discount curve.

The reason we also have to bootstrap the Swedish repo rate is that it is used as collateral rate in most CSA agreements.

21.1.10.2 Different Problems with Bootstrapping

The Choices of Instruments

The first problem is related to the availability and choice of liquid instruments. Let us assume that we have an OIS curve (obtained from the previous method) and we need to find a forward curve (e.g. USD Libor 3m curve). In the single-curve world, we typically bootstrap from combinations of cash (1w up to 1y), futures or FRAs (typically most liquid up to 3y or 5y) and swaps (1y up to 50y). However, in the dual-curve world, each Libor has its own risk, which means we cannot use spot Libor cash rates like 1w, 2w, 3w, 1m, 2m, 4m, 5m, 6m, etc., to strip the 3m Libor curve. Therefore, one possible choice could be to use FRAs for short term and swaps for long term. Note that the use of FRAs is subject to a few problems. In the dual-curve world, the Libor becomes risky, which means that the old approach would use the wrong discounting curve.

The problem here is that trading parties can only put daily collateral until the start time of the FRA period. Therefore, the period between the start time and the first maturity date of the FRA is only for theoretical Libor tenor, and there won't be any collateral transactions. This simple approach would assume that Libor forward is risky when it is settled, though any such settlement amount would still be discounted at the risk-free rate (e.g. continuously full collateralization with zero MTF). For futures, it is even trickier in view of the convexity adjustment.

The Libor Rates

In the single-currency world, the standard curve construction would be to make a discount curve and then an implied forward curve. This implies that the Libor forward depends on the choice of discounting. However, intuitively Libor is a floating index that should not depend on the choice of a discount curve. Different players may have different funding curves, and CSA may also require discounting other than OIS discounting which really refers to cash collateral within that single-currency market.

Mathematically, the curve construction depends on the discounting method in our typical risk-neutral environment. This is because the probability measure changes when the discounting method is changed. Prior to the crisis, the "standard" discount curve was the "same" as the Libor forward curve in USD. Now, the "standard" discounting method has shifted into OIS discounting. OIS discounting and Libor discounting would produce nearly the same price for a 3m Libor forward.

Furthermore, Libor fixings too have shifted to OIS discounting, although dealers/brokers may not report their quotes consistently. As CSA regulation tends to standardize collateral agreements and as the LCHuses OIS discounting for clearing swaps, the swap quotes and Libor fixings would be more reliable than during the crisis period, when Libor was criticized due to various reasons including collateral/funding discounting and liquidity.

Collateral

Another concern is collateral currency. This is actually a fairly common problem in the real world. Suppose a EUR-based bank is trading collateralized vanilla US IRS with a US bank. Assume further that most of the EUR-bank's business is in the EUR and that the EUR-based bank owns a lot of EUR treasury bonds and cash in EUR. Therefore, it would be very natural for the EUR bank to put collateral in a EUR cash or EUR treasury bonds, rather than put collateral in USD even if the trade is a USD single currency. In this situation, we are faced with the dilemma of how to discount the value of the trade price and the risks. This is the "cheapest to deliver" problem.

Cross-Currency swaps

There are two popular types of CCS.

- Float-float (commonly used for major currency pairs like USD/EUR, USD/JPY, EUR/JPY)
- Fix-float (commonly used for minor currency pairs like USD/TWD).

We will focus mostly on float-float CCS (CCBS). Using USD vs. JPY as a simple example to illustrate arbitrage-free relationships in interest rate and FX markets ("interest rate parity"), let us suppose a JPY investor has a future cash flow of 100 million in USD ($100 MM) at future date T. She/he could either discount it using USD discounting back to today $t = 0$ and convert it into JPY using the spot FX rate (FX(0)), or she/he could convert the future cash flow at T through FX rate (FX(T)) and then discount back to today using JPY discounting. If there were no-arbitrage opportunities, these two methods would yield the same amount. Such parity relations can be visualized in Fig. 21.17.

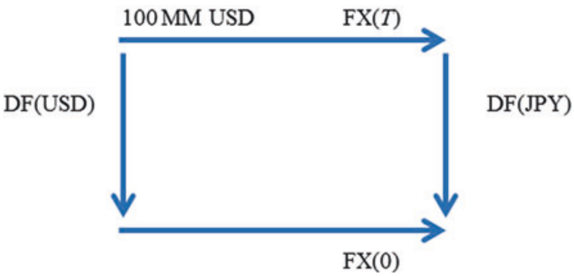


Fig. 21.17 A parity relation on cross-currency

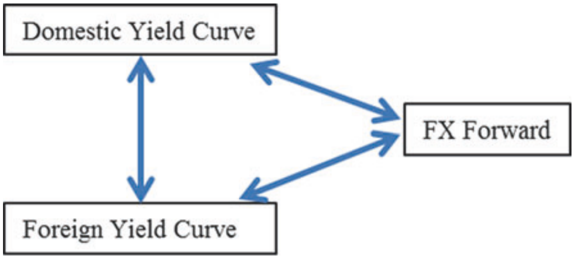


Fig. 21.18 With two known the third can be solved

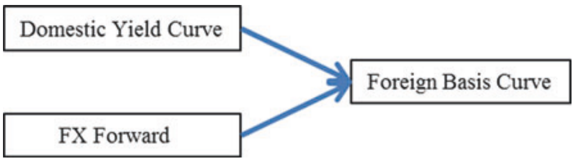


Fig. 21.19 Bootstrap of an implied foreign yield curve

Therefore, in theory, we should see a perfect triangle relationship betweeen the domestic yield curve, the foreign yield curve and the FX forward. As we illustrate in Fig. 21.18, one can solve for one out of three given the other two.

However, both interest rates (domestic and foreign) and FX forwards are independent actively traded markets. Therefore, the triangle won't be "perfect" even in normal market situations due to differences in liquidity among those three markets.

If one needs to value a cross-currency trade under the domestic measure, then a typical way would be to bootstrap the "implied" foreign yield curve (the foreign basis curve) from a set of FX forwards given a domestic yield curve as in Fig. 21.19.

Note that such a foreign basis curve typically won't be same as the foreign local yield curve (e.g. bootstrapped from cash, FRA/future, swap). This reflects the fact that the foreign investor's (domestic currency's) funding interest rate (discounting) at the local (foreign currency) market typically will be different from that of a local investor. Such a basis curve would satisfy

- Re-pricing each node of FxFwd back to par.

So far, everything is still consistent. Such a relationship implies that the cross-currency basis is zero (which is what "perfect" means here). However, if we look at CCBS quotes, they are not zero! This implies that the CCBS actually breaks the previous interest rate parity due to many reasons, supply and demand factors, including liquidity premium, currency strength, currency country's credit profile differentials, etc.

Prior to the crisis, such bases were very small, and many practitioners simply ignored them . However, after the crisis these bases became much larger (50–100 bps, compared to 1–5 bps pre-crisis). Therefore, such a basis can no longer be ignored in derivatives pricing and risk management. As with FX forward, one can also bootstrap an "implied" foreign basis curve from a set of CCBS given both domestic yield curve and foreign yield curve (which makes Libor forward consistent with local Libor forward, see Fig. 21.20).

Such a foreign basis curve should satisfy

- re-pricing each node of CCBS back to par;
- matching local Libor forward.

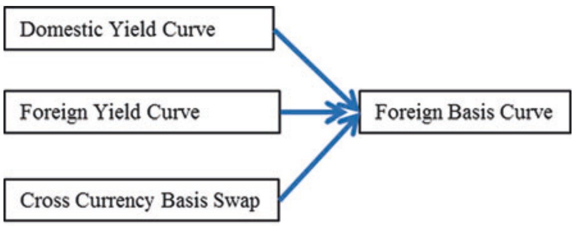


Fig. 21.20 Bootstrap an "implied" foreign basis curve from a set of cross-currency basis swaps

In short, we solve for the discount curve while keeping the prevailing index forward unchanged. Therefore, the Libor definition is changed slightly due to different discounting methods. From a practical point of view, this problem can often be ignored due to the tiny effect on forward change which means one can still assume that the basis swap's Libor forward in foreign currency is the same as Libor forward in the domestic market. As in the single-currency world, even if we change from single curve Libor to a very different discount curve (e.g. the OIS curve) when pricing a vanilla IRS, the Libor forwards in the IRS would be almost the same (maximum about 1–3 bps for 30y).

Furthermore, FX forwards may only last up to 5 years, depending on the currencies, while cross-currencies basis swaps would last up to 30 years. Therefore, from a liquidity point of view, one may strip the foreign basis curve from a combination of FX forward (for short term) and CCBS (for long term) (see Fig. 21.21).

OIS curve

We now turn our attention from the traditional problem in the cross-currency world to the problems arising in a multi-curve environment.

Recall that within the single-currency framework, the dual-curve logic is to bootstrap the OIS curve, then solve for the index forward curve while keeping the OIS curve unchanged. In the cross-currency world, the logic is to solve for the implied foreign discounting curve while keeping the foreign Libor forward unchanged. Thus, in a CCBS there are four curves involved:

1. Domestic discount curve (e.g. OIS curve).
2. Domestic forward curve (e.g. 3m Libor curve).

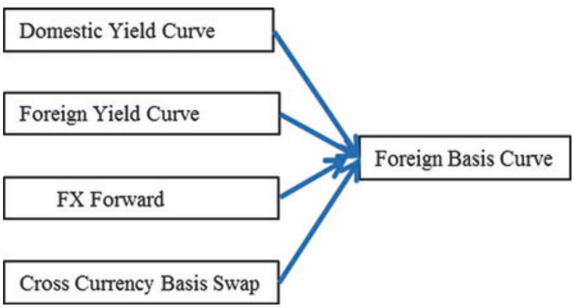


Fig. 21.21 A foreign basis curve stripped from a combination of four sources

3. Foreign discount curve.
4. Foreign forward curve (e.g. 3m Libor).

Still the goal here is to solve for a "cross-currency implied" foreign basis discount curve (or say a foreign OIS basis curve) where discounting and forward projections are decoupled.

The foreign OIS basis curve can be bootstrapped:

1. From FX forward and the domestic OIS curve
2. Using domestic and foreign OIS curves and CCY OIS basis swaps
3. Using CCY Libor basis swaps

In point 1 we assume that both domestic and foreign FX forward markets move to OIS discounting. Point 3 is more complex due to the market conventions. The common practice would be the following steps (using USD 3m Libor vs. JPY 3m Libor as an example):

1. Construct the domestic OIS discount curve.
2. Construct the domestic index forward curve (e.g. USD 3m Libor).
3. Construct the foreign OIS discount curve (e.g. JPY OIS curve from Tonar).
4. Construct the foreign index forward curve (e.g. JPY 6m Libor).
5. Construct the foreign index forward curve (e.g. JPY 3m Libor from Libor basis swap).
6. Solve for implied foreign basis curve (or, say, foreign OIS basis curve) from CCBS given in 1, 2 and 5.

Cheapest to Deliver

CTD is driven by the collateral agreement and CSA standardization. In CSA, the two counterparties may have rights to choose collateral on the fly among a few predefined currencies. This will create a huge headache for modelling, hedging and risk management.

Most dealers agree that the discount rate should – in theory – be based on the CTD collateral. Market practice is altogether different, with even the major dealers taking a variety of approaches to pricing trades based on multi-currency CSAs.

"Theoretically, it is not difficult to put together a model – it would be an extension of a stochastic basis model where you have more than one basis," says Vladimir Piterbarg, global head of quantitative research at Barclays Capital in London. "The huge question is whether you are able to execute the hedging strategy required."

Such a disorder in CTD modelling and hedging would have much bigger model risks. CSA standardization and simplification become critical while the market is moving towards a consistently multi-curve approach.

21.1.10.3 Collateral Choice

To compute the integral in the mapping

$$T \mapsto D_{c^i}(0,T) \cdot \exp\left\{ \int_0^T \max_j \left(y^{i,f_j}(s) \right) ds \right\},$$

we assume for each i and f_j that $y^{i,f_j}(t)$ is a piecewise constant function of t. This implies that the integral can be written as

$$\int_0^T \max_j \left(y^{i,f_j}(s) \right) ds = \sum_{k=1}^{K} \max_j \left(\int_{T_{k-1}}^{T_k} y^{i,f_j}(s) ds \right),$$

where $0 = T_0 < T_1 < \ldots < T_K = T$ is the approximation scheme where all the $y^{i,f_j}(t)$ are piecewise constant. Moreover, from the definition of the discount curve we can see that

$$\int_{T_{k-1}}^{T_k} y^{i,f_j}(s) ds = \ln\left(D_{c^i}(0,T_k)\right) - \ln\left(D^i_{f_j}(0,T_k)\right) - \ln\left(D_{c^i}(0,T_{k-1})\right) + \ln\left(D^i_{f_j}(0,T_{k-1})\right)$$

and hence the computation is now straight forward given the relevant discount curves.

21.1.10.4 Stripping in HUF as an Example of Illiquid Currencies

For example, say we are looking for curves in HUF (Hungarian Forint) rates. Then we can use local swaps, that is, HUF3M vs. fixed and CCS HUF3M vs. EUR3M. At the same time, we have to consider the collateral assumptions behind this.

We then use vanilla swap HUF3M vs. fixed, quarterly HUF3M (3M BUBOR) annually fixed and CCBS EUR3M vs. HUF3M where the EUR floating leg is quarterly EUR3M (EURIBOR3M) and HUF floating leg, quarterly HUF3M + spread (Tables 21.6 and 21.7).

When we do simultaneously stripping of the discount curves and the projection curves we use the following stripping equation for the 1Y point:

$$\begin{cases} Q \cdot \delta \cdot D_4 = \delta_1 \cdot F_1 \cdot D_1 + \delta_2 \cdot F_2 \cdot D_2 + \delta_3 \cdot F_3 \cdot D_3 + \delta_4 \cdot F_4 \cdot D_4 \\ \delta_1 \cdot (F_1 + s) \cdot D_1 + \delta_2 \cdot (F_2 + s) \cdot D_2 + \delta_3 \cdot (F_3 + s) \cdot D_3 + \delta_4 \cdot (F_4 + s) \cdot D_4 \\ \qquad = \dfrac{EUR_{Leg}}{X(0)} \end{cases}$$

where

D_i = Discount factors
F_i = HUF3M forwards
δ_i = Year fractions
(0) = Spot EUR/HUF exchange rate
Q = Quoted 1Y par swap rate
s = Quoted 1Y EUR3M/HUF3M basis spread

We also assume that the EUR curves are already stripped (e.g. from EONIA swaps and vanilla EUR3M swaps). So the unknowns in the

Table 21.6 Vanilla swap HUF3M vs. fixed

	Discount curve	Projecting curve
Fixed leg	HUF^{disc} curve	N/A
Floating leg	HUF^{disc} curve	HUF3M curve

Table 21.7 Cross-currency basis swap EUR3M vs. HUF3M

	Discount curve	Projecting curve
EUR leg	EONIA curve	EUR3M curve
HUF leg	HUF^{disc} curve	HUF3M curve

previous formula are only D_4 and D_4. The intermediate D_s and F_s (if unknown) are computed via interpolation. This can also be handled by a general solver.

In a practical situation, things might need to be done differently. The payment dates of the local vanilla swaps and those of the CCBS might be misaligned (due to differing day count conventions), or simply the quoted maturities for one set of instruments might be different from the quoted maturities for the other set of instruments. Thus performing a bootstrap might not be the best solution. Instead, we could use a global solver for all quoted instruments simultaneously. This is slower but produces more stable results. We could still perform intermediate passes using the quotes up to some fixed maturities in order to find good initial guesses for the later passes.

Simultaneous stripping produces two curves, a discount curve HUF^{disc} and a projection curve HUF3M. By §construction, using these two curves as the discount curve and the projection curve, respectively, we will price at par both the vanilla swaps HUF3M vs. fixed and the CCBS HUF3M vs. EUR3M.

Now, we might ask how this *curve stripping does* fit into a general model for derivatives pricing. So far we have only considered "linear" instruments and defined **formally** discount factors and forwards. Can these be used to price something else but swaps? Is this backed by a theory where the curves get back their usual meaning?

Consider an economy with two currencies: a domestic (Dom) and foreign one (For). Assume that collateral can be posted in any of the two currencies. The choice of the collateral currency holds for the whole lifetime of the derivative (i.e. assuming there is no option to switch collateral), so the domestic collateral earns foreign collateral earnings c_f. A pricing theory can be constructed rigorously with the help of the following replication argument.[10]

Denote the price of a collateralized **domestic** derivative by V. Then, in the domestic collateral

$$V(t) = E_t\left[\exp\left\{-\int_t^T c_d(s)ds\right\}\cdot V(T)\middle|\mathcal{F}_t\right].$$

[10] See Piterbarg (2010).

In the foreign currency we have

$$V(t) = E_t\left[\exp\left\{-\int_t^T c_d(s) + h(s)ds\right\} \cdot V(T)|\mathcal{F}_t\right].$$

Similarly, a collateralized foreign derivative V^f is given by

$$V^f(t) = E_t^f\left[\exp\left\{-\int_t^T \left[c_f(s) - h(s)\right] ds\right\} \cdot V^f(T)|\mathcal{F}_t\right]$$

and

$$V^f(t) = E_t^f\left[\exp\left\{-\int_t^T c_f(s)ds\right\} \cdot V^f(T)|\mathcal{F}_t\right].$$

The fact that the spread when computing from the foreign point of view is $-h$ follows from the domestic-foreign "parity" condition. If we fix the collateral currency and then compute the price of a contingent claim through foreign or through domestic yields we must get the same result in order to exclude arbitrage (see Fig. 21.22).

This implies that the **drift of the FX rate** X (in the domestic measure E) should be $r_{d,} = c_d - c_f + h$.

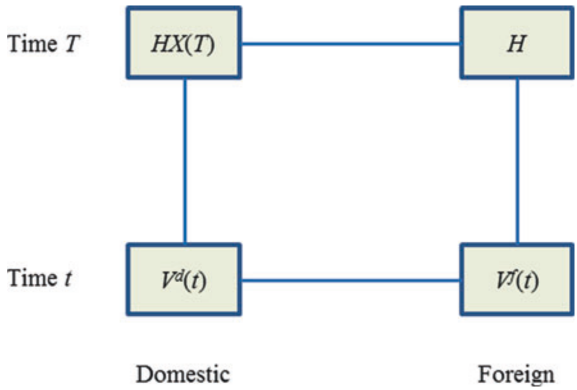


Fig. 21.22 Foreign or domestic yields we must give the same result to exclude arbitrage

Table 21.8 Pricing in classical theory and collateral

	Domesticrate	Foreign rate	FX rate drift
Classical theory	r_d	r_f	$r_d - r_f$
Domestic collateral	c_d	$C_{fd} = c_f - h$	$r_{df} = C_d - C_f + h$
Foreign collateral	$C_{df} = C_d + h$	C_f	$r_{df} = C_d - C_f + h$

Table 21.9 Vanilla swap HUF6M vs. fixed

	Discount curve	Projecting curve
Fixed leg	HUF^{disc} curve	N/A
Floating leg	HUF^{disc} curve	HUF6M curve

Table 21.10 Tenor basis swap HUF6M vs. HUF3M

	Discount curve	Projecting curve
Fixed leg	HUF^{disc} curve	HUF3M curve
Floating leg	HUF^{disc} curve	HUF6M curve

Table 21.11 Cross-currency basis swap EUR3M vs. HUF3M

	Discounting curve	Projecting curve
EUR leg	EONIA curve	EUR3M curve
HUF leg	HUF^{disc} curve	HUF6M curve

Once the collateral currency has been chosen, pricing under a CSA is in some way the same as pricing in the classical theory, as long as the appropriate curves are used.

Hence we have a pricing theory that is consistent and extends the formal swap pricing theory based on stripping where the stripping produces the initial term structures of the rates, that is, today's values of the curves used for discounting and forwarding. As seen before, the choice of the collateral is reflected in the curves that are used for discounting.

A problem will occur if the market does not directly quote 3M swaps. If we have quotes for 3M vs. 6M tenor basis swaps then we can simultaneously strip three curves (Tables 21.9, 21.10 and 21.11).

- Local swaps: HUF6M vs. fixed
- Local TS: HUF6M vs. HUF3M
- CCS: HUF3M vs. EUR3M

In the aftermath of the financial crisis in 2007 and 2008, the market has turned towards OIS discounting as the new standard. The G5 currencies and a few others have liquid OIS markets which provide the quotes from which the OIS curves in those currencies can be stripped. But for the other currencies the OIS markets are more often than not inexistent or illiquid. This renders it impossible to strip out any OIS curve. There is no magical solution, but one can contemplate using available cross-currency quotes to infer a number of curves. This requires the simultaneous stripping of single-currency and cross-currency instruments. The discounting method is intimately tied to the choice of the collateral; hence there is a trade-off: not having a curve at all vs. using curves based on a different collateral assumptions.

21.1.10.5 The Future – Standard CSA (SCSA)

The problem with the aforementioned method is the embedded optionality in CSA discounting to post collateral in many currencies and the choice of instrument to post. This leads to the following problems:

- no price transparency
- difficult to do unwinding and novation of trades
- difficult to hedge

Many market actors believe in using the Standard Credit Support Annex (SCSA), that is, a single-currency CSA. Here one will probably use the following:

- 17 currency silos
- Emerging currencies will use the multi-currency USD silo. For these currencies, we have no domestic OIS market. Therefore, we will use a simultaneous calibration of USD-collateralized domestic IRS and CIRS.

This would eliminate the currency switch options and we could use standardization on OIS in order to get better alignment with Central Clearing Partners. The institutions can make the move from CSA to SCSA but not vice versa. The switch to SCSA will not be mandatory. A greater collateralization will reduce the counterparty risk but also increase the funding costs.

21.1.11 General Pricing in the New Environment with Funding Value Adjustments

We will now describe general pricing in the new environment, with collateral and funding. Here the FVA can be considered as the difference between the PV of the trades between the "real-world" and the theoretical CSA: FVA(t) = PV(real-word CSA) – PV(theoretical CSA). This is the cost above the risk-free price to fund the trades in the presence of a real-world CSA. We can also consider the FVA as the cost to fund the trades above that of the risk-free price

$$FVA(t) = PV(\text{including funding costs}) - PV(\text{risk-neutral}).$$

There is still, in 2014, an ongoing debate on how to calculate and report the FVA and how to pass this cost to the counterparty. Another unresolved question is whether the FVA should be included in MtM prices or just be included in the unrealized profit/loss (P&L). In addition, it is not clear what the regulators will decide.

Market practitioners recognize that collateral has been successfully introduced in order to mitigate CVA. However, since collateral affects the funding of the trade, it also affects profitability. There is no argument that the FVA charge is real, and passed onto traders by funding desks.

In the classical theory we have postulated the existence of a single rate $r(t)$ where all tradable assets grow at this rate under a risk-neutral measure. The portfolio value $v(t)$ satisfied the standard Black-Scholes partial definition equation (PDE) (or term-structure equation)

$$\mathrm{L}(t)v(t) = r(t)v(t),$$

where $\mathrm{L}(t)$ is the evolution operator of the model drivers. All payments are discounted with the single rate $r(t)$. The modern market on the other hand has multiple rates, for example:

r_C the collateral rate (which is almost risk free)
r_R the interest rate for asset secured borrowing (the repo rate)
r_F the interest rate for unsecured funding

For example, a fully collateralized payment is discounted with r_C. Now, we will study what happens if the collateralization is only partial and how this theory is constructed. The evolution of the aforementioned rates is modelled by correlated stochastic processes and the portfolio may consist of general instruments, both vanillas and exotics, with known future payouts as function of the rates. Furthermore, the collateral is a known function of the portfolio value.

The FVA is defined as the difference between the modern price and the base one. The base price is often related to a fully collateralized portfolio. This is the only difference between the modern and the base models; the payments/indexes coincide. Only modern price is important; the base can be chosen from pragmatically numerical reasons.

Piterbarg[11] calculated a modern portfolio value by replication. This replication gives a modified pricing PDE (with respect to the classical theory) and is a unique way to determine the price. However, at this stage, the default risk (of the bank or the counterparty) is not taken into account.

Let $V(t)$ be the portfolio price and $C(t)$ the collateral. The collateral is a known function of the portfolio (e.g. $C(t) = (V(t))^+$). Then by replication we have

$$\mathrm{L}(t)V(t) = r_C C(t) + r_F(V(t) - C(t)) = r_F V(t) + C(t)(r_C - r_F),$$

where $\mathrm{L}(t)$ is the evolution operator corresponding to the rate processes. We also assume that the equity grows with the repo rate $r_R(t)$ minus dividends. In the right-hand side we divide the portfolio value in two parts $V = C + (V - C)$ where the part under collateral, C, is discounted with the collateral rate r_C and the residual part, not covered by collateral, $V - C$, is discounted with the funding rate r_F. One can generalize the theory to different rates for positive/negative parts of the collateral and funding (see Pallavicini et. al.). This would reflect a distinction between borrowing and lending.

[11] Vladimir Piterbarg (2010), "Funding beyond discounting: collateral agreements and derivatives pricing", RISK, Feb.

The solution to the PDE between payment/exercise dates is obtained via the Feynman-Kaè theorem

$$V(t) = E^Q\left[\exp\left\{-\int_t^T r_F(s)ds\right\}\cdot V(T)\middle|\mathcal{F}_t\right] + E^Q\left[\int_t^T \exp\left\{-\int_t^T r_F(s)ds\right\}(r_F(u) - r_C(u))\,C(u)du\middle|\mathcal{F}_t\right].$$

or equivalently

$$V(t) = E^Q\left[\exp\left\{-\int_t^T r_C(s)ds\right\}\cdot V(T)\middle|\mathcal{F}_t\right] + E^Q\left[\int_t^T \exp\left\{-\int_t^T r_C(s)ds\right\}(r_C(u) - r_F(u))\,(V(u) - C(u))\,du\middle|\mathcal{F}_t\right]$$

The previously mentioned PDE is non-linear (except when collateral is a linear function of the price V). We also have some special cases:

For $C = 0$ we get a solution for a non-collateralized deal

$$V(t) = E^Q\left[\exp\left\{-\int_t^T r_F(s)ds\right\}\cdot V(T)\middle|\mathcal{F}_t\right].$$

For $C = V$ we get a solution for the fully collateralized situation

$$V(t) = E^Q\left[\exp\left\{-\int_t^T r_C(s)ds\right\}\cdot V(T)\middle|\mathcal{F}_t\right].$$

Notice here that a fully collateralized security is discounted with the collateralized savings account, while the uncollateralized security is discounted with the pure funding account. Note that both conditional expectations are taken with respect to the same measure. Also, some conditions must be imposed on the coefficients in the evolution operator L(t) in order to avoid explosive solutions.

Of course, one can change the pricing measure using, for example, the Radon-Nikodym derivative

$$M(t) = D(t, T)/B_C(t),$$

where $D(t, T)$ is a collateralized zero-coupon bond

$$D(t, T) = E^Q\left[\exp\left\{-\int_t^T r_c(s)ds\right\}|\mathcal{F}_t\right].$$

The collateralized deal propagation leads to a standard formula

$$\begin{aligned}
V_C(t) &= E^Q\left[\exp\left\{-\int_t^T r_C(s)ds\right\} \cdot V_C(T)|\mathcal{F}_t\right] \\
&= M(t) \cdot E^{T_C}\left[\exp\left\{-\int_t^T r_C(s)ds\right\} \cdot V_C(T) \cdot \frac{1}{M(T)}|\mathcal{F}_t\right] \\
&= D(t, T) \cdot E^{T_C}\left[\frac{V_C(T)}{D(T, T)}|\mathcal{F}_t\right] \\
&= D(t, T) \cdot E^{T_C}\left[V_C(T)|\mathcal{F}_t\right]
\end{aligned}$$

However, for the pure funding security, such a change of measure gives

$$\begin{aligned}
V_F(t) &= E^Q\left[\exp\left\{-\int_t^T r_F(s)ds\right\} \cdot V_F(T)|\mathcal{F}_t\right] \\
&= M(t) \cdot E^{T_C}\left[\exp\left\{-\int_t^T r_F(s)ds\right\} \cdot V_F(T) \cdot \frac{1}{M(T)}|\mathcal{F}_t\right]
\end{aligned}$$

and will not lead to a standard numeraire-deflated form.

21.1.11.1 Portfolio FVA

One way to calculate the portfolio FVA can be summarized as follows: First, compute future values (exposures) for each instrument in the portfolio using the single-curve model and then aggregate of the instruments at the portfolio (netting set) level and compute the FVA with the following approximate formula:

$$FVA(t) = E\left[\int_0^T \left(r_{eff}(t, V(t) - r(t)\right) \exp\left\{-\int_0^t r_{eff}\,(s, V(s))ds\right\} \cdot V(t)dt\right].$$

Here $r_{eff}(t, V(t) - r(t)$ is the effective funding spread over the OIS rate and $V(t)$ the future exposure (MtM) of the portfolio at time t. The last integral represents an effective funding discount factor. The effective rate is given by

$$r_{eff}(t, V(t)) \cdot V(t) = r_C C(t) + r_F(V(t) - C(t)).$$

This gives us the price, including the funding costs as

$$V(t) = E_t^Q\left[\exp\left\{-\int_t^T r_{eff}\,(s, V(s))\,ds\right\} \cdot V(T)|\mathcal{F}_t\right].$$

The important point to note in the previous formulas is that $V(t)$ is the **uncollateralized exposure**, and not the total exposure of the derivative. The average is taken over all possible future paths where both the derivative and the collateral change due to market movements.

Intuitively you can think of the FVA formula as averaging the amount of the derivative trade that cannot be funded by the collateral. Also, in order to ensure that you will be able to meet your future cash flows, you must have access to funding at your external or market funding rate, which will typically be higher than the OIS rate prescribed in the CSA.

We now define the *effective funding rate* as

$$r_{eff}(t) = \varphi_C(t) \cdot r_C(t) + (1 - \varphi_c(t)) \cdot r_F(t),$$

where φ_c is the fraction of traded value in the collateral account and 1 – $\varphi_c$the fraction of traded value with credit exposure. With 100% collateral φ_c = 1 and r_{eff} the collateral (OIS) rate. With 100% funding φ_c= 0 and r_{eff} the funding rate.

The effective rate is a path-dependent quantity, due to the collateral changing depending on the prevailing rate environment – or if a security is used as collateral – due to the market fluctuations of the collateral itself.

Over the lifetime of the trade, you can calculate the "effective funding rate" on each path, which is a weighted average of the collateral rate (OIS) and the firm's market funding rate, weighted by the amount inside and outside the collateral account respectively. Again, the intuition is that the collateral account cannot be 100% relied upon to ensure that you could meet all future cash flow obligations, due to the imperfect nature of the collateral arrangement with CTD options, MTM, thresholds, etc. The amount needed above the amount present in the collateral account will have to be funded in the market. This effective rate gives the average funding rate taking into account both sources of funding.

21.1.11.2 The Law of One Price

If a corporate (price-taker) gets bids from a number of banks, he/she will certainly get different quotes. These quotes may differ up to 100 bps or thereabouts. Therefore **FVA is a real charge to traders** reflecting their diverse funding costs. Traders are passed this charge from their treasury/funding desk.

Traders will attempt to pass this charge onto the purchaser as an upfront fee as a spread over the floating rate. However, this will be difficult in competitive markets. The FVA will certainly affect the profitability of the trade and all traders must know this charge.

In the interbank market, where the price is determined by the law of supply and demand, the prices on average should not include any charge for FVA.

In 2013, four banks (RBS, Lloyds, Goldman and Barclays) reported funding valuation in their financial statements.

22

CVA and DVA

22.1 Credit Value Adjustments and Funding

For years, a practice in the financial industry has been to mark derivatives portfolios to market without considering counterparty risk. All cash flows were discounted using the LIBOR or another "risk-free" curve. However, the true portfolio value must incorporate the possibility of losses due to counterparty default. This observation has gained wider recognition following the high-profile defaults of 2008. The credit value adjustment (CVA) is by definition the difference between the risk-free portfolio and the true portfolio value which should take into account the possibility of counterparty defaults. In other words, CVA represents the monetized value of the CCR.

22.1.1 Definitions of CVA and DVA

When reporting the fair value of any derivative position, we also need to consider counterparty credit risk (CCR). This is done by an adjustment to the value, known as **CVA**.

A pure definition can be written as

$$CVA = Discounted\ expected\ exposure \times Default\ probability \\ \times Loss\ given\ default.$$

For symmetry reason, we also need to consider the bilateral nature of CCR. This means that an institution would calculate a CVA under the assumption that they, as well as their counterparty, may default. A

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_22

defaulting institution "gains" on any outstanding liabilities that cannot be paid in full. This component is often referred to as **debt value adjustment (DVA)**.

The justification for using DVA is that a bank could buy back its debt cheaply to realize DVA gains, but no firm has actually done this.

22.1.2 Standard Approach

To study CVA and DVA we consider the simplest deal we can think of, a loan where the premium paid for an amount K at time T is equal to the risk-free price minus CVA. This is

$$P = e^{-rT}K - CVA_L,$$

where

$$CVA_L = E\left[e^{-rT}K \cdot 1_{\{\tau(B)\leq T\}}\right] = e^{-rT}K \cdot Q[\tau(B) \leq T] = e^{-rT}K \cdot \left[1 - e^{-\pi(B)T}\right].$$

Here, we denote the lender with L. $\tau(B)$ is **time to default** for the borrower (B), and $\pi(B)$ the **credit default swap (CDS) spread** for the borrower. Here we used the definition of the **default probability**

$$\begin{aligned}
&Q_D(\tau(B) > T) = E\left[1_{\{\tau(B)>T\}}\right] = e^{-\pi(B)T} \\
&\Leftrightarrow \\
&Q_D(\tau(B) \leq T) = E\left[1_{\{\tau(B)\leq T\}}\right] = 1 - e^{-\pi(B)T}.
\end{aligned}$$

If we have a certain recovery rate R, the previous formula should be

$$Q_D(\tau(B) \leq T)E\left[1_{\{\tau(B)\leq T\}}\right] = \frac{1 - e^{-\pi(B)T}}{1 - R}.$$

For the borrower we have

$$P = -e^{-rT}K + DVA_B.$$

The present values of all cash flows are given by

$$\begin{aligned}
V_L &= e^{-rT}K - CVA_L - P; \\
V_B &= -e^{-rT}K + DVA_B + P.
\end{aligned}$$

To get an agreement between the borrower and the lender we need $V_L = V_B$. We then have

$$2P = 2e^{-rT}K - CVA_L - DVA_B.$$

With $DVA_L = CVA_B$ we get

$$P = e^{-rT}K - e^{-rT}\left[1 - e^{-\pi(B)T}\right] = K \cdot e^{-(r+\pi(B))T}.$$

This is the agreed price between the borrower and the lender. But something is missing here. We need to know about the liquidity. The lender needs to finance the amount until maturity at a **funding spread** $S(L)$ but the borrower can reduce his funding cost with P. The borrower should therefore see a funding benefit, and the lender should see that the fair value of the claim reduced by the funding cost. Therefore, the effect of this funding cost seems to be missing in the earlier formula.

22.1.3 Approach Including Liquidity

One method is to introduce DVA as a **liquidity cost** by adjusting the discounting term and introduce the possibility of defaulting on the payoff. This means that we need to replace the term $e^{-rT}K$ with $e^{-(r+s(L))T}K \cdot 1_{\{\tau(B)>T\}}$ and get

$$\begin{aligned} V_L &= E\left[e^{-(r+s(L))T}K \cdot 1_{\{\tau(B)>T\}}\right] - P = E\left[e^{-(r+\pi(L)+\gamma(L))T}K \cdot 1_{\{\tau(B)>T\}}\right] - P \\ &= e^{-(r+\pi(L)+\gamma(L))T} \cdot K \cdot e^{-\pi(B)T} - P \equiv e^{-(r+\pi(L)+\pi(B)+\gamma(L))T} \cdot K - P. \end{aligned} \tag{22.1}$$

Here $\gamma(L)$ is the **deterministic default intensity** for the lender, L

$$s(X) = \pi(X) + \gamma(X); X = \{B, L\}.$$

Similarly, for the borrower B we have

$$\begin{aligned} V_B &= -E\left[e^{-(r+s(B))T} \cdot K \cdot 1_{\{\tau(B)>T\}}\right] + P \\ &= -E\left[e^{-(r+\pi(B)+\gamma(B))T} \cdot K \cdot 1_{\{\tau(B)>T\}}\right] + P \\ &= -e^{-(r+\pi(B)+\gamma(B))T} \cdot K \cdot e^{-\pi(B)T} + P \equiv -e^{-(r+2\pi(B)+\gamma(B))T} \cdot K + P \end{aligned} \tag{22.2}$$

When comparing this result with the previous section, it is convenient to confine ourselves to the simplest situation where the lender is default-free with no liquidity spread, while the borrower is defaultable with the minimum liquidity spread allowed, the CDS spread. In this case we have ($s(L) = 0$, $s(B) = \pi(B) > 0$) and we get

$$V_L = e^{-(r+\pi(B))T} K - P$$
$$V_B = -e^{-(r+2\pi(B))T} K + P.$$

Here we observe two bizarre aspects. First, even in a situation where we have assumed no liquidity spread the two counterparties cannot agree on the simplest transaction with default risk. A day-one profit should be accounted for by borrowers in all transactions with CVA. This contradicts market reality.

Secondly, the explicit inclusion of the DVA term results in a duplication of the funding benefit for the party that assumes the liability. Against all evidence the formula implies that the funding benefit is remunerated twice. If this were correct then a consistent accounting of liabilities at fair value would require pricing zero-coupon bonds by multiplying twice their risk-free present value by their survival probabilities.

22.1.4 How to Make It Right

To solve this in a right way, we do not calculate the liquidity by the adjusted discounting approach as in Equations (22.1) and (22.2). Instead, we generate the liquidity costs and the benefits by modelling explicitly the **funding strategy.** Here we take into account how the companies capitalize and discount money with the risk-free rate r, and then add or subtract the actual credit and funding costs that arise in the deal.

This allows us to introduce explicitly both credit and liquidity and to investigate more precisely, where credit/liquidity gains and losses are financially generated. We take into account that the aforementioned deal has two legs.

If we consider the lender, one leg is the **deal leg**, with net present value

$$E\left[-P + e^{-rT}\Pi\right].$$

Here Π is the payoff at T, including a potential default indicator. The other leg is the **funding leg** with the net present value

$$E\left[+P - e^{-rT} F\right],$$

where F is the funding payback at T, including a potential default indicator. When there is no default risk or liquidity cost involved, this funding leg can be overlooked because it has a value of $E\left[+P - e^{-rT} e^{rT} P\right] = 0$. In the general case the total net present value is

$$V_L = E\left[-P + e^{-rT}\Pi + P - e^{-rT} F\right] \equiv E\left[e^{-rT}(\Pi - F)\right].$$

Thus, the premium at time 0 cancels out with its funding and we are left with the discounting of a total payoff including the deal's payoff and the liquidity payback. An analogous relationship applies for the borrower, as will be described later.

When we continue, we work under the hypothesis that all liquidity management happens in the cash market. Then, the funding is made by issuing bonds and excess funds are used to reduce or to avoid increasing the stock of bonds. This is the most natural assumption since it is similar to the assumption that banks make in their internal liquidity management, namely, what the treasury desk assumes in charging or rewarding trading desks.

22.1.4.1 Risky Funding with DVA for the Borrower

The borrower has a liquidity advantage from receiving the premium P at time zero, as it allows him/her to reduce the funding requirement by an equivalent amount. The amount P generates a negative cash flow at T, when funding must be paid back. This is equal to

$$-P \cdot e^{rT} e^{s(B)T} \cdot 1_{\{\tau(B)>T\}}.$$

This future outflow equals P, capitalized at the funding cost, times a default indicator. We need to include a default indicator in case of default and zero recovery. During default, the borrower does not pay back the borrowed funding and there is no outflow. Thus reducing the funding by P corresponds to receiving at T a positive amount equal to

$$\begin{aligned} P \cdot e^{rT} e^{s(B)T} \cdot 1_{\{\tau(B)>T\}} &= \{s(B) = \pi(B) + \gamma(B)\} \\ &= P \cdot e^{rT} e^{\pi(B)T} e^{\gamma(B)T} \cdot 1_{\{\tau(B)>T\}}. \end{aligned} \tag{22.3}$$

Thus, the total payoff at $T$is

$$P \cdot e^{rT} e^{\pi(B)T} e^{\gamma(B)T} \cdot 1_{\{\tau(B)>T\}} - K \cdot 1_{\{\tau(B)>T\}}.$$

Taking discounted expectation, we get

$$V_B = e^{-\pi(B)T} \cdot P \cdot e^{\pi(B)T} e^{\gamma(B)T} - e^{-rT} \cdot K \cdot e^{-\pi(B)T} = P \cdot e^{\gamma(B)T} - K \cdot e^{-(r+\pi(B))T}.$$

Compared with Equation (22.2), $(P - K \cdot e^{-(r+2\pi(B)+\gamma(B))T})$ we have no unrealistic double-accounting of default probability. Also notice that

$$V_B = 0 \Rightarrow P_B = K \cdot e^{-rT} e^{-2\pi(B)T} e^{\gamma(B)T}, \tag{22.4}$$

where P_B is the **breakeven premium** for the borrower, in the sense that the borrower will find this deal convenient as long as $V_B \geq 0 \Rightarrow P \geq P_B$. Compared with Equation (22.2) i.e. the standard DVA with liquidity where we observe the double counting

$$V_B = 0 \Rightarrow P_B = K \cdot e^{-rT} e^{-2\pi(B)T} e^{\gamma(B)T}.$$

As before, assuming

$$P_B = K \cdot e^{-(r+\pi(B))T}$$

we may conclude that, in this case, the computation from the standard CVA is correct, because it is taking into account the probability of default in the valuation of the funding benefit, which removes any liquidity advantage for the borrower.

Equation (22.4) shows what happens when, in addition, there is a **pure liquidity basis** component in the funding cost. On the other hand, charging liquidity costs with an adjusted funding spread cannot be naturally extended to the case where we want to observe explicitly the possibility of default events in our derivatives.

In writing the payoff for the borrower we have not explicitly considered the case in which the deal is interrupted by the default of the lender, since we can replace the deal with a new counterparty. This keeps V_B independent of the default time of the lender, consistently with the reality of bond and deposit markets.

22.1.4.2 Risky Funding with CVA for the Lender

If the lender pays P at time 0, he incurs a liquidity cost. In fact, he needs to finance (borrow) P until T. At T, the lender will pay back the borrowed money with interest, but only if he has not defaulted. Otherwise, he gives back nothing, so the outflow for the lender is

$$P \cdot e^{rT} e^{s(L)T} \cdot 1_{\{\tau(L)>T\}} = P \cdot e^{rT} e^{\pi(L)T} e^{\gamma(L)T} \cdot 1_{\{\tau(L)>T\}}$$

while he receives in the deal $K \cdot 1_{\{\tau(B)>T\}}$. The total payoff at T is therefore

$$-P \cdot e^{rT} e^{\pi(L)T} e^{\gamma(L)T} \cdot 1_{\{\tau(L)>T\}} + K \cdot 1_{\{\tau(B)>T\}}.$$

Taking discounted expectation we get

$$\begin{aligned} V_L &= -P \cdot e^{-\pi(L)T} e^{\pi(L)T} e^{\gamma(L)T} \cdot 1_{\{\tau(L)>T\}} + e^{-rT} e^{-\pi(B)T} K, \\ &= -P \cdot e^{\gamma(L)T} \cdot 1_{\{\tau(L)>T\}} + e^{-(r+\pi(B))T} K. \end{aligned}$$

The condition that makes the deal convenient for the lender is

$$\begin{aligned} &V_L \geq 0 \Rightarrow P \leq P_L, \\ &P_L = K \cdot e^{-rT} e^{-\pi(B)T} e^{-\gamma(L)T} \equiv K \cdot e^{-(r+\pi(B)+\gamma(L))T}, \end{aligned}$$

where P_L is the break-even premium.

It is interesting to note that when the lender computes the value of the deal and takes into account all future cash flows as they are seen from the counterparties, the valuation does not include a charge to the borrower for the component $\pi(L)$. This term, the cost of funding would be associated with his own risk of default. This term is cancelled by the fact that funding is not given back in case of default.

In terms of relative valuation of a deal, this fact about the lender is exactly symmetric to the fact that for the borrower, the inclusion of the DVA eliminates the liquidity advantage associated with $\pi(B)$. In terms of managing cash flows, instead, there is an important difference between the parties, which is discussed next. For reaching an agreement in the market, we need

$$V_L \geq 0, V_B \geq 0$$

which implies

$$\begin{aligned}
&P_L \geq P \geq P_B \quad \Leftrightarrow \\
&K \cdot e^{-rT} e^{-\pi(B)T} e^{-\gamma(L)T} \geq P \geq K \cdot e^{-rT} e^{-\pi(B)T} e^{-\gamma(B)T} \\
&\Rightarrow \\
&\gamma(B) \geq \gamma(L).
\end{aligned}$$

This solves the problem, and shows that, if we only want to guarantee a positive expected return from the deal, the liquidity cost that needs to be charged to the counterparty of an uncollateralized derivative transaction is just the liquidity basis, rather than the bond spread or the CDS spread.

This is in line with what actually happened during the liquidity crisis in 2007–2009, when the bond-CDS basis exploded. Next we also show how the picture changes when we look at the possible realized cash flows (as opposed to the expected cash flows).

22.1.4.3 Positive Recovery

Now, we study what happens if we relax the assumption of zero recovery. The discounted payoff for the borrower is now

$$\begin{aligned}
&e^{-rT}\left(P \cdot e^{rT} e^{\pi(B)T} e^{\gamma(B)T} - K\right) \cdot 1_{\{\tau(B)>T\}} \\
&\quad + R_B \cdot e^{-rT}\left(P \cdot e^{rT} e^{\pi(B)T} e^{\gamma(B)T} - K\right) \cdot 1_{\{\tau(B)\leq T\}} \\
&\Rightarrow \\
&e^{-rT}\left(P \cdot e^{rT} e^{\pi(B)T} e^{\gamma(B)T} - K\right) \cdot \left((1 - 1_{\{\tau(B)\leq T\}}) + R_B \cdot 1_{\{\tau(B)\leq T\}}\right), \\
&\Rightarrow \\
&\left(P \cdot e^{\pi(B)T} e^{\gamma(B)T} - e^{-rT} K\right) \cdot \left(1 - (1 - R_B)(1 - 1_{\{\tau(B)>T\}})\right),
\end{aligned}$$

where the **recovery** is a fraction, R_B of the present value of the claims at the time of default of the borrower, consistent with standard derivative documentation. Notice that the borrower acts as a borrower both in the deal and in the funding leg, since we represented the latter as a reduction of the existing funding of the borrower. By taking the expectation at time 0 we obtain

$$V_B = (1 - (1 - R_B)S_B(T)) \cdot \left(P \cdot e^{\pi(B)T} e^{\gamma(B)T} - e^{-rT} K\right),$$

where $S_B(T)$ is the survival probability of B. Using $\pi(B) = \lambda(B)(1 - R_B)$, we can apply a first order approximation

$$1 - e^{-\pi(B)T} \approx (1 - R_B) \cdot \left(1 - e^{-\lambda(B)T}\right).$$

Here $\lambda(B)$ is the deterministic default intensity. Then

$$V_B \approx e^{-\pi(B)T} \cdot \left(P \cdot e^{\pi(B)T} e^{\gamma(B)T} - e^{-rT} K\right) = P \cdot e^{\gamma(B)T} - K \cdot e^{-(r+\pi(B))T}.$$

Rather surprisingly, this is the same formula we found when we were studying the risky funding with DVA for the borrower!

Similar arguments apply to the value of the claim for the lender, which acts as a lender in the deal and as a borrower in the funding leg. For the lender we have

$$V_L = -P \cdot e^{\gamma(L)T} \cdot 1_{\{\tau(L)>T\}} + e^{-(r+\pi(B))T} K$$

is recovered as a first-order approximation of

$$V_L = -(1 - (1 - R_L)S_L(T)) \cdot P \cdot e^{\pi(L)T} e^{\gamma(L)T} + (1 - (1 - R_B)S_B(T)) \cdot e^{-rT} K.$$

22.1.4.4 Can DVA Become a Funding Benefit?

One of the most controversial aspects of DVA concerns the consequences for the accounting of liabilities on the balance sheet. In fact, DVA enables the borrower to condition future liabilities on survival and create a benefit on default. However, liabilities are already reduced by risk of default in the case of bonds when banks use the *fair value option* according to international accounting standard, and when banks mark the bond liabilities at historical cost.

So, what is the meaning of DVA? Can we really observe a benefit in case of our own default?

In order to answer this question we need to study what happens if the borrower pretends to be default-free, thereby ignoring DVA.

The borrower can perform valuation for accounting purposes using an accounting credit spread $\pi(B)$ that may be different from the market spread and an accounting liquidity basis $\gamma(B)$ possibly different from the market one, albeit with the constraint that their sum $s(B)$ must match the market funding spread. In particular, when the party pretends to be default free, we have $\pi(B) = 0$ and $\gamma(B) = s(B)$, and there are no more indicators for our own default in the payoffs.

Assume that the borrower pretends, for accounting purposes, to have zero default risk. The premium P paid by the lender gives the borrower a reduction of the funding payback at T corresponding to a cash flow $P \cdot e^{rT} e^{s(B)T}$ at T, where there is no default indicator because the borrower is treating itself as default-free.

This cash flow must be compared with the payout of the deal at T, which is – K, again without indicator, i.e. without DVA. Thus the total payoff at T is

$$P \cdot e^{rT} e^{s(B)T} - K.$$

By discounting to time 0 we obtain an accounting value V_B such that

$$V_B = P \cdot e^{s(B)T} - K \cdot e^{-rT}$$

which yields an accounting break-even premium P_B for the borrower equal to the break-even in Equation (22.4),

$$P_B = K \cdot e^{-rT} e^{-\pi(B)T} e^{-\gamma(B)T} \equiv K \cdot e^{-(r+\pi(B)+\gamma(B))T},$$

where now $\pi(B)$ and $\gamma(B)$ are those provided in the market. So in this case too the borrower B sees a funding benefit that actually takes into account its own market risk of default $\pi(B)$, plus an additional liquidity basis $\gamma(B)$, thereby matching the premium computed by the lender that includes the CVA/DVA term. But now this term is accounted for as a funding benefit and *not* as a benefit coming from the reduction of future expected liabilities, thanks to the risk of default.

This shows how the DVA term can be implemented. When a bank enters a deal in a borrower position, it is making funding for an amount as large as the premium. If this premium is used to reduce existing funding which is equally or more expensive, which in our setting means buying bonds or avoiding some issuance that would be necessary otherwise, this provides a real financial benefit that is enjoyed in the case of survival by a reduction of the payments at maturity. A bank can buy back its own bonds, which is like 'selling protection on itself' fully funded. When a sale of protection is funded, there is no counterparty risk and therefore no limit to whom can sell protection, which is different from the case of an unfunded CDS. In fact buying their own bonds is a standard and important activity of banks. The reduction is given by the difference in $P \cdot e^{rT} e^{s(B)T} - K$.

If the quantity of outstanding bonds is sufficiently high to allow the implementation of such a strategy, we have shown how the DVA term can be seen *not* as a **default benefit**, but rather a natural component of fair value whenever fair value mark-to-market takes into account counterparty risk and funding costs.

22.1.4.5 The Accounting View for the Lender

The previous results show that the borrower's valuation does not change if he considers himself default free by using an accounting credit spread $\pi(B) = 0$ and treating all the funding cost $s(B)$ he sees in the market as a pure liquidity spread $\gamma(B) = s(B)$. Do we have a similar property also for the lender? Not at all.

If the lender computes the break-even premium using an accounting credit spread $\pi(L)$ and an accounting liquidity spread $\gamma(L) = s(L) - \pi(L)$ different from those provided by the market, he gets a different breakeven premium, because

$$P_L = K \cdot e^{-(r+\pi(B)+\gamma(L))T}.$$

Thus, the breakeven premium and the agreement that will be reached in the market depend crucially on $\gamma(L)$. In Fig. 22.1, for a sample deal, we show how P_L varies when, holding $s(L)$ fixed, we vary $\kappa(L) = \gamma(L)/s(L)$, which we call the **liquidity ratio** of the lender. This

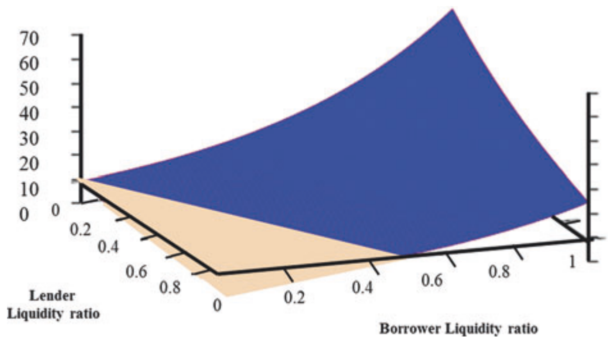


Fig. 22.1 Break-even premium for L, P_L as a function of the liquidity cost

is not the only difference between the situation for the borrower and the lender. Notice that the borrower's net payout at maturity T is given by

$$P \cdot e^{(r+\pi(B)+\gamma(B))T} \cdot 1_{\{\tau(B)>T\}} - K \cdot 1_{\{\tau(B)>T\}}$$

and is non-negative in all states of the world if provided we keep $P \geq P_B$, although the latter condition was designed only in order to guarantee that the expected payout is non-negative. For the lender instead the payout at maturity is given by

$$-P \cdot e^{(r+\pi(L)+\gamma(L))T} \cdot 1_{\{\tau(L)>T\}} + K \cdot 1_{\{\tau(B)>T\}}.$$

The condition $V_L \geq 0$ does not imply the non-negativity of the earlier expression. In particular, we can have a negative carry even if we assume that both counterparties will survive until maturity.

Figure 22.1 shows break-even premium for the lender P_L as a function of the liquidity cost ratios $\kappa(L)$ and $\kappa(B)$ when $s(L) = 005$, $s(B) = 01$, $T = 20$, $K = 100$, $r = 002$. The xyplane crosses the z-axis at the break-even premium for the borrower P_B. A deal is possible only in the blue region.

If we want to guarantee a non-negative carry at least when nobody defaults we need $\pi(L) \leq \pi(B)$. Otherwise, the lender is exposed to liquidity shortages and a negative carry even if the deal is convenient for him. Liquidity shortages when no one defaults can be excluded by imposing for each deal $\pi(L) \leq \pi(B)$, or, with a solution working for whatever deal with whatever counterparty, by working as if the lender were default-free. Only if the lender pretends for accounting purposes to be default-free will the condition for the convenience of the deal based on expected cash flows be

$$P \leq K \cdot e^{-(r+\pi(B)+s(L))T} = K \cdot e^{-(r+\pi(B)+\gamma(L)+\pi(L))T}.$$

This clearly implies that

$$-P \cdot e^{(r+\pi(L)+\gamma(L))T} \cdot 1_{\{\tau(L)>T\}} + K \cdot 1_{\{\tau(B)>T\}}.$$

should be non-negative. On the other hand, the lender's assumption to be default-free makes a market agreement more difficult, since

$$K \cdot e^{-(r+\pi(B)+\gamma(B))T} \leq P \leq K \cdot e^{-(r+\pi(B)+\gamma(L)+\pi(L))T}.$$

implies

$$\gamma(B) \geq \gamma(L) + \pi(L)$$

rather than $\gamma(B) \geq \gamma(L)$. Under this assumption, uncollateralized payoffs should be discounted at the full funding cost also in our simple setting. Let us consider a bank X that pretends to be default-free and thus works under the assumption $\kappa(B) = \kappa(X) = 1$ when the bank X is a net borrower and $\kappa(L) = \kappa(X) = 1$ when X is in a lender position. When the bank is in the borrower position, we have

$$P_B = P_X = K \cdot e^{-(r+s(X))T}.$$

When it is in a lender position with respect to a counterparty the breakeven premium will be given by

$$P_L = K \cdot e^{-(r+s(X))T} = P_B = P_X.$$

and the discounting at the funding rate $r + s(X)$ is recovered for both positive and negative exposures.

22.1.5 Final Conclusions

The previous discussion showed how a consistent framework for the joint pricing of liquidity costs and counterparty risk can be formulated. This was accomplished by explicitly modelling the funding components of a simplified derivative where both counterparties might default. We saw how bilateral counterparty risk adjustments (CVA and DVA) could be combined with liquidity/funding costs without any unrealistic double counting effects.

We also found that DVA has a meaningful representation in terms of funding benefit for the borrower, so that a bank can take into account DVA and find an agreement with lenders computing CVA even when it neglects its own probability of default. On the other hand, the lender's cost of funding includes a component that is associated with his own risk of default, but this component cancels out with his default probability, so that only his liquidity spread (or equivalently his bond-CDS basis) contributes as a net funding cost to the value of the transaction.

We also discussed how the situations of the borrower and the lender were different; in particular, the lender could have negative carry upon no default even if the value of the deal was positive for him.

Thus, while the debate appears to be focused on the impact of accounting choices on the valuation of liabilities, the previous discussion illustrated that it is rather on the valuation of assets that such choices make a difference.

23
Market Models

23.1 The LIBOR Market Model

One of the general disadvantages of short rate and HJM models is that they focus on unobservable instantaneous interest rates. The so-called market models, that was developed in the late 90's[1] tries to overcome this problem by instead modelling observable market rates such as LIBOR and swap rates. These models can be calibrated to the market and have gained a widespread acceptance from practitioners.

The first market models were developed in the HJM framework where the dynamics of instantaneous forward rates are used to determine the dynamics of zero-coupon bonds. The dynamics of zero coupon bond prices were then used to determine the dynamics of LIBOR. Market models are therefore not inconsistent with HJM models discussed in Chapter 16.

23.1.1 Introduction

The term structure models studied in the previous chapters have involved assumptions about the evolution in one or more continuously compounded interest rates, either the short rate $r(t)$ or, as in the Heath-Jarrow-Morton (HJM) framework, the instantaneous forward rates $f(t, T)$. One drawback of these models is that they are expressed

[1] See Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997), Jamshidian (1997) and Musiela and Rutkowski (1997).

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_23

in terms of interest rates that are not directly observed in the market. Another disadvantage in HJM is that all model parameters must be recalibrated at each point in time; there is no mechanism for sequential updating. Also it is difficult to calibrate the model to actively traded prices in the market. However, many securities traded in money markets (e.g. caps, floors, swaps and swaptions) depend on discretely compounded interest rates such as spot LIBOR rates $l(t, t+\delta)$, forward LIBOR rates $L(t, T, T+\delta)$, spot swap rates $l_{swap}(t, \delta)$ and forward swap rates $L_{swap}(t, T, \delta)$. For the pricing of these securities it seems more appropriate to apply models that are based on assumptions about the LIBOR rates directly or spot and forward swap rates.

We use the term market models for models based on assumptions about discretely compounded interest rates. Market models take the currently observed term structure of interest rates as a given and are therefore to be classified as relative pricing or pure no-arbitrage models. Consequently, they offer no insights into the determination of the current interest rates. LIBOR market models (LMM) are based on assumptions about the evolution of forward LIBOR rates. Similarly, swap market models are based on assumptions about the evolution of forward swap rates. By construction, market models are not suitable for the pricing of futures and options on government bonds and similar contracts that do not depend on money market interest rates.

In the recent literature various market models have been developed, but most attention has been given to the so-called lognormal LMM. In such models the volatilities of a relevant selection of the forward LIBOR rates are assumed to be proportional to the level of the forward rate so that the distribution of the future forward LIBOR rates is lognormal. Lognormally distributed continuously compounded interest rates have unpleasant consequences, but Sandmann and Sondermann (1997) showed that models with lognormally distributed, discretely compounded rates are not subject to the same problems. Later, we will demonstrate that a lognormal assumption on the distribution of forward LIBOR rates implies that pricing formulas for caps and floors identical to Black's pricing formulas can be derived. Hence, the lognormal market models provide some support for the widespread use of Black's formula for fixed income securities.

We have to be aware that lognormal models can't handle negative rates. So we here suppose we have a market situation with strictly positive interest rates.

23.1.2 General LIBOR Market Models

In this section we will introduce a general **LMM**, also referred to as the **BGM/J model** (Brace, Gatarek, Musiela and Jamshidian), describe some of the model's basic properties, and discuss how derivative securities can be priced within the framework of the model.

23.1.2.1 Model Description

As was described in Section 4.1.14, a cap is a contract that protects a floating rate borrower against paying an interest rate higher than some given rate K, the so-called cap rate. We let $T_1, \ldots, T_n$ denote the payment dates and assume that $T_i - T_{i-1} = \delta$ for all i. In addition we define $T_0 = T_1 - \delta$. At each time $T_i (i = 1, \ldots, n)$ the cap gives a payoff of

$$C^i(T_i) = N\delta \max\{L(T_i, T_i-\delta)-K, 0\} = N\delta \max\{L(T_i-\delta, T_i-\delta, T_i)-K, 0\},$$

where N is the face value of the cap. A cap can be considered as a portfolio of caplets, namely one caplet for each payment date with payoffs described by the previous formula.

The definition of the forward martingale measure in Chapter 17 implies that the value of the aformentioned payoff can be found as the product of the expected payoff computed under the T_i-forward martingale measure and $p(t, T_i)$ the current discount factor for time T_i payments, that is,

$$C^i(t) = N\delta p(t, T_i)E_t^{Q^{T_i}}[\max\{L(T_i - \delta, T_i - \delta, T_i) - K, 0\}]; \quad t < T_i - \delta.$$

The price of a cap can therefore be determined as the sum of the value of the caplets

$$C(t) = N\delta \sum_{i=1}^{n} p(t, T_i)E_t^{Q^{T_i}}[\max\{L(T_i - \delta, T_i - \delta, T_i) - K, 0\}]; \quad t < T_0.$$

For $t \geqslant T_0$ the first-coming payment of the cap is known so that its present value is obtained by multiplication by the risk-less discount factor, while the remaining payoffs are valued as the previous one.

The price of the corresponding floor is

$$F(t) = N\delta \sum_{i=1}^{n} p(t,\ T_i) E_t^{Q^{T_i}} [\max\{K - L(T_i - \delta,\ T_i - \delta,\ T_i),\ 0\}]; \quad t < T_0.$$

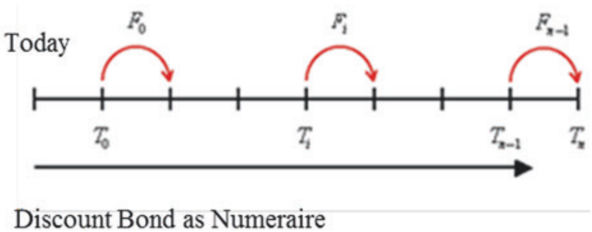
In order to compute the cap price, we need to know of the distribution of $L(T_i - \delta, T_i - \delta, T_i)$ under the T_i-forward martingale measure Q^{Ti} for each $i = 1, \ldots, n$. For this purpose it is natural to model the evolution of $L(t, T_i - \delta, T_i)$ under Q^{Ti} . The following argument shows that under the Q^{Ti} probability measure the drift rate of $L(t, T_i - \delta, T_i)$ is zero, that is, $L(t, T_i - \delta, T_i)$ is a Q^{Ti} martingale. The simple compounded forward rate at time t spanning the future period $[T_1, T_2], L(t, T_1, T_2)$ is defined by

$$\frac{p(t,\ T_2)}{p(t,\ T_1)} = \frac{1}{1 + L(t,\ T_1,\ T_2)(T_2 - T_1)}.$$

We rewrite this as

$$L(t,\ T_i - \delta,\ T_i) = \frac{1}{\delta}\left(\frac{p(t,\ T_i - \delta)}{p(t,\ T_i)} - 1\right),$$

where $\delta = T_2 - T_1$. The following diagram illustrates a set of forward rates spanning the set of dates T_i



Under the T_i-forward martingale measure Q^{Ti} the ratio between the price of any asset and the numeraire, that is, the zero-coupon bond price $p(t, T_i)$ is a martingale. In particular, the ratio $p(t, T_i - \delta)/p(t, T_i)$ is a Q^{Ti} martingale so its expected change over any time interval is equal to zero under the Q^{Ti} measure. From the previous formula it follows that the expected change (over any time interval) in the periodically compounded forward rate $L(t, T_i\delta, T_i)$ also is zero under Q^{Ti} . We summarize the result in the following theorem

Theorem 23.1. *The forward rate $L(t, T_i - \delta, T_i)$ is a Q^{Ti} martingale.*

Consequently, a LMM is fully specified by the number of factors (i.e. the number of standard Brownian motions) that influence the forward rates and the forward rate volatility functions. For simplicity, we focus on the one-factor models

$$dL(t, T_i - \delta, T_i) = \beta\left(t, T_i - \delta, T_i, L(t, T_j, \delta)|_{T_j \geqslant t}\right) dz(t, T_i), \quad i = 1 \ldots, n,$$

where $z(t, T_i)$ is a one-dimensional standard Brownian motion under the T_i-forward martingale measure Q^{Ti}. The fourth argument in the volatility function β indicates that the volatility at time t can depend on the current values of all the modelled forward rates. In the lognormal LMMs we will study later, one assumes that volatility of each forward rate is proportional to the current level of the same forward rate

$$\beta\left(t, T_i - \delta, T_i, L(t, T_j, \delta)|_{T_j \geqslant t}\right) = \gamma(t, T_i - \delta, T_i) L(t, T_i - \delta, T_i)$$

for some deterministic function γ. However, for now we continue to discuss the more general specification. We see from the general cap pricing formula that the cap price also depends on the current discount factors $p(t, T_1), p(t, T_2), \ldots, p(t, T_n)$. These discount factors can be determined by $p(t, T_0)$, and the current values of the modelled forward rates (i.e. $L(t, T_0, T_1), L(t, T_1, T_2)), \ldots, L(t, T_{n-1}, T_n)$,. Similar to the HJM model, the LMMs take the currently observable values of these rates as given.

23.1.2.2 The Dynamics of All Forward Rates Under the Same Probability Measure

The basic specification of the LMM involves n different forward martingale measures. In order to better understand the model and to simplify the computation of some derivative prices we will describe the evolution of the relevant forward rates under a common probability measure. As discussed below, Monte Carlo simulation is often used to compute prices of certain derivatives in LMMs. It is much simpler to simulate the evolution of the forward rates under a common probability measure than to simulate the evolution of each forward rate under a different martingale measure associated with the respective forward rate. One possibility is to choose one of the n different forward martingale measures used in the assumption of the model. Note that the

T_i-forward martingale measure only makes sense up to time T_i. Therefore, it is appropriate to use the forward martingale measure associated with the last payment date (i.e. the T_n-forward martingale measure Q^{Tn}), since this measure applies to the entire relevant time period. In this context Q^{Tn} is sometimes referred to as *the terminal measure*. Another obvious candidate for the common probability measure is *the spot martingale measure*. Let us look at these two alternatives in more detail.

23.1.2.3 The Terminal Probability Measure

We wish to describe the evolution of all modelled forward rates under a common probability measure – here the T_n-forward martingale measure. For that purpose we shall apply the following theorem that outlines how to shift between the different forward martingale measures of the LMM.

Theorem 23.2. *Assume that the evolution in the LIBOR forward rates $L(t, T_i - \delta, T_i)$ for $i = 1, \ldots, n$, where $T_i = T_{i-1} + \delta$, is given by*

$$dL(t, T_i - \delta, T_i) = \beta\left(t, T_i - \delta, T_i, L(t, T_j, \delta)|_{T_j \geq t}\right) dz(t, T_i), \quad i = 1 \ldots, n.$$

Then the processes $z(T_i - \delta)$ and $z(T_i)$ are related as follows:

$$dz(t, T_i) = dz(t, T_i - \delta) + \frac{\delta\beta\left(t, T_i - \delta, T_i, L(t, T_j, \delta)|_{T_j \geqslant t}\right)}{1 + \delta L(t, T_i - \delta, T_i)} dt.$$

Using this repeatedly, we get that

$$dz(t, T_n) = dz(t, T_i) + \sum_{j=i}^{n-1} \frac{\delta\beta\left(t, T_j, T_{j+1}, L(t, T_j, \delta)|_{T_j \geqslant t}\right)}{1 + \delta L(t, T_j, T_{j+1})} dt.$$

Consequently, for each $i = 1, \ldots, n$, we can write the dynamics of $L(t, T_i - \delta, T_i)$ under the Q^{Tn}-measure as

$$\begin{aligned} dL(t, T_i - \delta, T_i) &= \beta\left(t, T_i - \delta, T_i, L(t, T_j, \delta)|_{T_j \geqslant t}\right) dz(t, T_i) \\ &= \beta\left(t, T_i - \delta, T_i, L(t, T_j, \delta)|_{T_j \geqslant t}\right) \\ &\quad \times \left[dz(t, T_n) - \sum_{j=1}^{n-1} \frac{\delta\beta\left(t, T_j, T_{j+1}, L(t, T_j, \delta)|_{T_j \geqslant t}\right)}{1 + \delta L(t, T_j, T_{j+1})} dt \right], \end{aligned}$$

$$= -\sum_{j=i}^{n-1} \frac{\delta\beta\left(t,\, T_i - \delta,\, T_i, L(t,\, T_k,\, \delta)|_{T_k \geqslant t}\right) \beta\left(t,\, T_j,\, T_{j+1}, L(t,\, T_k,\, \delta)|_{T_k \geqslant t}\right)}{1 + \delta L(t,\, T_j,\, T_{j+1})} dt$$
$$+ \beta\left(t,\, T_i - \delta,\, T_i, L(t,\, T_k,\, \delta)|_{T_k \geqslant t}\right) dz(t,\, T_n).$$

Note that the drift may involve some or all of the other modelled forward rates. Therefore, the vector of all the forward rates $(L(t, T_0, T_1), \dots, L(t, T_{n-1}, T_n))$ will follow an n-dimensional diffusion process so that an LMM can be represented as an n-factor diffusion model. Security prices are hence solutions to a partial differential equation (PDE), but in typical applications the dimension n (i.e. the number of forward rates) is so big that neither explicit nor numerical solution of the PDE is feasible. For example, in order to price caps, floors, and swaptions that depend on 3-month interest rates and have maturities of up to 10 years, one must model 40 forward rates so that the model is a 40-factor diffusion model! However, Andersen and Andreasen (2000) introduce a trick that may reduce the computational complexity considerably.

Next, let us consider an asset with a single payoff at some point in time $T \in [T_0, T_n]$. The payoff $N(T)$ may in general depend on the value of all the modelled forward rates at and before time T. Let $V(t)$ denote the value of this asset at time t (measured in monetary units, e.g. dollars). From the definition of the T_n-forward martingale measure Q^{T_n} it follows that

$$V(t) = p(t,\, T_n) E^{Q^{T_n}}\left[\left.\frac{N(T)}{p(T,\, T_n)}\right| \mathcal{F}_t\right].$$

In particular, if T is one of the time points of the tenor structure, say $T = T_k$, we get

$$V(t) = p(t,\, T_n) E^{Q^{T_n}}\left[\left.\frac{N(T_k)}{p(T_k,\, T_n)}\right| F_t\right].$$

From

$$L(t,\, T_i - \delta,\, T_i) = \frac{1}{\delta}\left(\frac{p(t,\, T_i - \delta)}{p(t,\, T_i)} - 1\right),$$

we have that

$$\frac{1}{p(T_k, T_n)} = \frac{p(T_k, T_k)}{p(T_k, T_{k+1})}\frac{p(T_k, T_{k+1})}{p(T_k, T_{k+2})}\cdots\frac{p(T_k, T_{n-1})}{p(T_k, T_n)}$$
$$= [1 + \delta L(T_k, T_k, T_{k+1})][1 + \delta L(T_k, T_{k+1}, T_{k+2})]\ldots[1 + \delta L(T_k, T_{n-1}, T_n)]$$
$$= \prod_{j=k}^{n-1}[1 + \delta L(T_k, T_j, T_{j+1})]$$

so that the price can be rewritten as

$$V(t) = p(t, T_n)E^{Q^{T_n}}\left[N(T_k)\prod_{j=k}^{n-1}[1 + \delta L(T_k, T_j, T_{j+1})]\middle|\mathcal{F}_t\right].$$

The right-hand side may be approximated using Monte Carlo simulations in which the evolution of the forward rates under Q^{Tn} is used, as outlined earlier. If the security matures at time T_n, the price expression is simpler

$$V(t) = p(t, T_n)E^{Q^{T_n}}[N(T_n)|\mathcal{F}_t].$$

In that case it suffices to simulate the evolution of the forward rates that determine the payoff of the security.

The Spot LIBOR Martingale Measure

The spot martingale measure Q, which we defined before, is associated with the use of a bank account earning the continuously compounded short rate as the numeraire. However, the LMM does not at all involve the short rate so the traditional spot martingale measure does not make sense in this context. The LIBOR market counterpart is a roll over strategy in the shortest zero-coupon bonds. To be more precise, the strategy is initiated at time T_0 by an investment of one dollar in the zero-coupon bond maturing at time T_1, which allows for the purchase of $1/p(T_0, T_1)$ units of the bond. At time T_1 the payoff of $1/p(T_0, T_1)$ dollars is invested in the zero-coupon bond maturing at time T_2, etc. Let us define

$$I(t) = min\{i \in \{1, 2, \ldots, n\} : T_i \geqslant t\}$$

so that $T_{I(t)}$ denotes the next payment date after time t. In particular, $I(T_i) = i$ so that $T_{I(T_i)} = T_i$. At any time $t \geqslant T_0$ the strategy consists of holding

$$M(t) = \frac{1}{p(T_0, T_1)} \frac{1}{p(T_1, T_2)} \cdots \frac{1}{p(T_{I(t)-1}, T_{I(t)})}$$

units of the zero-coupon bond maturing at time $T_{I(t)}$. The value of this position is

$$\begin{aligned} A^*(t) = p(t, T_{I(t)})M(t) &= p(t, T_{I(t)}) \prod_{j=0}^{I(t)-1} \frac{1}{p(T_j, T_{j+1})}, \\ &= p(t, T_{I(t)}) \prod_{j=0}^{I(t)-1} [1 + \delta L(T_j, T_j, T_{j+1})], \end{aligned}$$

where the last equality follows from

$$L(t, T_i - \delta, T_i) = \frac{1}{\delta} \left(\frac{p(t, T_i - \delta)}{p(t, T_i)} - 1 \right).$$

Since $A^*(t)$ is positive, it is a valid numeraire. The corresponding martingale measure is called the **spot LIBOR martingale measure** and is denoted by Q^* .

Let us look at a security with a single payment at a time $T \in [T_0, T_n]$. The payoff $N(T)$ may depend on the values of all the modelled forward rates at and before time T. Let us by $V(t)$ denote the dollar value of this asset at time t. From the definition of the spot LIBOR martingale measure Q^* it follows that

$$\frac{V(t)}{A^*(t)} = E^{Q^*} \left[\frac{N(T)}{A^*(T)} \middle| \mathcal{F}_t \right]$$

and hence

$$V(t) = E^{Q^*} \left[\frac{A^*(t)}{A^*(T)} N(T) \middle| \mathcal{F}_t \right].$$

From the calculation

$$\frac{A^*(t)}{A^*(T)} = \frac{p(t, T_{I(t)})}{p(T, T_{I(T)})} \frac{\prod_{j=0}^{I(t)-1} [1 + \delta L(T_j, T_j, T_{j+1})]}{\prod_{j=0}^{I(T)-1} [1 + \delta L(T_j, T_j, T_{j+1})]}$$

$$= \frac{p(t, T_{I(t)})}{p(T, T_{I(T)})} \prod_{j=I(t)}^{I(T)-1} [1 + \delta L(T_j, T_j, T_{j+1})]^{-1}$$

we get that the price can be rewritten as

$$V(t) = p(t, T_{I(t)}) E^{Q^*} \left[\frac{N(T)}{p(T, T_{I(T)})} \prod_{j=I(t)}^{I(T)-1} [1 + \delta L(T_j, T_j, T_{j+1})]^{-1} \middle| \mathcal{F}_t \right].$$

In particular, if T is one of the dates in the tenor structure, say $T = T_k$, we get

$$V(t) = p(t, T_{I(t)}) E^{Q^*} \left[N(T_k) \prod_{j=I(t)}^{I(T)-1} [1 + \delta L(T_j, T_j, T_{j+1})]^{-1} \middle| \mathcal{F}_t \right]$$

since $I(T_k) = k$ and $p(T_k, T_{I(T_k)}) = p(T_k, T_k) = 1$.

In order to compute (typically by simulation) the expected value on the right-hand side, we need to know the evolution of the forward rates $L(t, T_j, T_{j+1})$ under the spot LIBOR martingale measure Q^* . It can be shown that the process z^* defined by

$$dz^*(t) = dz^{T_i}(t) - \left[\sigma^{T_{I(t)}}(t) - \sigma^{T_i}(t)\right]$$

is a standard Brownian motion under the probability measure Q^*. As usual, $\sigma^T(t)$ denotes the volatility of the zero-coupon bond maturing at time T. Repeated use of the previous theorem yields

$$\sigma^{T_{I(t)}}(t) - \sigma^{T_i}(t) = \sum_{j=I(t)}^{i-1} \frac{\delta\beta \left(t, T_j, T_{j+1}, L(t, T_k, \delta)|_{T_k \geqslant t}\right)}{1 + \delta L(t, T_j, T_{j+1})}$$

so that

$$dz^*(t) = dz(t, T_i) - \sum_{j=I(t)}^{i-1} \frac{\delta\beta \left(t, T_j, T_{j+1}, L(t, T_k, \delta)|_{T_k \geqslant t}\right)}{1 + \delta L(t, T_j, T_{j+1})} dt.$$

Substituting this relation into

$$dL(t, T_i - \delta, T_i) = \beta \left(t, T_i - \delta, T_i, L(t, T_j, \delta)|_{T_j \geq t}\right) dz(t, T_i), \quad i = 1 \ldots, n.$$

we can rewrite the dynamics of the forward rates under the spot LIBOR martingale measure as

$$
\begin{aligned}
dL(t, T_i - \delta, T_i) &= \beta\left(t, T_i - \delta, T_i, L(t, T_k, \delta)|_{T_k \geqslant t}\right) dz(t, T_i) \\
&= \beta\left(t, T_i - \delta, T_i, L(t, T_k, \delta)|_{T_k \geqslant t}\right) \\
&\quad \times \left[dz^*(t) - \sum_{j=I(t)}^{i-1} \frac{\delta\beta\left(t, T_j, T_{j+1}, L(t, T_k, \delta)|_{T_k \geqslant t}\right)}{1 + \delta L(t, T_j, T_{j+1})} dt \right], \\
&= - \sum_{j=I(t)}^{i-1} \frac{\delta\beta\left(t, T_i - \delta, T_i, L(t, T_k, \delta)|_{T_k \geqslant t}\right) \beta\left(t, T_j, T_{j+1}, L(t, T_k, \delta)|_{T_k \geqslant t}\right)}{1 + \delta L(t, T_j, T_{j+1})} dt \\
&\quad + \beta\left(t, T_i - \delta, T_i, L(t, T_k, \delta)|_{T_k \geqslant t}\right) dz^*(t).
\end{aligned}
$$

Note that the drift in the forward rates under the spot LIBOR martingale measure follows from the specification of the volatility function β and the current forward rates. The relation between the drift and the volatility is the market model counterpart to the drift restriction of the HJM models.

23.1.2.4 Consistent Pricing

As indicated earlier, the model can be used for the pricing of all securities that only have payment dates in the set $\{T_1, T_2, \dots, T_n\}$, and where the size of the payment only depends on the modelled forward rates and no other random variables. This is true for caps and floors on δ-period interest rates of different maturities where the price can be computed as before. The model can also be used for the pricing of swaptions that expire on one of the dates $T_0, T_1, \dots, T_{n-1}$, and where the underlying swap has payment dates in the set $\{T_1, \dots, T_n\}$ and is based on the δ-period interest rate. For European swaptions the price can be written as

$$
V(t) = p(t, T_{I(t)}) E^{Q^*} \left[N(T_k) \prod_{j=I(t)}^{I(T)-1} [1 + \delta L(T_j, T_j, T_{j+1})]^{-1} | \mathcal{F}_t \right].
$$

For Bermuda swaptions that can be exercised at a subset of the swap payment dates $\{T_1, \dots, T_n\}$, one must maximize the right-hand side over all feasible exercise strategies. See Andersen (2000) for details and a description of a relatively simple Monte Carlo-based method for the approximation of Bermuda swaption prices.

The LMM is built on assumptions about the forward rates over the time intervals $[T_0, T_1], [T_1, T_2], \ldots, [T_{n-1}, T_n]$. However, these forward rates determine the forward rates for periods that are obtained by connecting succeeding intervals. For example, the forward rate over the period $[T_0, T_2]$ is uniquely determined by the forward rates for the periods $[T_0, T_1]$ and $[T_1, T_2]$ since

$$\begin{aligned} L(t, T_0, T_2) &= \frac{1}{T_2 - T_0}\left(\frac{p(t, T_0)}{p(t, T_2)} - 1\right) = \frac{1}{T_2 - T_0}\left(\frac{p(t, T_0)}{p(t, T_1)}\frac{p(t, T_1)}{p(t, T_2)} - 1\right), \\ &= \frac{1}{2\delta}([1 + \delta L(t, T_0, T_1)][1 + \delta L(t, T_1, T_2)] - 1), \end{aligned}$$

where $\delta = T_1 - T_0 = T_2 - T_1$ as usual. Therefore, the distributions of the forward rates $L(t, T_0, T_1)$ and $L(t, T_1, T_2)$ implied by the LMM determine the distribution of the forward rate $L(t, T_0, T_2)$. A LMM based on 3-month interest rates can hence also be used for the pricing of contracts that depend on 6-month interest rates, as long as the payment dates for these contracts are in the set $\{T_0, T_1, \ldots, T_n\}$. More generally, in the construction of a model, one is only allowed to make exogenous assumptions about the evolution of forward rates for non-overlapping periods.

23.1.3 The Lognormal LIBOR Market Model

23.1.3.1 Black's Model

The standard model for valuing OTC interest rate options, caps, floors and European swaptions is the Black model. The Black model is used by traders in the market to price these derivatives, and as will be seen later on, the analytical Black formulas will play a key role when calibrating the LMM.

The basic assumptions under the Black model are the following:

- The underlying forward rate or swap rate is a lognormally distributed stochastic variable.
- The volatility of the underlying is constant.
- Prices are arbitrage free.
- There is continuous trading in all instruments.

In Black's world we denote the forward/futures price with expiry T on an underlying with expiry T^* as $\Phi(T, T^*)$. The price is lognormally distributed with standard deviation $\sigma\sqrt{T-t}$. It is further assumed that on expiry, the expected futures price is equal to the current futures price

$$E^Q[\Phi(T, T^*)|\mathcal{F}_t] = \Phi(t, T^*).$$

For a European call option in Black's model we have

$$C_t = e^{-r(T-t)}\left\{\Phi(t, T^*) \cdot N\left(d_1\left[t, \Phi(t, T^*)\right]\right) - K \cdot N\left(d_2\left[t, \Phi(t, T^*)\right]\right)\right\},$$

where

$$d_2 = \frac{\ln(\Phi(t, T^*)/K) - (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

23.1.3.2 Bond Options

The unique no-arbitrage value at time t of a forward contract with delivery at time T of a zero-coupon bond maturing at time S at delivery price K is given by

$$V(t, T, S) = p(t, S) - K \cdot p(t, T).$$

The unique no-arbitrage forward price on the zero-coupon bond is

$$F(t, T, S) = \frac{p(t, S)}{p(t, T)}.$$

Next, we consider a forward contract on a coupon bond where we assumed to yield payments at $T_1 < T_2 < \ldots < T_n$ in time where $T < T_n$. We denote the coupon payments as $c_i, i = 1, 2, \ldots, n$. At time t the value of the bond is therefore given by

$$P(t) = \sum_{T_i > t} c_i p(t, T_i).$$

Let $V^{cpn}(t,T)$ denote the time t value of this forward contract. Then we have a no-arbitrage value if the forward given by

$$V^{cpn}(t,\ T) = \sum_{T_i>t} c_i p(t,\ T_i) - K \cdot p(t,\ T) = P(t) - \sum_{t<T_i<T} c_i p(t,\ T_i) - K \cdot p(t,\ T).$$

The no-arbitrage forward price is given by

$$\begin{aligned} F^{cpn}(t,\ T) &= \frac{\sum_{T_i>T} c_i p(t,\ T_i)}{p(t,\ T)} = \frac{P(t) - \sum_{t<T_i<T} c_i p(t,\ T_i)}{p(t,\ T)} \\ &= \sum_{T_i>T} c_i F(t,\ T,\ T_i). \end{aligned}$$

Consider between time t and delivery time T, the two portfolios.

1. A forward contract, K zero-coupon bonds maturing at T and for each T_i with $t < T_i < T, c_i$ zero-coupon bonds maturing at T_i
2. The underlying coupon bond.

These portfolios have exactly the same payments. At time T, the first portfolio equals $P(T) - K + K = P(T)$, which is identical to the value of the second portfolio. Therefore the absence of arbitrage implies

$$V^{cpn}(t,T) + K \cdot p(t,T) + \sum_{t<T_i<T} c_i \cdot p(t,T_i) = P(t).$$

The expected payoff of the forward contract is then given by

$$\begin{aligned} E^Q\left[\max(p(T) - K, 0) \,|\, \mathcal{F}_t\right] = F^{cpn}(t,T) \cdot N\left(d_1\left[t, F^{cpn}t, T\right]\right) \\ - K \cdot N\left(d_2\left[t, F^{cpn}(t,T)\right]\right) \end{aligned}$$

where

$$d_2 = \frac{\ln(F^{cpn}(t,T)/K) - (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

If we multiply the expected payoff with the relevant discount factor (e.g. the zero-coupon bond price $p(t,T)$), we get **Black's formula for**

a European call option on a bond

$$C^{cpn}(t,T,K) = p(t,T)\begin{Bmatrix} F^{cpn}(t,T)\cdot N\left(d_1\left[t,F^{cpn}(t,T)\right]\right) \\ -K\cdot N\left(d_2\left[t,F^{cpn}(t,T)\right]\right) \end{Bmatrix},$$
$$= \left(P(t) - \sum_{t<T_i<T} c_i\cdot p(t,T_i)\right) N\left(d_1\left[t,F^{cpn}(t,T)\right]\right)$$
$$-K\cdot p(t,T)N\left(d_2\left[t,F^{cpn}(t,T)\right]\right).$$

Similarly, for a put option we have

$$P^{cpn}(t,T,K) = K\cdot p(t,T)N\left(-d_2\left[t,F^{cpn}(t,T)\right]\right)$$
$$-\left(P(t) - \sum_{t<T_i<T} c_i\cdot p(t,T_i)\right) N\left(-d_1\left[t,F^{cpn}(t,T)\right]\right).$$

23.1.3.3 Caps and Floors

For a caplet, with a payoff given by

$$C^i_{T_i} = N\delta\max\{L(T_i, T_i-\delta) - K, 0\}$$

we obtain the Black's price as

$$C^i(t) = N\delta p(t,T_i)\begin{Bmatrix} L(t,T_i-\delta,T_i)\cdot N\left(d_1^i\left[t,L(t,T_i-\delta,T_i)\right]\right) \\ -K\cdot N\left(d_2^i[t,L(t,T_i-\delta,T_i)]\right) \end{Bmatrix},$$

where $t < T_i < \delta$ and

$$d_2^i = \frac{\ln(L(t,T_i-\delta,T_i)/K) - (\sigma_i^2/2)(T_i-\delta-t)}{\sigma\sqrt{T_i-\delta-t}} = d_1 - \sigma_i\sqrt{T_i-\delta-t}.$$

We have assumed that the forward rates $F(t,T_i-\delta,T_i)$ in a risk-free world follow the process

$$dL(t,T_i-\delta,T_i) = \sigma_i L(t,T_i-\delta,T_i)dV(t).$$

The price of the cap and the floor is given by

$$C(t) = N\delta \sum_{i=1}^{n} p(t, T_i) \left\{ \begin{array}{l} L(t, T_i - \delta, T_i) \cdot N\left(d_1^i\left[t, L(t, T_i - \delta, T_i)\right]\right) \\ -K \cdot N\left(d_2^i\left[t, L(t, T_i - \delta, T_i)\right]\right) \end{array} \right\},$$

$$F(t) = N\delta \sum_{i=1}^{n} p(t, T_i) \left\{ \begin{array}{l} K \cdot N\left(-d_2^i\left[t, L(t, T_i - \delta, T_i)\right]\right) \\ -L(t, T_i - \delta, T_i) \cdot N\left(-d_1^i\left[t, L(t, T_i - \delta, T_i)\right]\right) \end{array} \right\}.$$

23.1.3.4 LMM Model Description

The traditional derivation of Black's formula is based on some inappropriate assumptions.

First, the assumed lognormality of bond prices and interest rates is doubtful. For several reasons the price of a bond cannot follow a geometric Brownian motion throughout its life. We know that the price converges to the terminal payment of the bond as the maturity date approaches. Furthermore, the bond price is limited from above by the sum of the future bond payments under the appropriate assumption that all forward rates are non-negative. When the bond price approaches its upper limit or the maturity date approaches, the volatility of the bond price has to go to zero. The volatility of the bond price will therefore depend on both the level of the price and the time to maturity. The lognormality assumption can at most be a locally relevant approximation to the true distribution. In addition, the forward price and the futures price on a bond are not necessarily equal when the interest rate uncertainty is taken into account. It is less clear whether it is reasonable to assume that future interest rates are lognormally distributed, and that the expected changes in the forward rates and the forward swap rates are zero in a risk-neutral world.

Second, the multiplication of the current discount factor and the risk-neutral expectation of the payoff do not lead to the correct price. In fact, this is true only if we take the expectation under the appropriate forward martingale measure instead of the risk-neutral measure.

Third, simultaneous applications of Black's formula to different derivative securities are mutually inconsistent. If, for example, we apply Black's formula to the pricing of a European option on zero-coupon bond, we must assume that the price of the zero-coupon bond is lognormally distributed. If we also apply Black's formula for the pricing of a European option on a coupon bond, we must assume that the price of the coupon bond is lognormally distributed. Since the price of the coupon bond is a weighted average of the prices of zero-coupon bonds and a sum of lognormally distributed random variables is not lognormally distributed, these assumptions are inconsistent. Similarly, the swap rate is a linear combination of forward rates. When Black's formula is applied for the pricing of caplets, it is implicitly assumed that the relevant forward rates are lognormally distributed. Then the swap rate will not be lognormally distributed, so that it is inconsistent to use Black's formula for swaptions also. Furthermore, lognormality assumptions for both interest rates and bond prices are inconsistent.

Several research papers suggest other models for bond option pricing that are also based on specific assumptions on the evolution of the price of the underlying bond. The most prominent examples are Ball and Torous (1983) and Schaefer and Schwartz (1987). A critical analysis of such models can be seen in Rady and Sandmann (1994). A problem in applying these models is that the assumptions about the price dynamics for different bonds may be inconsistent, and hence the option pricing formula obtained in the model will only be valid for options on one particular bond.

To ensure consistent pricing of different fixed income securities we must model the evolution of the entire term structure of interest rates. In many of the consistent term structure models we shall discuss in the following sections, we will obtain relatively simple and internally consistent pricing formulas for many of the popular fixed income securities. As we shall see in this section, it is in fact possible to construct consistent term structure models in which Black's formula is the correct pricing formula for some securities, but, even in those models, applications of Black's formula for different classes of securities are inconsistent.

The lognormal LMM provides a more reasonable framework in which the Black cap formula is valid. The model was originally developed by Miltersen, Sandmann, and Sondermann (1997), while Brace, Gatarek, and Musiela (1997) sorted out some technical details and introduced an explicit, but approximate, expression for the prices of European swaptions in the lognormal LMM. Whereas Miltersen, Sandmann, and Sondermann derive the cap price formula using PDEs, we will follow the approach taken by Brace, Gatarek, and Musiela and use the forward martingale measure technique, since this simplifies the analysis considerably.

In the development of Black's cap pricing formula, we assumed among other things that the forward rate $L(t, T_i - \delta, T_i)$ was a martingale under the spot martingale measure Q and that the future value $L(T_i - \delta, T_i - \delta, T_i)$ was lognormally distributed under Q. However, as was seen in the previous theorem this forward rate is a martingale under the T_i-forward martingale measure Q^{Ti} and will therefore not be a martingale under the Q-measure. (Remember: an equivalent change of measure corresponds to changing the drift rate.) Looking at the general cap pricing formula

$$C(t) = N\delta \sum_{i=1}^{n} p(t, T_i) E_t^{QTi} \left[max\{L(T_i) - \delta, T_i - \delta, T_i - K, 0\} \right] \quad t < T_0$$

it is clear that we can obtain a pricing formula of the same form as Black's formula by assuming that $L(T_i - \delta, T_i - \delta, T_i)$ is lognormally distributed under the T_i-forward martingale measure Q^{Ti}. This is exactly the assumption of the lognormal LMM

$$dL(t, T_i - \delta, T_i) = L(t, T_i - \delta, T_i)\gamma\,(t, T_i - \delta, T_i)\,dz(t, T_i), \quad i = 1, 2 \ldots, n.,$$

where $\gamma(t, T_i\delta, T_i)$ is a bounded, deterministic function. Here we assume that the relevant forward rates are only affected by one Brownian motion, but below we shall briefly consider multi-factor lognormal LMMs.

A familiar application of Itô's lemma implies that

$$d\left[\ln L(t, T_i - \delta, T_i)\right] = -\frac{1}{2}\left(\gamma\,(t, T_i - \delta, T_i)\right)^2 dt + \gamma\,(t, T_i - \delta, T_i)\,dz(t, T_i)$$

from which we see that

$$\ln L(T_i - \delta, T_i - \delta, T_i) = \ln L(t, T_i - \delta, T_i) - \frac{1}{2}\int_t^{T_i-\delta} \left(\gamma\,(u, T_i - \delta, T_i)\right)^2 du$$
$$+ \int_t^{T_i-\delta} \gamma\,(u, T_i - \delta, T_i)\; dz(u, T_i).$$

Because γ is a deterministic function, it follows that

$$\int_t^{T_i-\delta} \gamma\,(u, T_i - \delta, T_i)\,dz(u, T_i) \sim N\left[0, \int_t^{T_i-\delta} \left(\gamma\,(u, T_i - \delta, T_i)\right)^2 du\right]$$

under the T_i-forward martingale measure. Hence,

$$\ln L(T_i-\delta, T_i-\delta, T_i) = N\left[\begin{array}{l} \ln L(t, T_i-\delta, T_i) - \frac{1}{2}\int\limits_t^{T_i-\delta} (\gamma\,(u, T_i-\delta, T_i))^2\, du, \\ \int\limits_t^{T_i-\delta} (\gamma\,(u, T_i-\delta, T_i))^2\, du \end{array}\right]$$

so that $T_i-\delta$ is lognormally distributed under Q^{Ti}. The following result should not now come as a surprise.

Theorem 23.3. *Under the assumption*

$$dL(t, T_i-\delta, T_i) = L(t, T_i-\delta, T_i)\gamma\,(t, T_i-\delta, T_i)\, dz(t, T_i), \quad i = 1, 2 \ldots, n.,$$

the price of the caplet with payment date T_i at any time $t < T_i-\delta$ is given by

$$C^i(t) = N\delta p(t, T_i)\left\{L(t, T_i-\delta, T_i)\cdot N\left(d_1^i\right) - K\cdot N\left(d_2^i\right)\right\},$$

where

$$\begin{cases} d_1^i = \dfrac{\ln(L(t, T_i-\delta, T_i)/K)}{v_L(t, T_i-\delta, T_i)} + \dfrac{1}{2}v_L(t, T_i-\delta, T_i) \\ d_2^i = d_1^i - v_L(t, T_i-\delta, T_i) \end{cases}$$

and

$$v_L(t, T_i-\delta, T_i) = \sqrt{\int\limits_t^{T_i-\delta} (\gamma\,(u, T_i-\delta, T_i))^2\, du}.$$

Note that $v_L(t, T_i-\delta, T_i)^2$ is the variance of $\ln[L(T_i-\delta, T_i-\delta, T_i)]$ under the T^i-forward martingale measure given the information available at time t. The previous caplet price is identical to Black's formula if we insert

$$\sigma_i = \frac{v_L(t, T_i-\delta, T_i)}{\sqrt{T_i-\delta-t}}.$$

An immediate consequence of the previous theorem is the following cap pricing formula in the lognormal one-factor LMM.

Theorem 23.4. *Under the assumption*

$$dL(t, T_i - \delta, T_i) = L(t, T_i - \delta, T_i)\gamma\,(t, T_i - \delta, T_i)\, dz(t, T_i), \quad i = 1, 2 \ldots, n.$$

the price of a cap at any time $t < T_0$ is given as

$$C(t) = N\delta \sum_{I=1}^{N} p(t, T_i) \left\{L(t, T_i - \delta, T_i) \cdot N\left(d_1^i\right) - K \cdot N\left(d_2^i\right)\right\},$$

where d_1^i and d_2^i are as above.

For $t \geqslant T_0$ the first upcoming payment of the cap is known and is therefore to be discounted with the relevant discount factor, while the remaining payments are to be valued as shown previously.

Analogously, the price of a floor is

$$F(t) = N\delta \sum_{I=1}^{N} p(t, T_i) \left\{K \cdot N\left(-d_2^i\right) - L(t, T_i - \delta, T_i) \cdot N\left(-d_1^i\right)\right\}.$$

The deterministic function $\gamma(t, T_i - \delta, T_i)$ remains to be specified. We will discuss this matter below.

If the term structure is affected by n exogenous standard Brownian motions, the assumption on $dL(t, T_i - \delta, T_i)$ earlier is replaced by

$$dL(t, T_i - \delta, T_i) = L(t, T_i - \delta, T_i) \sum_{j=1}^{n} \gamma_j\,(t, T_i - \delta, T_i)\, dz_j(t, T_i),$$

where all $\gamma_j(t, T_i - \delta, T_i)$ are bounded and deterministic functions. Again, the cap price is given by the previous formula with the small change that $v_L(t, T_i - \delta, T_i)$ is to be computed as

$$v_L(t, T_i - \delta, T_i) = \sqrt{\sum_{j=1}^{n} \int_t^{T_i - \delta} \left(\gamma_j\,(u, T_i - \delta, T_i)\right)^2 du}.$$

23.1.3.5 Pricing of Other Securities

No exact explicit solution for European swaptions has been found in the lognormal LIBOR market setting. In particular, Black's formula for swaptions is not correct under the assumption on $dL(t, T_{i-p}-\delta, T_i)$. The reason is that when the forward LIBOR rates have volatilities proportional to their level, the volatility of the forward swap rate will not be proportional to the level of the forward swap rate. The swaption price can be approximated by a Monte Carlo simulation, which is often quite time-consuming. Brace, Gatarek, and Musiela (1997) derived the following Black-type approximation to the price of a European payer swaption with expiration date T_0 and exercise rate K under the lognormal LMM assumptions

$$P(t) = N\delta \sum_{I=1}^{N} p(t, T_i) \left\{L(t, T_i - \delta, T_i) \cdot N\left(d_1^i\right) + K \cdot N\left(d_2^i\right)\right\} \quad t < T_0,$$

where d_1 and d_2 are quite complicated expressions involving the variances and covariance of the time T_0 values of the forward rates involved. These variances and covariance are determined by the γ-function. This approximation delivers the price much faster than a Monte Carlo simulation. Brace, Gatarek, and Musiela provide numerical examples in which the price computed using the previous approximation is very close to the correct price (computed using Monte Carlo simulations). Of course, a similar approximation also applies to the European receiver swaption.

Under the assumptions of the lognormal LMM Miltersen, Sandmann, and Sondermann (1997) derived an explicit pricing formula for European options on zero-coupon bonds, but only for options expiring at one of the time points $T_0, T_1, \ldots, T_{n-1}$, and where the underlying zero-coupon bond matures at the following date in this sequence. In other words, the time distance between the maturity of the option and the maturity of the underlying zero-coupon bond must be equal to δ. Representing the exercise price by K, the pricing formula for a European call option is

$$C^i(t, K, T_i-\delta, T_i) = (1-K)\, p(t, T_i) N\left(e_1^i\right) - K \cdot \left[p(t, T_i - \delta) - p(t, T_i)\right] N\left(e_2^i\right),$$

where

$$\begin{cases} e_1^i = \dfrac{1}{v_L(t, T_i - \delta, T_i)} \ln\left(\dfrac{(1-K)p(t, T_i)}{K \cdot \left[p(t, T_i - \delta) - p(t, T_i)\right]}\right) + \dfrac{1}{2} v_L(t, T_i - \delta, T_i) \\ e_2^i = e_1^i - v_L(t, T_i - \delta, T_i) \end{cases}$$

and

$$v_L(t, T_i - \delta, T_i) = \sqrt{\int_t^{T_i-\delta} (\gamma\,(u, T_i - \delta, T_i))^2\,du}$$

or

$$v_L(t, T_i - \delta, T_i) = \sqrt{\sum_{j=1}^{n} \int_t^{T_i-\delta} \left(\gamma_j\,(u, T_i - \delta, T_i)\right)^2\,du.}$$

The price of the corresponding European put option follows from the put-call parity.

23.1.3.6 Further Remarks

De Jong, Driessen, and Pelsser (2001) investigated the extent to which different lognormal LIBOR and Sap market models can explain empirical data consisting of forward LIBOR interest rates, forward swap rates, and prices of caplets and European swaptions. The observations are from the US market in 1995 and 1996. For the lognormal one-factor LMM they find that it is empirically more appropriate to use a γ-function that is exponentially decreasing in the time-to-maturity $T_i - \delta - t$ of the forward rates,

$$\gamma(t, T_i - \delta, T_i) = \gamma e^{-\kappa(T_i-\delta-t)},$$

than to use a constant, $\gamma(t, T_i - \delta, T_i) = \gamma$. This is related to the well-documented mean reversion of interest rates that makes long-term interest rates relatively less volatile than shorter-term interest rates. They also calibrate two similar model specifications perfectly to observed caplet prices, but find that in general the prices of swaptions in these models are further from the market prices than are the prices in the previous time homogeneous models. In all cases the swaption prices computed using one of these lognormal LMM exceed the market prices; that is, the lognormal LMMs overestimate the swaption prices. All their specifications of the lognormal one-factor LMM give a relatively inaccurate description of market data and are rejected by statistical tests. De Jong, Driessen and Pelsser also show that two-factor lognormal LMMs are not significantly better than the one-factor models and conclude that the lognormality assumption is probably inappropriate. Finally, they present similar results for lognormal swap market models and find that these models are even worse than the lognormal LMMs when it comes to fitting the data.

23.1.4 Calibrating the LIBOR Market Model

In this section we describe how to calibrate LMM to market data. A basic assumption is that we have chosen the forward-rate-based LMM. This is the natural approach when pricing caps and floors both not when pricing swaptions.

Let the tenor structure be $0 = T_0 < T_1 < \ldots < T_{n-1} < T_n$ and i an integer ranging over the reset dates of the rates (e.g. $1 \leqslant i \leqslant \quad n$).

We define $\eta(t)$ as the unique index such that $T_{\eta(t)}$ is the next tenor date after t. The (one factor) model is given by the following SDE for the underlying rates (swap or forward)

$$\frac{df_i}{f_i} = \mu_i\left(f(t), t\right) dt + \sigma_i(t) dz(t),$$

where

f_i	=	forward/swap rate at time i
μ_i	=	drift term
σ_i	=	volatility of rate i
$z(t)$	=	is a Wiener process

The solution to this SDE is

$$f_i(T) = f_i(T) \exp\left(\int_0^T \left(\mu_i(u) - \frac{1}{2} s_i^2(u)\right) du + \int_0^T s_i(u)\, dz(u)\right).$$

The drift terms depend on the choice of numeraire and can be determined by applying the assumption of no arbitrage. Suppose we have forward rates as the underlying rates and choose $p(T_0, T_1)$ as the numeraire. Then the drift terms become

$$\mu_i = \sigma_i \sum_{k=1}^{i} \frac{\sigma_k f_k (T_{k+1} - T_k) \rho_{ik}}{1 + f_k (T_{k+1} - T_k)}.$$

Determining the time-dependent forward rate volatilities is equivalent to calibrating the model. How the calibration is performed is explained in a section below.

Although it is not necessary it will always be assumed here that the (instantaneous) volatilities σ_i of the rates are deterministic (not stochastic) functions of time.

A one-factor model means that all the forward rates are perfectly instantaneously correlated. In this case, a single Wiener process is sufficient to evolve the rates. This is not often a reasonable assumption, and eliminates one of the advantages of employing the LMM. An m-factor model is one where m-independent Wiener processes are used to evolve the rates. In this case the equation becomes

$$\frac{df_i}{f_i} = \mu_i\left(f(t), t\right) dt + \sum_{k=1}^{m} \sigma_{i,k}(t) dz_k(t), 1 \leqslant k \leqslant m.$$

This is solved for

$$f_i(T) = f_i(T) \exp\left(\int_0^T \left[\mu_i(u) - \frac{1}{2}\sigma_i^2(u)\right] du + \int_0^T \sum_{k=1}^{m} \sigma_{i,k}(u)\, dz_k(u)\right).$$

The loadings $\sigma_{i.k}(u)$ can be interpreted as the sensitivities at time u of the ith forward rate to the kth shock provided by the Wiener process z_k. They must satisfy

$$\sigma_i^2(t) = \sum_{k=1}^{m} \sigma_{i,k}^2(t).$$

All that remains before we can start to analyse how the rates will evolve is to specify the instantaneous volatilities σ_i's and their loadings $\sigma_{i,k}$'s. This can be done in many different ways. One choice is presented below.

23.1.4.1 Volatility Calibration

Volatility calibration deals with the determination of the σ_i's (the instantaneous volatility of the forward rate with reset at T_i). This is done differently for caps and swaptions. Since the cap volatility calibration is a first step in the swaption volatility calibration, we start by describing the cap volatility calibration.

23.1.4.2 Cap Volatility Calibration

In the Black model for caplets it is assumed that the underlying rate has a lognormal distribution with variance equal to $T \cdot \sigma^2_{Black}$ where T is the reset date of the underlying forward rate. In the LMM, this lognormal assumption is also made for each rate separately. The instantaneous volatility at reset for each rate is related to the aforementioned expression in the following way:

$$\int_0^{T_i} \sigma_i^2(t)dt = T_i\sigma^2_{Black}.$$

In other words, the instantaneous volatility at reset for each underlying rate is equal to the implied Black volatility, which can be read from the market. Although not necessary, we make the assumption that the σ_i's are deterministic functions of time only. There are (infinitely) many solutions to these equations, and our goal is to pick one that fits our needs. We follow the approach suggested by Rebonato (2002). Let

$$\sigma(t) = (a + b\,t)\,e^{-ct} + d$$

and

$$\sigma_i\,(t) = k_i\sigma\,(T_i - t)\,.$$

This form is flexible and can by varying the constants a, b, c and d take many different shapes. The constants k_i are rate specific and are used to assure that the caplet prices are exactly recovered. How the k_i's are set should become clear below.

1 Find values on the constants a, b, c and d such that the forward rates

$$\frac{df_i}{f_i} = \mu_i\,(f(t), t)\,dt + \sigma_i(t)dz(t)$$

fit as close as possible. We use both the Broyden-Fletcher-Goldfarb-Shannon dimensional variable metric method and the Levenberg-Marquardt method[2] in parallel to solve this problem and pick the best solution.

[2] The Broyden-Fletcher-Goldfarb-Shannon dimensional variable metric sometimes gives adjusting factors with the property that the first ones (i.e. $i = 1, 2, \ldots$) have a higher deviation from unity than the rest. The Levenberg-Marquardt method yields adjusting factors with a more constant deviation from unity.

2 Set values on the k_i's by computing

$$k_i = \sqrt{\frac{\sigma_{Black}^2 T_i}{\int_0^{T_i} \sigma_i^2(t)\,dt}}$$

The second step ensures equality for the forward rates; that is, the instantaneous volatility and the implied Black volatility are set to be equal at each reset. This completes the volatility calibration for caps.

Before we end the cap calibration section, we shall discuss the issue of deciding the implied Black volatility, for example, σ_{Black}, in the case when we are calibrating to an exotic cap with path-dependent strikes. For example, consider a ratchet cap where each caplet has a strike given by $K_i = f_{i-1} + X$ where K_i and X denote the strike for the i:th caplet and a spread respectively. Recall that σ_{Black} of a caplet is a function of the maturity of the caplet and the strike.

The fact that σ_{Black} depends on the strike gives us some problems. To see this, note that in order to get σ_{Black} of a caplet we must know its strike. But if the strike is path-dependent, as it is in a ratchet cap, we cannot know the strike beforehand. To solve this problem the following approach is taken:

1. Make good guesses on the start values of the strikes.
2. Get the σ_{Black} of the caplets by using the strikes from the first step.
3. Evolve a small sample of the rates (e.g. 1024 Monte-Carlo simulations). Then compute the average rates for each caplet.
4. Compute new strikes by using the average rates.
5. Go to the second step with the newly computed strikes and repeat until some desirable convergence criterion is achieved.

Empirical results show that this scheme always (although not proven) converges and gives good estimates on the strikes. Let us now turn to the volatility calibration for swaptions.

23.1.4.3 swaption Volatility Calibration

We concentrate on the volatility calibration of a Bermudan swaption. The volatility calibration of a European swaption is a special case of this discussion. The basic set-up of the calibration is that we want to recover the implied Black volatilities of a set of co-terminal swap rates. Let $T_1, T_2, \ldots, T_n$ be the expiry dates of the co-terminal swaptions and SR_i denote the i:th swap rate (e.g. the swap rate for the swaption with expiry T_i). Recall that

$$SR_i = \sum_{j=i}^{n} w_j f_j(t),$$

where

$$w_j = \frac{B_{j+1}(T_{j+1} - T_j)}{\sum\limits_{k=i}^{n} B_{k+1}(T_{k+1} - T_k)}.$$

Furthermore, denote with $\sigma_{i\times n}(t)$ the instantaneous volatility of SR_i, for example, the instantaneous volatility of the swap rate with expiry at T_i and maturity at T_n. It can be shown that

$$\sigma_{i\times n}(t)^2 = \sum_{j=i}^{n}\sum_{k=i}^{n} \zeta_{jk}(t)\rho_{jk}\sigma_j(t)\sigma_k(t),$$

where

$$\zeta_{jk}(t) = \frac{w_k(t)f_k(t)w_j(t)f_j(t)}{\left(\sum\limits_{m=i}^{n} w_m(t)f_m(t)\right)^2}.$$

The last equation is an approximation. One of the main results is that

$$\sigma_{i\times n}(t)^2 \approx \sum_{j=i}^{n}\sum_{k=i}^{n} \zeta_{jk}(T_0)\rho_{jk}\sigma_j(t)\sigma_k(t).$$

This is the key point in the calibration and one ought to understand how this greatly simplifies our task (it is recommended to read Rebonato (2002), "Modern Pricing of Interest-Rate Derivatives", Princeton University Press for a discussion regarding this issue). In order to recover the swaption prices, we must have that

$$\sigma_{i\times n}^{Black}(t)^2 T_i = \int_0^{T_i} \sigma_{i\times n}(u)^2 du.$$

Our approach to achieving this is the following.

Find values of the constants a, b, c and d such that the equations for the forward rates fit as close as possible. We use both the Broyden-Fletcher-Goldfarb-Shannon dimensional variable metric method and the Levenberg-Marquardt method (for a description of these methods, refer to Press et. al, (2002) "Numerical Recipes in C") in parallel to solve this problem and pick the best solution. Note that this is the same first step as in the cap calibration.

Consider the last swaption in the set of co-terminal swaptions (e.g. with expiry T_n). This is simply a floating rate exchanged for a fixed rate (i.e. a caplet). Set

$$k_n = \sqrt{\frac{\sigma_{n\times n}^{Black}(t)^2 T_n}{\int_0^{T_n} \sigma_{n\times n}(u)^2 du}}$$

which makes

$$\sigma_i(t) = k_j \sigma(T_i - t)$$

for $i = n$ an equality.

Move a step back and consider the swaption with expiry T_{n-1}. Our goal is to set the value for k_{n-1} such that

$$\sigma_{n-1\times n}^{Black}(t)^2 T_{n-1} = \int_0^{T_{n-1}} \sigma_{n-1\times n}(u)^2 du.$$

Since we have already solved for k_n and are using the approximation of

$$\int_0^{T_i} \sigma_i^2(t)dt = T_i \sigma_{Black}^2,$$

k_{n-1} is the only unknown variable. Since k_{n-1} appears in squared form, we need to solve a second-degree equation. Although straightforward, the algebra becomes quite messy.

Next we continue to compute the values of the remaining k_i's in a similar fashion as in the previous step. Doing this yields values for the k_i's such that

$$\sigma_i(t) = k_i \sigma(T_i - t)$$

is fulfilled for $1 \leqslant i \leqslant n$.

Our approach to calibration of the instantaneous swap volatility cannot, as far as we know, be found in any published/academic/scientific article. This is a brief motivation why it seems to be a desirable procedure. In all but very rare cases, the first step gives values on a, b, c and d such that the instantaneous volatilities of the forward rates are time homogeneous. A problem might be that the constants a, b, c and d are not designed to recover the swaption prices. However, they should not be too far off since we are considering the same rates, namely the forward rates. Finally, the remaining steps make sure that we recover the swaption prices exactly.

23.1.4.4 Principal Component Analysis

We use principal component analysis to reduce the number of driving factors needed when valuing plain-vanilla caps, European swaptions, and Bermudan swaptions. In this section we describe how this is done. Note that we do not use principal component analysis for the valuation of path-dependent caps such as ratchet, sticky, momentum, flexi, and chooser. In this case, we use as many factors as there are rates. More information on why we do this can be found in Rebonato (2002) "Modern Pricing of Interest-Rate Derivatives", Princeton University Press.

Consider a cap with n caplets with resets at $T_1, T_2, \ldots, T_n$ respectively. Each caplet has an associated forward rate f_i for $1 \leqslant i \leqslant n$. We describe the principal component analysis in a simple case, namely when the rates $f_1, f_2, \ldots, f_n$ are evolved to the first reset date T_1. The complete picture can then be understood from this discussion. Now suppose we have an m-factor model where $m < n$ and let

$$\mathbf{Cov}_{ij} = \int_0^{T_1} \rho_{ij}\sigma_i(u)\sigma_j(u)du$$

be the $n \times n$ terminal covariance matrix where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$. Note that $\mathbf{Cov}_{ij}$ is symmetric.

Use the Jacobian transformations of a symmetric matrix method to find the n eigenvalues of Cov_{ij} and the corresponding normalized eigenvectors as described by Rebonato (2002, section 11.1). Denote the vector of eigenvalues with $\mathbf{e}_i = [e_1\, e_2\, \ldots e_n]^T$ and the corresponding normalized eigenvectors with $\mathbf{v}_i$. Furthermore, let $\mathbf{v}_{ij} = [\mathbf{v}_1\, \mathbf{v}_2 \ldots \mathbf{v}_n]$. Sort $\mathbf{e}_i$ in decreasing order and change the vectors in $\mathbf{v}_{ij}$ accordingly. Compute

$$\mathbf{B}_{ik} = \left[\sqrt{e_1}\mathbf{v}_1 \sqrt{e_2}\mathbf{v}_2 \ldots \sqrt{e_m}\mathbf{v}_m\right],$$

where $1 \leqslant k \leqslant m$. Compute

$$\mathbf{s}_i = \left[\sqrt{\frac{\mathbf{Cov}_{11}}{\sum_{l=1}^{m} e_l v_{1l}^2}} \sqrt{\frac{\mathbf{Cov}_{22}}{\sum_{l=1}^{m} e_l v_{2l}^2}} \cdots \sqrt{\frac{Cov_{nn}}{\sum_{l=1}^{m} e_l v_{nl}^2}} \right]^T = [\mathbf{s}_1\, \mathbf{s}_2 \ldots \mathbf{s}_n]^T .$$

and

$$\mathbf{B}_{ik}^{'} = \begin{bmatrix} s_1\sqrt{\boldsymbol{e}_1}\boldsymbol{v}_{11}\ \boldsymbol{s}_1\sqrt{\boldsymbol{e}_2}\boldsymbol{v}_{12} \ldots \boldsymbol{s}_1\sqrt{\boldsymbol{e}_m}\boldsymbol{v}_{1m} \\ s_2\sqrt{\boldsymbol{e}_1}\boldsymbol{v}_{21}\ \boldsymbol{s}_2\sqrt{\boldsymbol{e}_2}v_{22} \ldots \boldsymbol{s}_2\sqrt{e_m}\boldsymbol{v}_{2m} \\ \vdots \\ \boldsymbol{s}_n\sqrt{\boldsymbol{e}_1}\boldsymbol{v}_{n1}\ \boldsymbol{s}_n\sqrt{\boldsymbol{e}_2}v_{n2} \ldots \boldsymbol{s}_n\sqrt{\boldsymbol{e}_m}\boldsymbol{v}_{nm} \end{bmatrix}.$$

Finally, compute the model covariance $\mathbf{Cov}_{ij}^{'} = \mathbf{B}_{ik}^{'} \times \mathbf{B}_{ik}^{'T}$. In particular note that $\mathbf{Cov}_{ii}^{'} = \mathbf{Cov}_{ii}$ for all i's, that is, the variance of each rate is unchanged, and that $\mathbf{B}_{ik}^{'}$ is the principal component matrix. To put this in the context of the section "Libor Market Model", the equation for the drift terms

$$\mu_i = \sigma_i \sum_{k=1}^{i} \frac{\sigma_k f_k(T_{k+1} - T_k)\rho_{ik}}{1 + f_k(T_{k+1} - T_k)}$$

is rewritten as

$$\mu_i = \sum_{l=1}^{i} \frac{f_l(T_{l+1} - T_l)}{1 + f_l(T_{l+1} - T_l)} Cov_{il}^{'}$$

and in equation

$$\frac{df_i}{f_i} = \mu_i(f(t), t)dt + \sum_{k=1}^{m} \sigma_{i,k}(t)dz_k(t), \quad 1 \leq k \leq m$$

$\sigma_{i,k}$ correspond to $\mathbf{B}_{ik}^{'}$.

23.1.4.5 Correlation Matrix

The $n \times n$ correlation matrix

$$\mathbf{P}_{ij} = \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \\ \vdots & & \ddots & \\ \rho_{n1} & \cdots & & \rho_{nn} \end{bmatrix}$$

is user defined. After the user has input $\mathbf{P}_{ij}$, we use the heuristic of applying principal component analysis to it as described in the previous section before the calculations start. In other words, $\mathbf{P}_{ij}$ is exposed to the same transformation as $\mathbf{Cov}_{ij}$ was above.

23.1.5 Evolving the Forward Rates

We use two different approaches to evolve the forward rates – the short step and the long step. The short step approach evolves the "living" forward rates to each reset. To exemplify, suppose that the resets of the rates are $T_1, T_2, \ldots, T_n$. Then $f_1, f_2, \ldots, f_n$ are evolved to T_1, $f_2, \ldots, f_n$ are evolved to T_2, and so on. We do not attempt to describe the technical details of the approach here. However, the implications of the results are that we can evolve a forward rate to an appropriate point in time in one step, without hardly any loss of accuracy when compared to the short step approach.

We use both approaches when evolving the forward rates. In particular, we use the long step when valuing a cap (of any kind). This allows us to evolve each forward rate to its reset date in one step. When valuing a swaption (of any kind) we use the short step for all but the first time sensitive date (i.e. the first exercise date in the case of a Bermudan swaption) – to the first time sensitive date when we use the long step.

23.1.6 Pricing of Bermudan Swaptions

In this section we make a theoretical elaboration of the pricing procedure for a Bermudan swaption. This pricing procedure is the only one we need to describe; a European swaption is, as we will see, a special case of this discussion, and the pricing procedure of a cap follows from its definition.

A Bermudan swaption contract denoted by X-non-call-Y gives the holder the right to enter into a swap at a pre-specified strike rate "K" on a number of exercise opportunities. The first exercise opportunity in this case would be Y years after inception. The swap that can be entered into always has the same terminal maturity date, X. A Bermudan swaption entitling the holder the right to enter into a swap in which they pay the fixed rate is referred to as a *Payer's* otherwise *Receiver's*.

Therefore, as the owner of a Bermudan swaption one is in a position where at each exercise date it is necessary to use one's own judgement to determine if exercise is optimal or not. This makes the pricing of Bermudan swaptions a little more complicated than that of a European swaption. We have implemented Andersen's strategy that can be found in Andersen (1999) "A simple approach to the pricing of Bermudan swaptions in the multi-factor LIBOR Market Model", *The Journal of Computational Finance*, 3(2): 5–32, 1999/2000. Below we describe the approach and how we use it.

When pricing a Bermudan swaptions the most important question is how to determine the free exercise boundary. In other words, given that the world is in a certain state at one of the exercise dates, under what circumstances should one exercise the option?

Let

$S_{s,e}$ = The European payer's swaption maturing at time T_s and with a last cash flow at date T_e.

$S_{s,x,e}$ = The Bermudan swaption with lockout date (first exercise opportunity) T_s, last exercise date T_x and final swap maturity T_e.

The decision whether or not to exercise a Bermudan swaption at a date T_i will in general depend on the state of all forward rates $F_I(T_i)$. To simplify matters somewhat one could make the assumption that the strategy depends only on the intrinsic values of the underlying swap. Let $I(T_i)$, be the indicator function that equals one if exercise is optimal at dates T_i and zero otherwise. It is hence assumed that

$$I(T_i) = f\left(S_{i,e}, H(T_i)\right),$$

where f is a specified Boolean function with a possibly time-dependent parameter H. The relationship

$$I(T_x) = \begin{cases} 1 & S(T_x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

must of course be fulfilled. For the other exercise opportunities the following form of $I(T_i)$ is assumed:

$$I(T_i) = \begin{cases} 1 & S_{i,e}(T_i) > H(T_i) \\ 0 & \text{otherwise} \end{cases}.$$

In this strategy the option is exercised if the intrinsic value of the underlying swap is above the barrier H.

The next step is to determine the value of the function H at the dates $T_s, \ldots, T_x.H(T)$. The function H is characterized as being the function that maximizes the value of the Bermudan swaption.

A brute-force way of determining the values of the function H is hence to solve the multi-dimensional optimization problem. This could be done as follows. Store a Monte Carlo simulation in memory. Simultaneously find values on the function H at the dates $T_s, \ldots, T_x$ such that value of the Bermudan swaption is maximized. This would be an iterative process where in each iteration, first, new values on H at the dates $T_s, \ldots, T_x$ are set, then, the value of the Bermudan swaption is calculated. Whichever choices on H at the dates $T_s, \ldots, T_x$ that give the highest value on the Bermudan swaption are picked. Obviously, this way of finding H is tremendously slow and not applicable in practice. Fortunately, there is an another way of finding H, as proposed by Andersen (1999), which is much more efficient. It can be described as follows:

1. Set $\boldsymbol{H}(T_x) = 0$.
2. Compute an appropriate Monte Carlo simulation of the forward rates and store it in memory.
3. Consider the Bermudan swaption $S_{n-1,x,e}$. At time T_{n-1} the exercise strategy must be the same as for $S_{s,x,e}$ since at this time there is no difference between the options. $\boldsymbol{H}(T_x) = 0$ (ordinary European option) is known and determining the value of $\boldsymbol{H}(T_{n-1})$ is hence a one-dimensional optimization problem. We solve this optimization problem with the Golden Section Search in One Dimension (see section 10.1 of Press et. al, (2002) “Numerical Recipes in C” for a description of this algorithm).
4. Repeat in turn the previous step for $S_{x-2,x,e}, S_{x-3,x,e}, \ldots, S_{s+1,x,e}, S_{s,x,e}$.

When doing this, it is sufficient to store a single Monte Carlo session in memory and to re-use it over and over again. Having determined the exercise boundary in this way another Monte Carlo simulation is run to calculate the price. This Monte Carlo approach to determine the free exercise boundary produces a lower bound on Bermudan swaption prices that can be shown to be very tight for many realistic term structures.

Finally, note that valuing a European swaption is a special case of valuing a Bermudan swaption; a European swaption is a Bermudan swaption with one exercise event, T_x, and it follows that $\boldsymbol{H}(T_x) = 0$.

24

A Model for Exotic Instruments

24.1 Managing Exotics

In the following section, I refer to articles by Patrick Hagan. The adjustors described here are called "Hagan adjustors". This is a method of turning bad prices into good prices,

We will study the need of pricing and trading an exotic derivative, but because of limitations in our pricing systems, we cannot calibrate on the "natural set" of hedging instruments. Instead, we have to calibrate on some other set of vanilla instruments, which provide only a poor cash-flow replication of the exotic. Consequently, our prices are questionable, and if we are bold enough to trade on these prices, our hedges will be unstable, chewing up any profit as bid-ask spread. Here we discuss how to get out of these jams by using "adjusters", a technique for re-expressing the Vega risks of an exotic derivative in terms of its "natural hedging instruments". This helps to prevent unstable hedges and exotic deal mismanagement, and, as a side benefit, leads to significantly better pricing of the exotic. First, let us briefly discuss how we get in these jams.

During normal times, the pricing of fixed income derivatives depends on two key markets. First we have the swap market (or *delta market*), from where we get the yield curve. swap desks maintain current yield curves by continually stripping and re-stripping a set of liquid swaps, futures, and deposit rates throughout the day. This curve determines all current swap rates, FRA rates, forward swap rates, etc. The yield curve also shows how to hedge all interest rate risks by

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_24

buying and selling the same swaps, futures, and deposit rates used in the stripping process.

Second, we have the plain vanilla option market (or *Vega market*) for European swaptions, caps, and floors. Prices of these options are quoted in terms of the volatility σ, which is inserted into Black's 1976 formula to determine the dollar price of the option.

European swaptions are defined by three numbers:

1. the exercise date,
2. the *tenor* (length) and
3. the *strike* (fixed rate) of the swap received upon exercise.

Keeping track of this market requires maintaining a *volatility cube*, which contains the volatilities σ as a function of the three coordinates. However, the vast majority of swaptions are struck *ATM*, i.e. at strikes equalling the current swap rate of the underlying forward swap, so desks normally track this market by maintaining a volatility matrix containing the volatilities of *ATM* swaptions, and a set of auxiliary "smile" matrices showing how much to add/subtract to the volatility for strikes 50 bps, 100 bps, etc., above or below the current swap rate.

Alternatively, some swap desks determine the adjustment by using a smile model, such as the SABR or Heston models. In any case, desks are reasonably confident that they can trade the vanilla instruments at the indicated prices.

Now consider the typical management of an *exotic* interest rate derivative, such as a Bermudan swap or a callable range note. During the nightly mark-to-market, the deal will be priced by

- selecting an interest rate model, such as Hull-White or Black-Karasinski,
- selecting a set of vanilla swaptions and/or caplets as the calibration instruments,
- calibrating the interest rate model so that the model reproduces the market prices of these instruments, either exactly or in a least squares sense, and
- using the calibrated model to find the value of the exotic via finite difference methods, trees, or Monte Carlo.

The exotic's Vega risks will then be obtained by

- bumping each volatility in the matrix (or cube) one at a time,
- re-calibrating the model and re-pricing the exotic derivative for each bump and
- subtracting to obtain the difference in value for the bumped case versus the base (market) case.

This results in a matrix of Vega risks where each cell represents the deal's dollar gain or loss when the volatility of that particular swaption changes. These Vega risks are then hedged by buying or selling enough of each underlying swaption so that the total Vega risks are zero. Of course the desk first adds up the Vega exposure of all deals, and only hedges the net exposure.

Calibration is the only step in this procedure, which incorporates information about marketing the volatilities.

Under the typical nightly procedure *the exotic derivative will only have Vega risks to the set of vanilla swaptions and/or caplets used in calibration.* So regardless of the actual nature of the exotic derivative, the Vega hedges will be trying to mimic the exotic derivative as a linear combination of the calibration instruments. If the calibration instruments are "natural hedging instruments" which are "similar" to the exotic, then the hedges probably provide a faithful representation of the exotic. If the calibration instruments are dissimilar to the exotic, having the wrong expiries, tenors, or strikes, then the Vega hedges will probably be a poor representation of the exotic. This often causes the hedges to be unstable, which gets expensive as bid-ask spread is continually chewed up in re-hedging the exotic.

For example, consider a cancellable 10-year receiver swap struck at 7.50%, where the first call date is in 3 years (10NC3@7.50). Surely the natural hedging instruments for this Bermudan are the diagonal swaptions: the 3y into 7y struck at 7.50%, the 4y into 6y struck at 7.50%, . . ., and the 9y into 1y struck at 7.50%, since a dynamic combination of these instruments should be capable of accurately replicating the exotic. Indeed, if we do not calibrate on these swaptions, then our calibrated model would not produce the correct market prices of these swaptions, and if our prices for the 3y into 7y, the 4y into 6y, . . ., are incorrect, we don't have a prayer of pricing and hedging the callable swap correctly.

When feasible, best practice is to use auto-calibration for managing exotic books. For each exotic derivative on the books, auto-calibration first selects the "natural hedging instruments" of the exotic, usually based on some simple scheme of matching the expiries, tenors, and effective strikes of the exotic. It then re-calibrates the model to match these instruments to their market values, and then values the exotic. Auto-calibration then picks the next deal out of the book, selects a new set of natural hedging instruments, re-calibrates the model, and re-prices the exotic, and so on.

There are a variety of reasons why auto-calibration may not be feasible. If one's interest rate model is too complex, perhaps a several factor affair, one may not have the computational resources to allow frequent calibration. Or if one's calibration software is too "fractious," one may not have the patience to calibrate the model very often. In such cases one would generally calibrate to all swaptions in the volatility matrix in a least squares sense and the calibration would only include *ATM* swaptions. Alternatively, an interest rate model may be more easily calibrated on some instruments than others. For example, a multi-factor Brace-Gatarek-Musiela (BGM) model is much easier to calibrate to caplets than to swaptions. Finally, one's software may not be set up to calibrate on the "natural hedging instruments".

A callable range note provides an example. Consider a regular (non-callable) 10-year range note, which pays a coupon of, say, $1 each day Libor sets between 2.50% and 6.00%. Apart from minor date differences, the range note is equivalent to being long one digital call at 2.50% and short a digital call at 6.00% for each day over the next 10 years. Since digital calls can be written in terms of ordinary calls, a range note is very, very close to being a vanilla instrument, and can be priced exactly from the swaption volatility matrix (or cube). To price a *callable* range note, one would like to calibrate on the underlying daily range notes, for if we don't price the underlying range notes correctly, how could we trust our price for the callable range note? Yet many systems are not set up to calibrate on range notes.

24.1.1 At-The-Money Volatility Matrix

European swaptions are defined by the time-to-exercise (row), and length (column) and fixed rate (strike) of the swap received upon exercise. A volatility matrix (as opposed to a volatility cube) contains the

Table 24.1 A volatility matrix

σ(in %)	3m	1y	2y	3y	...	10y
1m	5.25	12.25	13.50	14.125	...	14.25
3m	7.55	13.00	14.125	14.375	...	14.50
6m	11.44	14.25	14.875	15.00	...	14.75
1y	16.20	16.75	16.375	16.125	...	15.50
2y	19.25	17.75	17.125	17.00	...	15.75
⋮	⋮	⋮		⋮		⋮
10y	14.00	13.50	13.00	12.50	...	11.00

volatilities of *ATM* swaptions, swaptions whose fixed rates are equal to the current forward swap rate of the underlying swap. Linear interpolation is used for the volatilities in between grid points. The 3m column is the caplet column. Such a volalility matrix is shown in Table 24.1.

24.1.2 Migration of Risk

We will now describe a method for moving the Vega risk, either all of it, or as much as possible, to the natural hedging instruments. Suppose we have an exotic derivative v which has $h_1, h_2, \ldots, h_m$ as its natural hedging instruments. For example, for the 10NC3 Bermudan struck at 7.50%, the natural hedging instruments are just the 3y into 7y swaption struck at 7.50%, the 4y into 6y at 7.50%, ..., and the 9y into 1y at 7.50%. Suppose that for "operational reasons", one could not calibrate on $h_1, h_2, \ldots, h_m$, but instead were forced to calibrate on the swaptions and/or caplets $S_1, S_2, \ldots, S_n$. Let these instruments have market volatilities $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Then after calibrating the model, all prices obtained from the model are functions of these volatilities. So let

$$V^{mod} = V^{mod}(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

be the value of the exotic derivative v obtained from the model. Suppose we use the model to price the natural hedging instruments $h_1, h_2, \ldots, h_m$. Let

$$H_k^{\mathrm{mod}}\,(\sigma_1, \sigma_n, \ldots, \sigma_n) \quad k - 1, 2, \ldots, m$$

be the value of these instruments according to the calibrated model. Finally, let

$$H_k^{\mathrm{mar}} \quad k = 1, 2, \ldots, m$$

be the market price of the natural hedging instruments. Let us create an imaginary portfolio consisting of the exotic derivative and its natural hedging instruments,

$$\pi = v - \sum_{k=1}^{m} b_k h_k,$$

where the amounts b_k of the hedging instruments will be selected shortly. Using the calibrated model to price this portfolio yields

$$\Pi - V^{\text{mod}}(\sigma_1, \sigma_n, \ldots, \sigma_n) - \sum_{k=1}^{m} b_k H_k^{\text{mod}}(\sigma_1, \sigma_n, \ldots \sigma_n).$$

According to the calibrated model, this portfolio has the Vega risks

$$\frac{\partial \Pi}{\partial \sigma_j} = \frac{\partial V^{\text{mod}}}{\partial \sigma_j} - \sum_{k=1}^{m} bk \frac{\partial H_k^{\text{mod}}}{\partial \sigma_j}$$

to the calibration instruments.

In the next section we will show how to choose the amounts b_k so as to eliminate the Vega risks, either completely or as completely as possible. For the moment just suppose we have chosen the portfolio weights b_k. We add and subtract this portfolio of natural hedging instruments to write the exotic derivative v as

$$v = \left\{ v - \sum_{k=1}^{m} b_k h_k \right\} + \left\{ \sum_{k=1}^{m} b_k h_k \right\}.$$

We now use the calibrated model to value the instruments in the first set of braces, and use the market prices to evaluate the instruments in the second set of braces. This yields the adjusted price

$$\begin{aligned} V^{adj} &= \left\{ V^{\text{mod}} - \sum_{k=1}^{m} b_k H_k^{\text{mod}} \right\} + \left\{ \sum_{k=1}^{m} b_k H_k^{mar} \right\} \\ &= V^{\text{mod}} + \sum_{k=1}^{m} b_k \left(H_k^{mar} - H_k^{\text{mod}} \right). \end{aligned}$$

This procedure is generally known as "applying an adjuster". The terms inside {} are evaluated using the calibrated model, so they only have Vega risk to the volatilities of the calibration instruments

$\sigma_1, \sigma_2, \ldots, \sigma_n$.With the weights b_k chosen to eliminate these risks as nearly as possible, the adjusted price V^{adj} has little or no Vega risk to the calibration instruments. Instead, the Vega risks of the adjusted price come from the last term,

$$\sum_{k=1}^{m} b_k H_k^{mar}$$

which only contains the market prices of the natural hedging instruments. So, as claimed, the adjuster has moved the Vega risks from the calibration instruments to the natural hedging instruments. In fact, to hedge these risks one must take the opposite position

$$-\sum_{k=1}^{m} b_k h_k$$

in the natural hedging instruments of the exotic. For the 10NC3 Bermudan struck at 7.50%, for example, the resulting hedge is a combination of the 3y into 7y, the 4y into 6y, ..., and the 9y into 1y swaptions, all struck at 7.50%, regardless which set of instruments were used originally to calibrate the model.

The last part of the previous equation gives a different view. It shows the adjusted price as being the model price corrected for the difference between the market price and the model price of the natural hedging instruments.

24.1.3 Choosing the Portfolio Weights

We wish to choose the hedging portfolio weights b_k so as to minimize the model's Vega risks. This is an exercise in linear algebra. Define the matrix $\boldsymbol{M}$ and vectors $\mathbf{U_j}$ by

$$M_{jk} = \frac{\partial H_k^{\text{mod}}}{\partial \sigma_j}, \quad U_j = \frac{\partial \Pi}{\partial \sigma_j}$$

and let $\mathbf{b}$ be the vector of positions $(b_1, b_2, \ldots, b_m)^T$ so that the Vega risks to the calibration instruments are

$$\boldsymbol{U} - \boldsymbol{M}\boldsymbol{b}.$$

There are three cases to consider. First suppose that there are fewer hedging instruments than model calibration instruments. One cannot expect to eliminate n risks with $m < n$ hedging instruments, so one cannot eliminate all the Vega risks in this case. Instead one can minimize the sum of squares of the Vega risks:

$$\mathbf{b} = (\boldsymbol{M}^T \boldsymbol{M})^{-1} \boldsymbol{M}\, \mathbf{U} \quad (\text{if } m < n)$$

If there are exactly as many hedging instruments as calibration instruments, then we can expect to completely eliminate the risk entirely by choosing

$$\mathbf{b} = \boldsymbol{M}^{-1} \boldsymbol{U} \quad (\text{if } m = n)$$

Finally, if there are as many hedging instruments than model calibration instruments, then we can select the smallest hedge which completely eliminates the Vega risks to the calibration instruments. This yields

$$\mathbf{b} = \boldsymbol{M}^T (\boldsymbol{M}\boldsymbol{M}^T)^{-1} \mathbf{U} \quad (\text{if } m > n)$$

24.1.3.1 Examples

Consider once more the cancellable 10-year receiver swap struck at 7.50%, where the first call date is in 3 years. This derivative is normally booked as a straight 10-year swap, with a Bermudan option to enter into the opposite swap. Here we just price the Bermudan option, the option to enter a payer swaption at 7.50% on any coupon date starting on the third anniversary of the deal. For the purposes of this example, we assume a flat 5% yield curve, and use the Hull-White model with the USD volatility matrix from March 1999.

Clearly the natural hedging instruments are the 3y into 7y swaption struck at 7.50%, the 4y into 6y swaption at 7.50%, . . ., and the 9y into 1y swaption at 7.50%. Suppose we calibrate the Hull-White model to these "natural hedging instruments" and then use the calibrated model to price the Bermudan. This leads to a price of

$$V = 200.18 \text{ bps.}$$

This represents the best price available within the one factor Hull-White framework. Suppose we calibrate to the same "diagonal" swaptions as before, but instead of calibrating to swaptions struck at 7.50%, we calibrate to swaptions struck *ATM*, at 5.00%. This yields a much lower price,

$$V^{mod} = 163.31 \text{ bps.}$$

If we add in the adjustor, we obtain the price

$$V^{\text{mod}} + \sum_{k=1}^{m} b_k \left(H_k^{mar} - H_k^{\text{mod}} \right) = 163.31 + 39.18 = 202.49 \text{ bps}$$

a great improvement.

Alternatively, suppose we calibrate the Hull-White model to the caplets starting at 3 years, at 3.25 years, at 3.5 years, ..., and at 9.75 years, with all caplets struck at 7.50%. Now we have the correct strike, but the wrong tenors. The calibrated model yields the price

$$V^{mod} = 196.82 \text{ bps.}$$

If we add in the adjustor, we obtain a price of

$$V^{\text{mod}} + \sum_{k=1}^{m} b_k \left(H_k^{mar} - H_k^{\text{mod}} \right) = 196.82 + 9.12 = 199.94 \text{ bps,}$$

again a distinct improvement.

24.1.4 Nothing Is Free

At first glance, it appears that using an adjuster greatly increases the computational load. After all, to determine the adjustment requires computing the exotic derivative's Vega risk $\partial V^{mod}/\partial \sigma_j$ to all calibration instruments. These risks are usually found via finite differences, so evaluating these risks would seem to require model calibrations in $n + 1$ separate scenarios (base case, and each σj bumped separately). However, these Vega risks are needed for hedging purposes, and are nearly always computed as part of the nightly batch, even if one is *not* applying an adjustor. So computing the Vega matrix is usually free. The computational load does increase modestly, because for each natural hedging instrument, one has to calculate the model price $H^{mod}{}_k$ and its Vega derivatives $\partial H^{mod}{}_k/\partial \sigma_j$. This requires calculating the model price of m vanilla instruments $n + 1$ times. This is the same load as calculating the *calibration error* in each of the $n + 1$ scenarios, clearly much, much faster than actually *calibrating* the model in each of the $n + 1$ scenarios.

24.1.5 The SABR Volatility Model

The **SABR** model is a stochastic volatility model, which attempts to capture the volatility smile in derivatives markets. The name stands for *Stochastic Alpha, Beta, Rho*, referring to the parameters of the model.

24.1.5.1 Dynamics

The **SABR** model describes a single forward F, such as a LIBOR forward rate, a forward swap rate, or a forward stock price. The volatility of the forward F is described by a parameter σ. **SABR** is a dynamic model in which both F and σ are represented by stochastic state variables whose time evolution is given by the following system of SDEs

$$dF_t = \sigma_t F_t^{\beta} dW_t$$
$$d\sigma_t = \alpha \sigma_t dZ_t$$

with the prescribed time zero (currently observed) values F_0 and σ_0. Here, W_t and Z_t are two correlated Wiener processes with correlation coefficient $-1 < \rho < 1$. The constant parameters β, α satisfy the conditions $0 \leq \beta \leq 1, \alpha \geq 0$.

The aforementioned dynamics is a stochastic version of the constant elasticity of variance (CEV) model with the *skewness* parameter β: in fact, it reduces to the **CEV** model if $\alpha = 0$ The parameter α is often referred to as the *volvol*, and its meaning is that of the log-normal volatility of the volatility parameter σ.

24.1.6 Asymptotic Solution

We consider a European option (say, a call) on the forward price F struck at K, which expires T years from now. The value of this option is equal to the suitably discounted expected value of the payoff $\max(F_T - K, 0)$ under the probability distribution of the stochastic process for F_t.

Except for the special cases of $\beta = 0$ and $\beta = 1$, no closed form expression for this probability distribution is known. The general case can be solved approximately by an asymptotic expansion in the parameter $\varepsilon = T\alpha^2$. Under typical market conditions, this parameter is small and the approximate solution is actually quite accurate. Also significantly, this solution has a rather simple functional form; it is easy to implement in computer code and lends itself well to risk management of large portfolios of options in real time.

It is convenient to express the solution in terms of the implied volatility of the option. Namely, we force the SABR model price of the option into the form of the Black model valuation formula. Then the implied volatility, which is the value of the log-normal volatility parameter in Black's model that forces it to match the SABR price, is approximately given by

$$\sigma_{impl} = \alpha \frac{\ln(F_0/k)}{D(\varsigma)} \left\{ 1 + \left[\frac{2\gamma_2 - \gamma_1^2 + 1/F_{mid}^2}{24} \left(\frac{\sigma_0 C(F_{mid})}{\alpha} \right)^2 + \frac{\rho\gamma_1}{4} \frac{\sigma_0 C(F_{mid})}{\alpha} + \frac{2 - 3\rho^2}{24} \right] \varepsilon \right\},$$

where, for clarity, we have set $C(F) = F^\beta$. The value F_{mid} denotes a conveniently chosen midpoint between F_0 and K (such as the geometric average $\sqrt{F_0 K}$ or the arithmetic average $(F_0 + K)/2$). We have also set

$$\varsigma = \frac{\alpha}{\sigma_0} \int_K^{F_0} \frac{dx}{C(x)} = \frac{\alpha}{\sigma_0(1-\beta)} \left(F_0^{1-\beta} - K^{1-\beta} \right)$$

and

$$\gamma_1 = \frac{C'(F_{mid})}{C(F_{mid})} = \frac{\beta}{F_{mid}},$$

$$\gamma_2 = \frac{C''(F_{mid})}{C(F_{mid})} = -\frac{\beta(1-\beta)}{F_{mid}^2}.$$

The function $D(\zeta)$ entering the previous formula is given by

$$D(\varsigma) = \ln\left(\frac{\sqrt{1 - 2\rho\,\varsigma + \varsigma^2} + \varsigma - \rho}{1 - \rho} \right).$$

Alternatively, one can express the SABR price in terms of Black's normal model. Then the implied normal volatility can be asymptotically computed using the following expression:

$$\sigma^n_{impl} = \alpha \frac{F_0 - K}{D(\varsigma)} \left\{ 1 + \left[\frac{2\gamma_2 - \gamma_1^2}{24} \left(\frac{\sigma_0 C(F_{mid})}{\alpha} \right)^2 + \frac{\rho \gamma_1}{4} \frac{\sigma_0 C(F_{mid})}{\alpha} + \frac{2 - 3\rho^2}{24} \right] \varepsilon \right\}$$

It is worth noting that the normal SABR implied volatility is generally somewhat more accurate than the log-normal implied volatility.

24.1.7 Conversion Between Log Normal and Normal Volatility

We have seen in Section 4.1.15.2 how to convert between normal and log-normal volatility for ATM swaptions. This is when the strike rate K and the forward rate F are equal. We will now give a general formula of how to convert between volatilities. The subsequent formulas follow the articles by Hagan and Woodward (1998) and Hagan[1] where they used a singular perturbation expansion.

The Black's log-normal model is

$$dF = \sigma_B F dW_t \; F(0) = f,$$

where f is today's forward swap/caplet rate and where σ_B is the implied Black (log-normal) volatility. The Black's normal model is

$$dF = \sigma_N dW_t \; F(0) = f,$$

where σ_N is the "normal" or "absolute" or the annualized "basis point" volatility. For a swaption with strike (fixed rate) K, the normal volatility σ_N (which gives the same price of the option) as the log normal volatility σ_B is

$$\sigma_N = \sigma_B \frac{f - K}{\ln(f/K)} \cdot \frac{1}{1 + \frac{1}{24}\left(1 - \frac{1}{120}\left[\ln(f/K)\right]^2\right)\sigma_B^2 \tau + \frac{1}{5760}\sigma_B^4 \tau^2},$$

[1] Hagan, Volatility Conversion Calculators

where τ is the time to the exercise date in years. When $f- > K$, the aforementioned formula goes to a "0 over 0". To avoid this complication, we should use the alternative formula

$$\sigma_N = \sigma_B \sqrt{f \cdot K} \cdot \frac{1 + \frac{1}{24}\left[\ln(f/K)\right]^2}{1 + \frac{1}{24}\sigma_B^2 \tau + \frac{1}{5760}\sigma_B^4 \tau^2}$$

when

$$\left| \frac{f - K}{K} \right| \leq 0.001.$$

To calculate the Black log-normal volatility from the normal volatility, we apply the previous formulas in a Newton-Raphson solver (or similar).

Since Black's log-normal model, in a risk-neutral world, can be written as

$$\frac{dF}{F} = \sigma_B dW$$

and the normal Black as

$$dF = \sigma_N dW$$

we say that σ_B is a relative volatility while σ_N is an absolute volatility.

24.1.8 Conversion Between Normal and CEV Volatility

The CEV model is

$$dF = \alpha F^{\beta} dW_t \quad F(0) = f.$$

To convert the CEV volatility into a normal (absolute) volatility, one can use

$$\sigma_N = \alpha \frac{(1 - \beta)(f - K)}{f^{1-\beta} - K^{1-\beta}} \cdot \frac{1}{1 + \frac{1 - \frac{2 - 2\cdot\beta + \beta^2}{120} \cdot [\ln(f/K)]^2}{1 - \frac{(1-\beta)^2}{12} \cdot [\ln(f/K)]^2} \cdot \frac{\beta(2-\beta)}{24(f \cdot K)^{1-\beta}} \alpha^2 \tau}.$$

When f is very near K, or when β is very near 1, the previous formula is singular. To avoid this complication, we should use the alternative formula

$$\sigma_N = \alpha(f \cdot K)^{\beta/2} \cdot \frac{1+\frac{1}{24}[\ln(f/K)]^2}{1+\frac{(1-\beta)^2}{24}\cdot[\ln(f/K)]^2}$$
$$\cdot \frac{1}{1+\frac{1-\frac{2-2\cdot\beta+\beta^2}{120}\cdot[\ln(f/K)]^2}{1-\frac{(1-\beta)^2}{12}\cdot[\ln(f/K)]^2}\cdot\frac{\beta(2-\beta)}{24(f\cdot K)^{1-\beta}}\alpha^2\tau}$$

when

$$(1-\beta)\left|\frac{f-K}{K}\right| \leq 0.001.$$

To calculate the CEV volatility from the normal volatility, we apply the aforementioned formulas in a Newton-Raphson solver (or similar).

25

Modern Term Structure Theory

25.1 Term Structure Theory

From the bootstrapped Swap curve, see Section 6.1.4 we have a given yield curve. This curve is used for discounting all cash flows. We define the discount function $D(t)$ as

$$D(t) = \exp\left\{-\int_0^t f(0, T')dT'\right\}.$$

To price exotic instruments we use the following schema:

1. To examine the risk, select an arbitrage free model.
2. Calibrate the model by
   a. selecting some vanilla hedging instruments
   b. matching the current discount curve $D(T)$
   c. matching the vanilla prices (caps, floors, swaptions) by e.g. least square method.
3. Price the exotics via the model and interpolation of vanilla instruments.

You can use a global calibration process with known instruments or a local calibration with the most likely vanilla instruments.

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6_25

25.1.1 The Three Elements

We use three elements as described next:

1. A *numeraire* $N(t)$. Use a positive value of a coupon free security, e.g. the money market account

$$N(t) = \exp\left\{\int_0^t r(t')dt'\right\}$$

   or zero-coupon bonds

$$N(t) = p(t, T)$$

2. The *valuation* in a risk neutral world is then given by

$$V(t) = N(t)E\left[\frac{V(T)}{N(T)}|\mathcal{F}_t\right].$$

3. We also use a random evolution of the *interest rates*.

There exist three approaches to model the interest rates:

1. Heath-Jarrow-Morton (HJM) and BGM (Brace Gatarek Musiela) sometimes also called the Libor market model (LMM),
2. A short rate model, such as Ho-Lee, Hull-White (HW), Black-Karasinsky (BK) etc.
3. A Markov model.

25.1.2 The BGM Model (Brace Gatarek Musiela)

The BGM model is a financial model of interest rates. It is used for pricing interest rate derivatives, especially exotic derivatives like Bermudan swaptions, ratchet caps and floors, target redemption notes, auto-caps, zero-coupon swaptions, constant maturity swaps and spread options, among many others.

The quantities that are modelled in BGM are a set of forward rates, which have the advantage of being directly observable in the market, and whose volatilities are naturally linked to traded contracts. Each forward rate is modelled by a log-normal process under its forward neutral martingale measure, i.e. a Black model leading to a Black formula for interest rate caps. This formula is the market standard to quote cap prices in terms of implied volatilities, hence the term "market model". The LMM may be interpreted as a collection of forward LIBOR dynamics for different forward rates with spanning tenors and maturities, each forward rate being consistent with a Black interest rate caplet formula for its canonical maturity. One can describe the dynamics of the different rates under a common pricing measure, for example, the forward-neutral measure for a preferred single maturity, in which case forward rates will not be log-normal under the unique measure in general, leading to a need for numerical methods such as Monte Carlo simulations.

The LMM models a set of n forward rates L_i, as log-normal processes

$$\frac{dL_i(t)}{L_i(t)} = \mu_i(\{L_i(t)\}, t)dt + \sigma_i(t)dW_i \quad i = 1, \ldots, n.$$

Here, L_i denotes the forward rate for the period $[T_i, T_{i+1}]$. For each single forward rate, the model corresponds to the Black model. The novelty is that, in contrast to the Black model, the LMM describes the dynamic of a whole family of forward rates under a common measure.

The valuation is given by

$$\tilde{V}(t, \vec{r}) = \frac{V(t, \vec{r})}{Z_0(t)} = E\left[\frac{V(T, \vec{R}(T))}{Z_0(T)} \vec{R}(T) = \vec{r}\right]$$

$$dR_k = \mu_k(t, \vec{R})dt + a_k(t, R_k)dW_k \quad k = 1, 2 \ldots,$$

where $\tilde{V}(t, \tilde{r})$ will satisfy

$$\frac{\partial \tilde{V}}{\partial t} + \sum_k \mu_k \frac{\partial \tilde{V}}{\partial r_k} + \frac{1}{2} \sum_j \sum_k \rho_{jk} a_j a_k \frac{\partial^2 \tilde{V}}{\partial r_j \partial r_k} = 0.$$

The function

$$\tilde{V}(t, \vec{r}) = \frac{Z_j(t, \vec{r})}{Z_0(t, \vec{r})} = \prod_{k=1}^{j} \frac{1}{1 + a_k r_k}$$

satisfies the PDE earlier, if and only if

$$\sum_{k=1}^{j} \mu_k \frac{a_k}{1 + a_k r_k} = \sum_{i=1}^{j} \sum_{k=1}^{j} \rho_{ik} a_i a_k \frac{a_i}{1 + a_i r_i} \frac{a_k}{1 + a_k r_k}$$

which gives the BGM drift condition

$$\mu_k(t, \vec{r}) = \sum_{j=1}^{k} \rho_{jk} a_j a_k \frac{a_j}{1 + a_j r_j}.$$

Compare this to the HJM drift condition. If we let R_m be the interest rate in the interval $[T_{m-1}, T_m]$ and $Z_m = Z(t, T_m)$, then we can use the numeraire

$$Z_m(t) = Z_0(t) \prod_{k=1}^{m} \frac{1}{1 + a_k R_k(t)}.$$

We now remember that a change of numeraire only changes the drift. We now have the valuation formula

$$\frac{V(t, \vec{r})}{Z_m(t)} = E\left[\frac{V(T, \vec{R}(T))}{Z_m(T)} \middle| \vec{R}(T) = \vec{r} \right]$$
$$dR_k = v_k(t, \vec{R})dt + a_k(t, R_k)dW_k \quad k = 1, 2 \ldots$$

If we replace Z_0 with Z_m in the preceding calculation

$$v_k(t, \vec{r}) = \left\{ \sum_{j=1}^{k} - \sum_{j=1}^{m} \right\} \rho_{jk} a_j a_k \frac{a_j}{1 + a_j r_j}$$

for the numeraire $Z_m(t)$. If we set $k = m$, we have no drift, i.e.

$$dR_m = a_m(t, R_m)dW_m$$

i.e. R_m is a martingale when $Z_m(t)$ is numeraire.

25.1.3 A Caplet in the BGM Framework

In this section we will study a caplet with strike R_f, a tenor α_m, and a FRA fair rate R_m over the interval $[T_{m-1}, T_m]$. Its payoff is given by

$$\mathit{payoff} = \alpha_m (R_m(t) - R_f)^+ Z_m(t) \quad t < T_{m-1} < T_m.$$

If we use

$$\frac{V_c(0,R_m^0)}{Z_m(0)} = E\left[\frac{V_c(T,R_m(T))}{Z_m(T)}|R_m(0) = R_m^0\right]$$
$$dR_m = a_k(t,R_m)dW_m$$

we get

$$V_c(0,R_m^0) = \alpha_m D(T_m)E\left[\left(R_m(t) - R_f\right)^+ |R_m(0) = R_m^0\right].$$

If we use Black's log-normal model: $a_m(t,R_m) = \sigma_m R_m$ we get

$$V_c = \alpha_m D(T_m)BS\left(t,R_m^0,R_f,\sigma_m\right).$$

If we use a local volatility model: $a_m(t,R_m) = A_m(R_m)R_m$ we get

$$V_c = \alpha_m D(T_m)BS\left(t,R_m^0,R_f,\sigma_m\right)$$
$$\sigma_m = A\left(\frac{1}{2}\left(R_m^0 + R_f\right)\{1 + \ldots\right.$$

Using a SABR (*Stochastic Alpha, Beta, Rho*) model we have

$$\begin{cases} dR_k = v_k(t,\vec{R})dt + \omega a_k R_k^\beta dW_k \quad k = 1,2\ldots \\ d\omega = \gamma\omega dW \end{cases},$$

where the correlation matrix is given by

$$\begin{cases} dR_k = v_k(t,\vec{R})dt + a_k R_k dW_k \\ dW_j W_k = \rho_{jk}dt = Corr\left\{dR_j, DR_k\right\}dt \end{cases}$$

and the volatilities $a_k(t,R_k)$ are determined by the volatility and smile of caplet k. The correlation matrix ρ_{ij} then determines swaption volatilities. High correlations give high swaption prices. The correlation matrix can be determined by

1. historical correlations
2. using factor analysis to write $\rho_{ij} = \lambda_1^2 q_{1j}q_{1k} + \lambda_2^2 q_{2j}q_{2k} + \ldots$ where only two or three factors, $q_1, q_2, \ldots$, are significant
3. adjusting factors to match swaption volatility as closely as possible.

The strength of the BGM model is that you can understand intuitively the relationships between caplet volatilities, correlations and the swaption volatility matrix.

The weaknesses are as follows:

- High dimensions, one per R_k. So, a 10-year deal has for example 40 rates and 40 dimensions!
- Limit valuation methods to Monte Carlo.
- Monte Carlo is very slow, is very noisy, gives bad hedges and is very challenging for multiple exercise deals.

So do not use Monte Carlo when there is an alternative.

25.1.4 Short Rate Models

The roll-over *numeraire* $N(t)$ is the money market given by

$$N(t) = \exp\left\{\int_0^t r(t')dt'\right\}.$$

The *valuation* in a risk neutral world is

$$V(t, r_0) = E\left[\int_0^t r(t')dr \cdot V(T, r(T)) \middle| r(t) = r_0\right].$$

Generalized Cross-Currency Interest Rate (CIR) models

$$dr = [\theta(t) - \kappa(t) \cdot r]\, dt + \sigma(t) r^{\beta} dW,$$

where β is a skew, $\beta = 1/2$ gives CIR and $\beta = 0$ HW. The generalized BK models can be written by

$$dr = [\theta(t) - \kappa(t) \cdot Y]\, dt + \sigma(t) dW,$$

where Y is Gaussian. $Y = \log r$ gives BK, $Y = r$ the HW and $Y = r^{1-\beta}$ the generalized BK.

We can also use a multi-factor generalization. The calibration is made on spot prices of zero-coupon bonds

$$p(t, Y_0; T) = E\left[\int_0^t r(t')dt' \cdot 1 \middle| Y(t) = Y_0\right].$$

With a given $\kappa(t)$ and $\sigma(t)$ adjust $\theta(t)$ to match the yield curve; $D(T) = p(0, Y_0; T)$ or use the Jamshidean's forward induction scheme and adjust $\kappa(t)$ and $\sigma(t)$ to match caps/floors/swaptions.

Short rate models are easy to work with. An n factor model has n dimensions. But the most serious valuations are done with either one or two factors. For $n = 1$ or $n = 2$ you can use tree models, lattice methods or finite difference methods to value deals with a fast and accurate result. The calibration process is often straightforward.

26

Pricing Exotic Instruments

26.1 Practical Pricing of Exotics

In this chapter we will give an introduction of valuation of exotic interest rate derivatives in a Gaussian framework and how to calibrate such models to market data of plain vanilla instruments.

26.1.1 Discount Factors, Zeroes and FRAs

Suppose at date t, one agrees to loan out \$1 at date T, and get repaid the next day

$$\begin{array}{rl} 1 & \text{paid at } T, \\ 1 + f(t,T)\Delta T & \text{received at } T + \Delta T \end{array}$$

By definition, the fair interest rate to charge is $f(t,T)$ = instantaneous forward rate for date T as seen at date t.

Now suppose at date t one agrees to loan out \$1 on T_{st}, with the money repaid on T_{end}. Economically this is equivalent to loaning out \$1 on T_{st}, getting repaid \$1 plus interest the next day, re-loaning out the \$1 plus interest, getting repaid \$1 plus interest plus interest on the interest, ... Clearly, if one agrees at date t to loan out

$$\begin{array}{ll} 1 & \text{paid at } T_{st} \end{array}$$

the agreement should specify getting repaid

$$\begin{array}{ll} e^{\int_{T_{st}}^{T_{end}} f(t,T')dT'} & \text{paid at } t \\ 1 & \text{received at} T_{end} \end{array}$$

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for the deal to be fair. Alternatively, we can rephrase this as

$$\begin{array}{ll} e^{-\int_{T_{st}}^{T_{end}} f(t,T')dT'} & \text{paid at } t \\ 1 & \text{received at } T_{end.} \end{array}$$

This type of single payment deal is equivalent to an FRA (forward rate agreement). Suppose we imagine that we are at date t, and we ask how much I would need to pay immediately to receive \$1 at date T . Clearly the fair amount is

$$\begin{array}{ll} e^{-\int_{t}^{T_{end}} f(t,T')dT'} & \text{paid at} T_{st} \\ 1 & \text{received at } T_{end.} \end{array}$$

By definition,

$$\hat{p}(t,T) = e^{-\int_t^T f(t,T')dT'}$$

i.e. the value at t of \$1 paid at T is the value of a zero-coupon bond for maturity T on date 0. Today is always $t = 0$ in our notation. Discount factors are today's values of the zero-coupon bonds

$$D(T) = \hat{p}(0,T) = e^{-\int_0^T f(0,T')dT'},$$

where $f(0,T)$ is today's instantaneous forward rate curve.

26.1.2 Swaps

We start by studying the *fixed leg*. Consider a swap with start date t_0, fixed leg pay dates $t_1, t_2, \ldots, t_n$, and fixed rate R^{fix}. The fixed leg makes the payments:

$$\begin{array}{rl} \alpha_i R^{fix} & \text{paid at } t_i, i = 1, 2, \ldots, n-1 \\ 1 + \alpha_n R^{fix} & \text{paid at } t_n, \end{array}$$

where

$$\alpha_i = cvg(t_{i-1}, t_i, \beta)$$

is the coverage (day count fraction) for interval i computed according to the appropriate day count basis β. Discounting the future payments with some given market discount factors $p(t, t_i)$ the current value of the fixed leg on any given day t will be

$$\hat{V}_{fix}(t) = R^{fix} \sum_{i=1}^{n} \alpha_i \hat{p}(t,t_i) + \hat{p}(t,t_n).$$

Then we study the *floating leg* of the swap. Floating legs usually have a different frequency than the fixed legs, so let the reset and pay dates of the floating leg be denoted by

$$t_0 = \tau_0, \tau_1, \ldots, \tau_m = t_n.$$

We suppose that the pay dates for the floating leg are $\tau_0,\tau_1, \ldots, \tau_m$. These will typically not be the same dates as the pay dates $t_1, t_2, \ldots, t_n$, for the fixed leg. Indeed, the number of pay dates for the fixed and floating leg may differ, often $m > n$. We do however require that the first and last pay dates for both legs coincide, i.e. $t_0 = \tau_0$, and $\tau_m = t_n \ldots$

The floating leg pays

$$\begin{aligned} \alpha_i' r_j \qquad & \text{paid at } \tau_j, j = 1, 2, \ldots, m-1 \\ 1 + \alpha_m' r_m \qquad & \text{paid at } \tau_m = t_n, \end{aligned}$$

where

$$\alpha'_{\ i} = cvg(\tau_{i-1}, \tau_i, \beta')$$

is the coverage for interval j computed according to the appropriate day count basis β'. Here r_j is generally the Libor or Euribor floating rate for interval j. This rate is set on the *fixing date*; for most floating legs, the fixing (reset) date is 2 London business days before the interval starts on τ_{j-1}.

We may consider the floating leg as a sequence of FRAs in which one lends 1 unit of the currency at τ_{j-1} and receives $1 + \alpha_j' r_j$ units back at τ_j. On the later date 1 unit of currency is lent again so the net cash flow of the series of FRAs will exactly duplicate the cash flow of the floating leg, i.e. $\alpha_j' r_j$ for $j < m$ and the final payment $1 + \alpha_m' r_m$, the notional does not relent on date τ_m . Taking a closer look at one of the FRAs we note that the current value of the lent amount must be equal to the current value of the repaid amount on each fixing date τ_j^{fix},. Using the relevant market discount factors we have

$$\hat{p}(\tau_j^{fix}, \tau_j) = (1 + \alpha_j' r_j)\ \dot{}\hat{p}(t, \tau_j).$$

From this formula we can solve for the discounted value of the interest payment $\alpha_j' r_j$ on each fixing day

$$V_j^{theor}(\tau_j^{fix}) = \hat{p}(\tau_j^{fix}, \tau_{j-1}) - \hat{p}(\tau_j^{fix}, \tau_j).$$

If the value of the floating rate payment is the difference between two freely tradable securities (two zero-coupon bonds) at the fixing time, then the value must equal this difference for all earlier times as well. So in principle, the value of the j^{th} floating interest rate payment is

$$V_j^{theor}(t) = \hat{p}(t, \tau_{j-1}) - \hat{p}(t, \tau_j) \text{ for t} < \tau_j^{fix}$$

for any date t, at least until the rate is fixed. At any date t, the forward fair or true rate $r_j^{true}(t)$ is defined so that the value of the interest payment exactly equals the theoretical value

$$\begin{aligned} \alpha_j' r_j^{true}(t) \cdot \hat{p}(t, \tau_j) &= \text{theoretical value of interest rate payment} \\ &= \hat{p}(t, \tau_{j-1}) - \hat{p}(t, \tau_j). \end{aligned}$$

So

$$r_j^{true}(t) = \frac{\hat{p}(t, \tau_{j-1}) - \hat{p}(t, \tau_j)}{\alpha'_j \hat{p}(t, \tau_j)}.$$

In practice, floating rates are not set at the fair rate; they are set at the fair rate plus a small offset s_j, the forward basis spread, due to credit considerations and supply and demand. The value of the (forward) basis spread depends on which index is used for the floating rate (3M USD Libor, 1M fed funds, 6M Euribor, etc.), and on the starting date. Taking the spread into consideration the value of the floating rate payment paid at τ_j is

$$\hat{V}_j(t) = \hat{p}(t, \tau_{j-1}) - \hat{p}(t, \tau_j) + \alpha'_j s_j \hat{p}(t, \tau_j).$$

By definition, the forward rate for the floating rate is defined by

$$\alpha'_j r_j^{fwd} \hat{p}(t, \tau_j) = \text{value of interest ratepayment}$$

so

$$r_j^{fwd}(t) r_j^{true}(t) + s_j = \frac{\hat{p}(t, \tau_{j-1}) - \hat{p}(t, \tau_j)}{\alpha'_j \hat{p}(t, \tau_j)} + s_j.$$

26.1.3 Basis Spread

Basis spread curves are obtained by stripping basis swaps. One can show that forward basis spreads are martingales with respect to the appropriate forward measures. Since they are very small, usually just 1-2 bps, and since they seldom vary, one always assumes they are constant. That is, one assumes that the gamma of the forward spread is inconsequential.

Summing these payments together, the value of the floating leg is

$$\hat{V}_{flt}(t) = \hat{p}(t, t_0) + \sum_{j=1}^{m} \alpha'_j s \hat{p}(t, \tau_j).$$

This is true regardless of the model used. The value of the receiver swap (receive the fixed leg, pay the floating leg) is

$$\hat{V}_{rec}(t) = R^{fix} \sum_{i=1}^{m} \alpha \hat{p}(t, t_i) + \hat{p}(t, t_n) - \hat{p}(t, t_0) - \sum_{j=1}^{m} \alpha'_j s \hat{p}(t, \tau_j).$$

The value of the payer swap (pay the fixed leg, receive the floating leg) is

$$\hat{V}_{pay}(t) = -\hat{V}_{rec}(t) = \hat{p}(t, t_0) + \sum_{j=1}^{m} \alpha'_j s \hat{p}(t, \tau_j) - R^{fix} \sum_{i=1}^{n} \alpha \hat{p}(t, t_i) - \hat{p}(t, t_n).$$

26.1.3.1 Handling the Basis Spread

Basis spreads are a nuisance. They are too big to be neglected (except for USD 3m Libor), yet small enough to be nearly irrelevant. One way of handling them is to treat them as another, very small, fixed leg of the swap. There is really nothing wrong with this approach, although one usually has twice as many fixed leg pay dates.

However, here we will use a second, common approach in which each interval's fixed rate is adjusted to account for the value of the basis spreads. If the basis spread is 0.625 bps in an interval, then we subtract 0.625 bps from the fixed rate instead of adding it to the floating leg. More precisely, including this adjustment, today's value of the swap is,

$$\begin{aligned}
\hat{V}_{rec}(0) &= R^{fix} \sum_{i=1}^{n} \alpha_i D(t_i) + D(t_n) - D(t_0) - \sum_{j=1}^{m} \alpha'_j s_j D(\tau_j), \\
&= \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i \right) D(t_i) + D(t_n) - D(t_0).
\end{aligned}$$

Here S_i is the basis spread expressed with the same frequency and day count basis as the fixed leg. If the floating leg frequency is the same or higher than the fixed leg frequency, then

$$S_i = \frac{\sum_{j \in I_i} \alpha'_j s_j D(\tau_j)}{\alpha_i D(t_i)},$$

where $j \in I_i$ represents the floating leg intervals which are part of the i^{th} fixed leg interval. That is, the floating leg intervals whose theoretical dates τ_j^{th} are contained in the i^{th} fixed leg theoretical interval $t_{j-1}^{th} \leqslant t_i^{th} \leqslant t_j^{th}$. If the fixed leg frequency is shorter than the floating leg frequency (this is rare), then the same S_i is used for all fixed leg intervals forming part of each floating leg interval. So,

$$S_i = \frac{\alpha'_j s_j D(\tau_j)}{\sum_{i = I_j} \alpha_j D(t_i)},$$

where $i \in I_j$ represents the fixed leg intervals i with $\tau_{j-1}^{th} \leqslant t_i^{th} \leqslant \tau_j^{th}$.

Either way, we may approximate the swap values as

$$\begin{cases} \hat{V}_{rec}(t) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right)\hat{p}(t, t_i) + \hat{p}(t, t_n) - \hat{p}(t, t_0) \\ \hat{V}_{pay}(t) = \hat{p}(t, t_0) - \hat{p}(t, t_n) - \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right)\hat{p}(t, t_i) \end{cases}$$

for all dates t, where the strike $R^{f\ ix}$ and effective spread S_i are known constants. We are neglecting any evolution of the basis spreads and any minor differences due to the differences between the legs' day count bases and frequencies. We will use this approach throughout. Computationally, it would be just as easy to modify the code to add another fixed leg, but this would make the formulas messier and debugging more difficult.

26.1.3.2 Swap Rate and Level

At any time t, the swap rate $R^{sw}(t)$ is defined to be the break even rate, the value of $R^{f\ ix}$ which would make the swap value equal to zero. Clearly,

$$R^{sw}(t) = \frac{\hat{p}(t, t_0) - \hat{p}(t, t_n) + \sum_{i=1}^{n} \alpha_i S \hat{p}(t, t_i)}{L(t)},$$

where the level $L(t)$ (also known as the PV01, the DV01, or the annuity) is

$$L(t) = \sum_{i=1}^{n} \alpha \hat{p}(t, t_i).$$

We can rewrite the swap values in terms of the swap rate and level as

$$\hat{V}_{rec}(t) = \left[\boldsymbol{R}^{fix} - \boldsymbol{R}^{sw}(t) \right] L(t) \quad \hat{V}_{pay}(t) = \left[\boldsymbol{R}^{sw}(t) - \boldsymbol{R}^{fix} \right] L(t).$$

In particular, today's swap rate and level are

$$R^0 = \frac{D(t_0) - D(t_n) + \sum_{i=1}^{n} \alpha_i S_i D(t_i)}{L^0}$$

$$L^0 = \sum_{i=1}^{n} \alpha_i D(t_i)$$

and the swap values are

$$\hat{V}_{rec}(t) = \left[\boldsymbol{R}^{fix} - \boldsymbol{R}^0 \right] \boldsymbol{L}^0 \quad \hat{V}_{pay}(t) \left[\boldsymbol{R}^0 - \boldsymbol{R}^{fix} \right] L^0.$$

Swaptions. A swaption is a European option on a swap. Consider a receiver swaption with notification date t^{ex}. If one exercises on this date, one obtains the receiver swap. Clearly

$$\hat{V}_{rec}^{opt}(t^{ex}) = \left[\boldsymbol{R}^{fix} - \boldsymbol{R}^{sw}(t^{ex}) \right]^+ L(t^{ex})$$

is the value of the receiver swaption on the exercise date. Swaption prices are almost always quoted in terms of Black's model. To introduce this model, suppose we choose the level $L(t)$ as our numeraire. (It is just the sum of a bunch of zero-coupon bonds, and hence is a tradable instrument.) The function $L(t)$ is sometimes called the forward annuity. There exists a probability measure in which the value of all tradable instruments (including the swaption) divided by the numeraire is a martingale. So

$$\hat{V}_{rec}^{opt}(t) = L(t) E \left[\frac{\hat{V}_{rec}^{opt}(T)}{L(T)} | \mathcal{F}_t \right].$$

If we evaluate the expected value at $T = t^{ex}$, we see that

$$\hat{V}_{rec}^{opt}(t) = L(t) E \left[\left[R^{fix} - R^{sw}(t^{ex}) \right]^+ | \mathcal{F}_t \right].$$

Moreover, the swap rate

$$R^{sw}(t) = \frac{\hat{p}(t,t_0)\hat{p}(t,t_n) + \sum i = 1n\alpha_i S\hat{p}(t,t_i)}{L(t)}$$

is clearly a tradable market instrument (a bunch of zero-coupon bonds) divided by the numeraire. So the swap rate is also a martingale. By the martingale representation model, then, we conclude that

$$dR^{sw} = A(t,*)dW,$$

where dW is Brownian motion, and $A(t,*)$ is some measureable coefficient. Fundamental theory can take us no further. We now have to model $A(t,*)$. Black proposed that $A(t,*) = \sigma R^{sw}$, so that the swap rate is log normal

$$dR^{sw} = \sigma R^{sw} dW.$$

Finding the expected value previously under this model yields Black's formula

$$\hat{V}_{rec}^{mkt}(t) = \left\{ \boldsymbol{R}^{fix} N(d_1) - \boldsymbol{R}^{sw}(t) N(d_2) \right\} L(t)$$

$$d_{1,2} = \frac{\log \frac{R^{fix}}{R^{sw}(t)} \pm \frac{1}{2}\sigma^2 (t_{ex} - t)}{\sigma\sqrt{t}_{ex} - t}.$$

Today's market price of the swaption is

$$\hat{V}_{rec}^{mkt}(0) = \left\{ \boldsymbol{R}^{fix} N(d_1^0) - \boldsymbol{R}^0 N(d_2^0) \right\} L_0$$

$$d_{1,2}^0 = \frac{\log \frac{R^{fix}}{R^0} \pm \frac{1}{2}\sigma^2 t_{ex}}{\sigma\sqrt{t}_{ex}}$$

$$R^0 = \frac{D_0 - D_n + \sum_{i=1}^n \alpha_i S_i D_i}{L_0} \qquad L_0 = \sum_{i=1}^n \alpha_i D_i.$$

Here $D_i = D(t_i)$ are today's discount factors. A payer swaption is a European option to pay the fixed leg and receiver the floating leg. The value of the payer swaption is obtained by reversing R^{fix} and R^0 in the aforementioned formulas.

If one analyses Black's formula, one discovers that the receiver and payer swaption values are both increasing functions of the volatility σ.

Instead of quoting swaption prices in terms of dollar values, one can just as well quote the price in terms of the value of σ that needs to be inserted into Black's formula to obtain the market price. This value of the volatility is known as the implied volatility.

26.1.4 Caplets and Floorlets

Consider a floorlet for the interval τ_0 to τ_1. The floating rate r for the interval is set on the fixing date t_{ex} two (London) business days before the interval starts at τ_0, and the floorlet pays the difference between the strike (fixed rate) and the floating rate at the end of its period, provided this difference is positive:

$$\alpha(R_{fix} - r)^+ \quad \text{paid at } \tau_1.$$

Here α is the coverage (day count fraction) of the interval $[\tau_0, \tau_1]$. As before, the value of the floating rate payment is

$$\hat{p}(t_{ex}, \tau_0) - \hat{p}(t_{ex}, \tau_1) + \alpha s_1 \hat{p}(t_{ex}, \tau_1),$$

on the fixing date, where s_1 is the (forward) basis spread for the interval. The floorlets payoff is

$$\hat{V}_{floorlet}(t) = \left[\alpha\left(R^{fix} - s_1\right)\hat{p}(t_{ex}, \tau_1) + \hat{p}(t_{ex}, \tau_1) - \hat{p}(t_{ex}, \tau_0)\right]^+.$$

This is the same payoff as a 1-period receiver swaption. Similarly, caplet payoffs are identical to the payoffs of 1-period payer swaptions.

The analysis of caplets and floorlets parallels the analysis for swaptions exactly. We define the forward or FRA rate as

$$R^{FRA}(t) = \frac{\hat{p}(t, \tau_0) - \hat{p}(t, \tau_n) + \sum_{i=1}^{n} \alpha_i s \hat{p}(t, \tau_i)}{\alpha \hat{p}(t, \tau_1)}$$

and choose the zero-coupon bond $\hat{p}(t, \tau_1)$ as our numeraire. The value of the floorlet is

$$\hat{V}_{floorlet}(t_{ex}) - \alpha \hat{p}(t_{ex}, \tau_1) E\left[\left[\boldsymbol{R}^{fit} - \boldsymbol{R}^{FRA}(t_{ex})\right]^+ \mid \mathcal{F}_t\right],$$

where the forward FRA rate is a martingale under this measure. Modelling this rate as log normal

$$dR^{F\ RA} = \sigma R^{F\ RA} dW$$

again yields Black's formula,

$$\begin{cases} \hat{V}_{floorlet}(t) = \left\{R^{fix}N(d_1) - R^{FRA}(t)N(d_2)\right\} \alpha\hat{p}(t, \tau_1) \\ d_{1,2} = \dfrac{\ln\left(R^{fix}/R^{FRA}(t)\right) \pm \frac{1}{2}\sigma^2(t_{ex} - t)}{\sigma\sqrt{t_{ex} - t}}. \end{cases}$$

Today's market price of the floorlet is

$$\begin{cases} \hat{V}^{mkt}_{floorlet}(0) = \left\{R^{fix}N(d^0_1) - R^0 N(d^0_2)\right\} \alpha D(\tau_1) \\ d^0_{1,2} = \dfrac{\ln\left(R^{fix}/R^0\right) \pm \frac{1}{2}\sigma^2(t_{ex})}{\sigma\sqrt{t_{ex}}} \end{cases}$$

$$R_0 = \frac{D_0 - D_n + \alpha s_1 D_1}{\alpha D_1}.$$

The value of the caplet is obtained by reversing R^{fix} and R^0 in the previous formulas

$$\begin{aligned} \hat{V}^{mkt}_{caplet}(0) &= \left\{\boldsymbol{R}^0 N(-d^0_2) - \boldsymbol{R}^{fix} N(-d^0_1)\right\} \alpha D(\tau_1) \\ &= \hat{V}^{mkt}_{floorlet}(0) - \left\{R^{fix} - R^0\right\} \alpha D(\tau_1). \end{aligned}$$

Note that the caplet and floorlet values are special cases of the payer and receiver swaptions with $n = 1$. As before, the implied volatility σ is the value of the volatility which makes the previously mentioned formulas match the actual market values of the floorlet and caplet.

26.1.5 Linear Gaussian Models

A modern interest rate model consists of 3 parts: a numeraire, a set of random evolution equations in the forward risk neutral world, and the matching martingale pricing formula. The one-factor Linear Gaussian Models (LGM) has a single state variable, X. It starts at today = 0 and satisfies

$$\begin{cases} d\hat{X}(t) = \alpha(t)d\hat{W}(t) \\ \hat{X}(0) = 0. \end{cases}$$

This is the evolution under the risk neutral measure induced by the numeraire, which will be named shortly. Clearly $\hat{X}(t)$ is Gaussian with the transition density

$$\varphi(t, x \,|\, T, X)dX = prob\left\{X < \hat{X}(T) \leqslant X + dX \middle| \hat{X}(t) = x\right\}$$

given by

$$\varphi(t, x|T, X) = \frac{1}{\sqrt{2\pi\,\Delta\varsigma}} e^{-\frac{1}{2}(X-x)^2/\Delta\zeta}$$

$$N(t, x) = \frac{1}{D(t)} e^{H(t)\cdot x + \frac{1}{2}H^2(t)\varsigma(t)}$$

$$\zeta(T) = \int_0^T \alpha^2(\tau)d\tau; \qquad \Delta\zeta = \zeta(T) - \zeta(t) = \int_t^T \alpha^2(\tau)d\tau,$$

where $\hat{N}(t, x)$ is the chosen numeraire. Note that in this case, the value of the numeraire is 1 today: $\hat{N}(0, 0) = 1$. The last part of the model is the martingale valuation formula. Suppose at time t the economy is in state $X(t) = x$. If $\hat{V}(t, x)$ is the value of any freely tradeable security, then (t, x) i s a martingale. So in the LGM model we get the valuation formula

$$\hat{V}(t, x) = \hat{N}(t, x)E\left[\frac{\hat{V}(T, X)}{\hat{N}(T, X)} \middle| \hat{X}(t) = x\right] = \frac{\hat{N}(t, x)}{\sqrt{2\pi\,\Delta\zeta}} \int \frac{\hat{V}(T, X)}{\hat{N}(T, X)} e^{-\frac{1}{2}(X-x)^2\Delta\zeta} dX.$$

If the security has intermediate cash payments, then we need to modify this formula appropriately. The prices in LGM model can be written in terms of the relative prices

$$V(t, x)\frac{\hat{V}(t, x)}{\hat{N}(t, x)}.$$

Since the value of the numeraire is 1 today, values and relative values are equal today, i.e. $\hat{V}(0, 0) = V(0, 0)$ As we shall see, we only have to calculate the relative prices $V(t, x)$ and never have to calculate the full prices $\hat{V}(t, x)$. This simplifies our formulas substantially. In terms of the relative prices $V(t,x)$, the LGM model is

$$V(t,x)\frac{1}{\sqrt{2\pi\,\Delta\zeta}}\int V(T,X)e^{-\frac{1}{2}(X-x)^2/\Delta\zeta}\,dX.$$

The value of a zero-coupon bond is

$$p(t,x,T)=\frac{\hat{p}(t,x,T)}{\hat{N}(t,x)}=\frac{1}{\sqrt{2\pi\,\Delta\zeta}}\int\frac{1}{\hat{N}(T,X)}e^{-\frac{1}{2}(X-x)/2\Delta\zeta}\,dX\quad\text{for any}T>t.$$

Substituting for the numeraire and carrying out the integration yields the zero-coupon price

$$p(t,x,T)=D(T)e^{-H(T)x-\frac{1}{2}H^2(t)\zeta(t)}.$$

At $t=0$, the state variable is $x=0$, by definition. Since $p(0,0,T)=D(T)$, the LGM model automatically matches today's discount curve $D(T)$.

At t,x the instantaneous forward rate for maturity T, namely $f(t,x,T)$, is defined via

$$\hat{p}(t,x,T)=p(t,x,T)\hat{N}(t,x,T)=e^{-\int_t^T f(t,x,T')dT'}.$$

Similarly, the discount factor can be written in terms of today's instantaneous forward rate $f_0(T)$ as

$$D(T)=e^{-\int_0^T f_0(T')dT'}.$$

This shows that for the LGM model,

$$f(t,x,T)=f_0(T)+H'(T)x+\left[H'(T)\right]^2\zeta(t).$$

The last term $[H'(T)]^2\zeta(t)$ is a small convexity correction; although it is needed for pricing, it does not affect the qualitative behaviour of the model. The other terms show that at any date t, the forward curve is made up of today's forward curve $f_0(T)$ plus an amount x of the curve $H'(T)$. The amount x of the shift is a Gaussian random variable with mean zero and variance $\zeta(t)$.

The curve $H'(T)$ is a model parameter; as we shall see, it replaces the mean reversion coefficient $\kappa(t)$ in the Hull-White (HW) model. The other model parameter is the variance $\zeta(t)$. It takes the place of the volatility $\sigma(t)$. As always, model parameters have to be set a priori during the calibration procedure by combining both theoretical reasoning (guessing) and calibration of vanilla instruments.

26.1.6 Hull-White

We continue to study LGM. The HW model is given by

$$dt = [\theta(t) - \kappa(t) \cdot r]\, dt + \sigma(t) dW$$
$$p(t, r_0; T) = E\left[\int_t^T r(\tau) d\tau \cdot 1 \,\middle|\, r(t) = r_0\right].$$

For given $\kappa(t)$ and $\sigma(t)$ we adjust $\theta(t)$ to match the yield curve; $D(T) = p(0, r_0; T)$. We can transform HW into a LGM model with

$$X(t) = \exp\left\{\int_0^t \kappa(\tau) d\tau\right\}\left\{r(t) - r(0) - \int_t^T \theta(\tau) \cdot \exp\left\{\int_0^\tau \kappa(\tau') d\tau'\right\} d\tau\right\}.$$

Using Itô we can write

$$dX = \alpha(t) dW$$
$$X(0) = 0,$$

where X is a Gaussian martingale. In the LGM model we get the valuation formula

$$\frac{V(t,x)}{N(t,x)} = E\left[\frac{V(T,X)}{N(T,X)} \,\middle|\, X(t) = x\right] = \int \frac{V(T,X)}{N(T,X)} \varphi(T, Xvert, t, x) dX,$$

where

$$\varphi(t, x \,|\, T, X) = \frac{1}{\sqrt{2\pi \Delta \varsigma}} e^{-\frac{1}{2}(X-x)^2/2 \cdot [\zeta(T) - \zeta(t)]}$$
$$N(t,x) = \frac{1}{D(t)} e^{H(t)\cdot x + \frac{1}{2} H^2(t) \varsigma(t)}$$
$$\zeta(T) = \int_0^T \alpha^2(\tau) d\tau; \qquad \Delta\zeta = \zeta(T) - \zeta(t) = \int_t^T \alpha^2(\tau) d\tau.$$

The model parameters are $\zeta(T)$, the accumulated volatility and $H(T)$, the mean revision. For a zero-coupon bond ($V(T,X) = 1$) we have

$$\frac{p(t,x,T)}{N(t,x)} = D(t) e^{-H(t)\cdot x - \frac{1}{2} H^2(t) \cdot \zeta(t)}$$

Table 26.1 The Hull-White and the LGM model functions

Hull-White	LGM
$\sigma(t) = H'(t)\sqrt{\delta'(t)}$ rate volatility	$\zeta^{(t)}$ accumulated volatility
$K(t) = -H''(t)/H'(t)$ reversion	$H(t)$ reversion
$\theta(t) = f_o'(t) + \kappa(t) \cdot f_o(t) + [H'(t)]^2 \zeta(t)$	

But

$$p(t,x,T) = e^{\int_t^T f(t,x,T')dT'}$$

gives

$$f(t,x,T) = f_0(T) + H'(T) \cdot x + H'(T)H(T) \cdot \zeta(t).$$

So in terms of LGM we can write the HW model as

$$\begin{aligned}
&dX = \alpha(t)dW, \qquad X(0) = 0\\
&f(t,x,T) = f_0(T) + H'(T) \cdot x + H'(T) \cdot H(T) \cdot \varsigma(t)\\
&r(t,x) = f(t,x,t) = f_0(t) + H'(t) \cdot x + H'(t) \cdot H(t) \cdot \varsigma(t).
\end{aligned}$$

With the money market numeraire we have

$$\begin{aligned}
&V(t,r_0) = E\left[e^{-\int_t^T r(\tau)d\tau} V(T,r(T)) \,\middle|\, r(t) = r_0 \right]\\
&dr = [\theta(t) - \kappa(t) \cdot r]\, dt + \alpha(t)dW.
\end{aligned}$$

To compare the models we use Table 26.1, where $\zeta(t)$ and $H(t)$ are directly measured!

The calibrated LGM prices are shown in Table 26.3 and their predicted implied volatility in Table 26.4. The error in the At-The-Money volatility with diagonal = 2% are shown in Table 26.5.

As we shall see, the value of any vanilla option depends only on the value of the variance at the exercise date, $\zeta(t_{ex})$, and on the mean reversion function $H(t_j)$ at the deal's pay dates t_j. Calibration determines the functions $\zeta(t)$ and $H(t)$ fairly directly. Obtaining the mean reversion parameter $\kappa(t)$ requires differentiating $H(t)$ twice, which is an inherently noisy procedure. Similarly, obtaining σ(t) also requires differentiating ζ(t). This is why calibrating directly on the HW model parameters (instead of the LGM formulation of the model) is often an inherently unstable procedure.

The LGM parameters can be written in terms of the HW parameters as

$$H(t) = A\int_0^t e^{-\int_0^{t'} \kappa(\tau)d\tau} dt' + B$$

$$\zeta(t) = \frac{1}{A^2}\int_0^t \sigma_1^2(t')e^{2\int_0^{t'} \kappa(\tau)d\tau} dt'.$$

where A and B are arbitrary positive constants. Since different A and B yield the same HW model, and thus yield the identical prices, the LGM model has 2 invariant representations.

First, all market prices remain unchanged if we change the model parameters by

$$H(t) \to C \cdot H(t), \quad \zeta(t) \to \zeta(t)/C^2$$

for any positive constant C. To prove this, note that if we make the previous transformation and then transform the internal variables x and X by

$$x \to x/C, X \to X/C,$$

we obtain the same transition probabilities and zero-coupon bond prices that we started with. Second, all market prices remain unchanged if

$$H(T) \to H(T) + K\zeta(T) \to \zeta(T)$$

for any constant K. To prove this, note that if we make the previous transformation, and then transform the internal variables $x$and X by

$$x \to x + K\zeta, X \to X + K\zeta,$$

we obtain the same transition probabilities and zero-coupon bond prices as before.

It is critical to pin down these invariances by arbitrarily choosing some value of $H(t)$ and of $\zeta(t)$) before calibration. Otherwise convergence would be infinitely slow, with numerical round-off determining which of the equivalent sets of model parameters is chosen.

26.1.6.1 Scaling

On average, interest rates in G7 countries change by ± 80 bps or so over the course of a year. Equivalently, the standard deviation of $H'(T)x$ should be about 1% or less each year. We choose to use the time scale of years (so $\Delta T = 1$ means an elapsed time of 1 year) and we scale $H(T)$ and $H'(T)$ to be $O(1)$. Then x is of order $O(1\% \times \sqrt{\mathrm{t}})$ at date t, and $\zeta(t)$ is of the order of $O(10^{-4}\mathrm{t})$. More precisely, suppose we have chosen $H(0) = 0$, and have scaled $H(T)$ so it increases by 1 or so every year. Then

$$
\begin{aligned}
&H'(T) \quad \sim O(1), H(T) \quad \sim O(T), \zeta(t) \quad \sim O(0.64 \times 10^{-4})t \\
&x, X \sim (0.8 \times 10^{-2})\sqrt{t} \\
&H'(T)H(T)\zeta(t) \sim (064 \times 10^{-4})tT.
\end{aligned}
$$

26.1.7 Summary of the LGM Model

The complete LGM model can be summarized as

$$
V(t,x) = \frac{1}{\sqrt{2\pi\,\Delta\zeta}} e^{\frac{1}{2}(X-x)/2\Delta\zeta}\,dX \quad \text{for any } T < t
$$

with $\Delta\zeta \equiv \zeta(T) - \zeta(t)$, and with the (reduced) zero-coupon bond formula being

$$
p(t,x,T) = D(T)e^{-H(T)x - \frac{1}{2}H(t)\zeta(t)}
$$

and with $x = 0$ at $t = 0$. Consequently, $\zeta(0) = 0$.

These equations are the only facts about the model we need to price any security. This model automatically reproduces the discount curve $D(T)$. The functions $H(T)$ and $\zeta(t)$ are model parameters, which are set during the calibration step, where the model prices are matched to the market prices of selected vanilla instruments, usually caplets and swaptions. Once the model is calibrated, $H(T)$ and $\zeta(t)$ are known functions, and the price of exotic deals can be determined from the aforementioned martingale formula, using the previous zero-coupon formula to calculate the payoffs. Later we will present the calibration and pricing steps in exquisite detail.

26.1.8 Calibration

26.1.8.1 Calibration and Hedging

Model calibration is the most critical step in pricing. It determines not only the price, obtained for an exotic deal, but also the hedges of the exotic. To see this, suppose we have some model $\mathcal{M}$. It invariably contains unknown mathematical parameters which are set by calibration.

To calibrate, one selects a set of vanilla instruments whose volatilities (prices) are known from market quotes. Let these volatilities be $\sigma_1, \sigma_2, \ldots, \sigma_n$.

The calibration procedure picks the model parameters by matching the model's yield curve to today's discount factors $D(T)$ and matching the model's price of the selected vanilla instruments to their market volatilities, either exactly or in a least squares sense.

The calibrated model $\mathcal{M}$' is a function of today's discount factors and these n volatilities. The calibrated model is then used to price the exotic deal. The only step in this procedure which uses market information is the calibration step. This means that the price of the deal is a function of today's yield curve $D(T)$ and the n volatilities $\sigma_1, \sigma_2, \ldots, \sigma_n$. The price of the deal depends on no other market information.

Consider what happens at the nightly mark-to-market. The model is calibrated and deals are priced as mentioned before. Next the Vega risks are calculated by bumping the volatilities in the volatility matrix (cube) one by one. After each bump, the deal is priced using the identical software, and the difference between the new price and the base price is the bucket Vega risk for the bumped volatility. Unless the bumped volatility is one of the n volatilities used in calibration, it has no effect on the calibration of the model, so it does not affect the price of the instrument. An exotic deal only has Vega risks to the n vanilla instruments used in calibration. After the Vega risks are calculated for all the deals on the books, enough of each vanilla instrument is bought/sold to neutralize the corresponding bucket Vega risk. This means that in the normal course of events, an exotic deal will be hedged by a linear combination of the vanilla instruments used during calibration. If the span of the vanilla instruments provides a good representation of the exotic, then the hedges should exhibit rock solid stability, with the day-to-day amounts of the hedges changing only as much as necessary to account for the actual changes in the market place. If the vanilla instruments do not provide a good representation of the exotic, then the hedges may exhibit instabilities, with day-to-day amounts of the hedges changing substantially even for relatively minor market changes. This latter is highly undesirable as the increased hedging costs gradually eliminate any initial profit from the exotic. (The nice term for this is leaking away your P&L.) Indeed, in practice, even small improvements in the algorithm for matching the hedges to the exotics pay off disproportionately in the adroitness of the hedging.

One can move most of, and possibly all, the Vega risk from the calibration instruments to a different (and presumably better) set of hedging instruments by using risk migration. This is also known as applying an external adjuster. As a side benefit, this method also improves the pricing, often dramatically. This technique will also be discussed as it pertains to the different exotics.

26.1.9 Exact Formulas for Swaption and Caplet Pricing

Under the LGM model, the reduced value of the swap is

$$V_{rec}(t,x) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) p(t,x,t_i) + p(t,x,t_n) - p(t,x,t_0)$$

$$V_{pay}(t,x) = p(t,x,t_0) - p(t,x,t_n) - \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) p(t,x,t_i).$$

where

$$p(t,x,t_i) = D_i e^{-H_i x - \frac{1}{2} H_i^2 \zeta(t)}$$

is the reduced value of the zero-coupon bonds. Here $D_i = D(t_i), H_i = H(t_i)$ are the discount factors and values of $H(t)$ at the swap's pay dates t_i.

Under the LGM model, the prices of vanilla swaptions, caplets, and floorlets depend on $\zeta(t)$ only through $\zeta(t_{ex})$, its value at the notification date. The swaption prices depend on $H(T)$ only through the differences $H(t_j) - H(t_0)$ for the pay dates t_j of the fixed leg. This will be the key to creating lightning fast and stable calibration schemes.

Under the one-factor LGM model, the exact pricing formulas for swaptions are

$$\hat{V}_{rec}(0,0) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i N \left(\frac{y + [H_i - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right)$$

$$+ D_n N \left(\frac{y + [H_n - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) - D_0 N \frac{y}{\sqrt{\zeta_{ex}}},$$

$$\hat{V}_{pay}(0,0) = D_0 N \frac{y}{\sqrt{\zeta_{ex}}} - \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i N \left(\frac{y + [H_i - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right)$$

$$- D_n N \left(-\frac{y + [H_n - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right).$$

$$= \hat{V}_{rec}(0,0) + D_0 - \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i - D_n$$

Here y is obtained by solving

$$\sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i e^{-(H_i - H_0)y - \frac{1}{2}(H_i - H_0)^2 \varsigma_{ex}} + D_n e^{-(H_n - H_0)y - \frac{1}{2}(H_n - H_0)^2 \varsigma_{ex}} = D_0.$$

Newton's method requires the derivatives of the prices with respect to the model parameters. We observe

$$\frac{\partial}{\partial H_i} \hat{V}_{rec}(0,0) = \frac{\partial}{\partial H_i} \hat{V}_{pay}(0,0)$$

$$= \sqrt{\varsigma_{ex}} \alpha_i \left(R^{fix} - S_i\right) D_i N' \left(\frac{y + [H_i - H_0]\,\varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right)$$

$$\frac{\partial}{\partial H_n} \hat{V}_{rec}(0,0) = \frac{\partial}{\partial H_n} \hat{V}_{pay}(0,0)$$

$$= \sqrt{\varsigma_{ex}} \left[1 + \alpha_i \left(R^{fix} - S_n\right) D_n\right] N' \left(\frac{y + [H_n - H_0]\,\varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right)$$

$$\frac{\partial}{\partial H_0} \hat{V}_{rec}(0,0) = \frac{\partial}{\partial H} \hat{V}_{pay}(0,0)$$

$$= -\sqrt{\varsigma_{ex}} \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i N' \left(\frac{y + [H_n - H_0]\,\varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right)$$

$$- \sqrt{\varsigma_{ex}} D_n N' \left(\frac{y + [H_n - H_0]\,\varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right)$$

$$
\begin{aligned}
\frac{\partial}{\partial\sqrt{\varsigma_{ex}}}\hat{V}_{rec}(0,0) &= \frac{\partial}{\partial\sqrt{\varsigma_{ex}}}\hat{V}_{pay}(0,0) \\
&= \sum_{i=1}^{n}\alpha_i\left(R^{fix}-S_i\right)D_i\left[H_i-H_0\right]N'\left(\frac{y+\left[H_i-H_0\right]\varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right) \\
&\quad + D_n\left[H_n-H_0\right]N'\left(\frac{y+\left[H_n-H_0\right]\varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right).
\end{aligned}
$$

The caplet/floorlet prices are given by

$$
\hat{V}_{caplet}(0,0) = D_0N(d_1) - \left(1+\tilde{\alpha}\left[R^{fix}-s_1\right]\right)D_1N(d_2)
$$

$$
d_{1,2} = \frac{\log\frac{1+\tilde{\alpha}\left[R^0-s_1\right]}{1+\tilde{\alpha}\left[R^{fix}-s_1\right]} \pm \frac{1}{2}(H_1-H_0)^2\zeta_{ex}}{(H_1-H_0)\sqrt{\zeta_{ex}}}
$$

and

$$
\begin{aligned}
\hat{V}_{floorlet}(0,0) &= \left(1+\tilde{\alpha}\left[R^{fix}-s_1\right]\right)D_1N(d_1^*) - D_0N(d_2^*) \\
&= \hat{V}^{opt}_{caplet}(0,0) + \left(1+\tilde{\alpha}\left[R^{fix}-s_1\right]\right)D_1 - D_0
\end{aligned}
$$

$$
d^*_{1,2} = \frac{\log\frac{1+\tilde{\alpha}\left[R^{fix}-s_1\right]}{1+\tilde{\alpha}\left[R^0-s_1\right]} \pm \frac{1}{2}(H_1-H_0)^2\zeta_{ex}}{(H_1-H_0)\sqrt{\zeta_{ex}}}.
$$

where

$$
R^0 = \frac{D(\tau_0)-D(\tau_1)}{\tilde{\alpha}D(\tau_1)} + s_1
$$

is the forward FRA rate. The caplet/floorlet prices are clearly Black's formulas for call/put prices for an asset with forward value D_0, strike $(1+\alpha_1 R^{fix}_{adj})D_1$, and implied volatility satisfying

$$
\sigma_{imp}\sqrt{t_{ex}} = (H_1-H_0)\sqrt{\zeta_{ex}}
$$

26.1.10 Approximation of Vanilla Pricing Formulas for the One-Factor LGM Model

It is practically useful to develop approximate formulas for the one-factor LGM model, even though exact closed form formulas are known. Recall that market prices for swaptions are usually quoted in terms of Black's formula

$$\hat{V}_{rec}^{mkt}(0) = \left\{R^{fix}N(d_1^0) - R^0N(d_2^0)\right\} L_0$$

$$\hat{V}_{pay}^{mkt}(0) = \left\{R^0N(-d_2^0) - R^{fix}N(-d_1^0)\right\} L_0 = \hat{V}_{rec}^{mkt}(0) - L_0 \left\{R^{fix} - R^0\right\}$$

where

$$d_{1,2}^0 = \frac{\log \frac{R^{fix}}{R^0} \pm \frac{1}{2}\sigma^2 t_{ex}}{\sigma\sqrt{t_{ex}}}$$

and

$$R^0 = \frac{D_0 - D_1 + \sum_{i=1}^n \alpha_i S_i D_i}{L_0}.$$

Here $D_i = D(t_i)$ are today's discount factors at the pay dates. By using equivalent volatility techniques (or direct asymptotic), one discovers that under the LGM model, the implied (Black) volatility of the swaption is approximately

$$\sigma_B\sqrt{t_{ex}} \approx \frac{\sqrt{\zeta_{ex}}}{\sqrt{R^{fix}R_0}} \frac{\sum_{i=1}^n \alpha_i \left(R^0 - S_i\right) D_i \left(H_i - H_0\right) + D_n \left(H_n - H_0\right)}{\sum_{i=1}^n \alpha_i D_i}.$$

This provides a good way to use market quotes of the implied volatility to obtain initial guesses for calibration. One can rewrite these quotes more simply in terms of the implied normal volatility. Under the Gaussian (normal) swap rate model, the value of the swaption is

$$\hat{V}_{rec}^{mkt}(0) = \left\{(R^{fix} - R^0)N\left(\frac{R^{fix} - R^0}{\sigma_N\sqrt{\tau_{ex}}}\right) - \sigma_N\sqrt{\tau_{ex}}G\left(\frac{R^{fix} - R^0}{\sigma_N\sqrt{\tau_{ex}}}\right)\right\} L_0$$

$$\hat{V}_{pay}^{mkt}(0) = \left\{(R^0 - R^{fix})N\left(\frac{R^0 - R^{fix}}{\sigma_N\sqrt{\tau_{ex}}}\right) - \sigma_N\sqrt{\tau_{ex}}G\left(\frac{R^0 - R^{fix}}{\sigma_N\sqrt{\tau_{ex}}}\right)\right\} L_0$$

$$= \hat{V}_{rec}^{mkt}(0) - L_0 \left\{R^{fix} - R^0\right\}$$

The equivalent volatility work shows that the implied normal or absolute volatility σ_N, is approximately

$$\sigma_B\sqrt{t_{ex}} \approx \frac{\sqrt{\zeta_{ex}}}{\sqrt{R^{fix}R^0}} \frac{\sum_{i=1}^{n} \alpha_i \left(R^0 - S_i\right) D_i \left(H_i - H_0\right) + D_n \left(H_n - H_0\right)}{\sum_{i=1}^{n} \alpha_i D_i}.$$

26.1.10.1 Forward Volatility

Forward volatility is a key concern of calibrated models. Suppose that we calibrate a model and then ask what the swaption volatilities will look like at a future date t in the future. If the volatilities are increasing with t, we may be buying future volatility at too dear a price, and if volatilities are decreasing with t, we may be selling future volatility too cheaply.

If we repeat the previous equivalent volatility analysis at a date t in the future, then we discover that the (normal) swaption volatility at that date is

$$\sigma_N \approx \frac{\sqrt{\zeta(\tau_{ex}) - \zeta(t)}}{\tau_{ex} - t} \frac{\sum_{i=1}^{n} \alpha_i \left(R^0 - S_i\right) D_i \left(H(t_i) - H(t_0)\right) + D_n(H(t_n) - H(t_0))}{\sum_{i=1}^{n} \alpha_i D_i}$$

where

$$R^0 = \frac{D_0 - D_1 + \sum_{i=1}^{n} \alpha_i S_i D_i}{L_0}.$$

If $H(t)$ is decreasing exponentially, then $\zeta(\tau_{ex})$ should be increasing exponentially to compensate.

26.1.10.2 Calibration Strategy

The most critical aspect of pricing is choosing the right set of vanilla instruments for calibrating the model. Even small improvements in matching the vanilla instruments to the exotic deals often lead to significant improvements in the price and the stability of the hedge. For each type of exotic, the best calibration strategy often cannot be determined from purely theoretical considerations. Instead, one needs to determine which method leads to the best (the most "market fit") prices and risks.

We will now briefly discuss calibration strategies, illustrating the different strategies with a simple Bermuda swap (Bermudan swaptions and callable swaps are considered much more carefully in a later section).

Consider a Bermudan receiver with start date t_0, end date t_n, and strike R^{fix}. Let the fixed leg dates be $t_0, t_1, \ldots, t_n$, and let the exercise dates be $\tau_1, \tau_2, \ldots, \tau_n$. If the Bermudan is exercised at τ_j, then the holder receives the fixed leg payments

$$\begin{array}{rl} \alpha_i R^{fix} & \text{paid at } t_i, i = j, j+1, \ldots, n-1 \\ 1 + \alpha_n R^{fix} & \text{paid at } t_n, \end{array}$$

where

$$\alpha_i = cvg(t_{i-1}, t_i, \beta)$$

is the coverage (day count fraction) for interval i computed according to the appropriate day count basis β. In return, the holder makes the floating leg payments, which are worth the same as

$$\begin{array}{ll} 1 & \text{paid at } t_{j-1} \\ \alpha_i S_j & \text{paid at } t_i, i = j, j+1, \ldots, m-1, \end{array}$$

where

$$\alpha_i' = cvg(\tau_{i-1}, \tau_i, \beta').$$

Here we have adjusted the basis spread to the fixed leg's frequency and day count basis as discussed earlier. Therefore, if the Bermudan is exercised at τ_j, one receives/makes the payments

$$\begin{array}{rll} -1 & \text{at} & t_{j-1} \\ \alpha_i \left(R^{fix} - S_i\right) & \text{at} & t_i \text{for } i = j, j+1, \ldots, n-1 \\ 1 + \alpha_n \left(R^{fix} - S_n\right) & \text{at} & t_n \end{array}$$

Clearly at any point t, x the j^{th} payoff is worth

$$\hat{V}_j^{pay}(t,x) = \sum_{i=j}^{n} \alpha_i \left(R^{fix} - S_i\right) \hat{p}(t,x,t_i) + \hat{p}(t,x,t_n) - \hat{p}(t,x,t_{j-1}).$$

The first step in calibration is to characterize the exotic, extracting its essential features.»> If the Bermudan is exercised on exercise date τ_j, one receives a swap worth

$$\hat{V}_j^{pay}(t,x) = \sum_{i=j}^{n} \alpha_i \left(R^{fix} - S_i\right) \hat{p}(t,x,t_i) + \hat{p}(t,x,t_n) - \hat{p}(t,x,t_{j-1})$$

The first step in calibration is to characterize the exotic, extracting its essential features. If the Bermudan is exercised on exercise date τ_j, one receives a swap worth

$$\sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) \hat{p}(\tau_j,x,t_i) + \hat{p}(\tau_j,x,t_n) - \hat{p}(\tau_j,x,t_{j-1}) \quad \text{, at } \tau_j.$$

Suppose we evaluate this swap using today's yield curve with a parallel shift of size γ,

$$\hat{p}(\tau_j,x,t_i) \to D(t_i)e^{-\gamma t_i} = D_i e^{-\gamma t_i}.$$

The shift γ_j at which the j^{th} swap is at the money is found by solving

$$\sum_{i=j}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i e^{-\gamma_j (t_i - t_j t_{-1})} + D_n e^{-\gamma_j (t_n - t_j t_{-1})} = D_{j-1}.$$

The Bermudan is characterized by

(a) the set of exercise dates $\tau_1, \tau_2, \ldots, \tau_n$;

(b) the set of parallel shifts γ_j for $j = 1, 2 \ldots, n$; and

(c) the length $t_n - t_0$ of the longest swap.

The second step is to select a calibration strategy and choose the calibration instruments. As we shall see, since we have 2 functions of time to calibrate, we can calibrate 2 separate series of vanilla instruments.

Under the LGM model, the prices of vanilla swaptions, caplets, and floorlets depend on $\zeta(t)$ only through $\zeta(t_{ex})$, its value at the notification date. The swaption prices depend on $H(T)$ only through the differences $H(t_j) - H(t_0)$ for the pay dates t_j of the fixed leg. This will be the key to creating lightning fast, stable calibration schemes.

The trick is to calibrate on vanilla instruments whose pay dates line up exactly. We now go through the various calibration strategies for this Bermudan.

26.1.11 swaptions

From

$$\frac{V(t,x)}{N(t,x)} = E\left[\frac{V(T,X)}{N(T,X)}\middle| X(t) = x\right] = \int \frac{V(T,X)}{N(T,X)}\phi(T,X|t,x)dX$$

we have today

$$V(0,0) = \int \frac{V(T,X)}{N(T,X)}\phi(T,X|0,0)dX.$$

The payoff at t_{ex} is given by (Fig. 26.1)

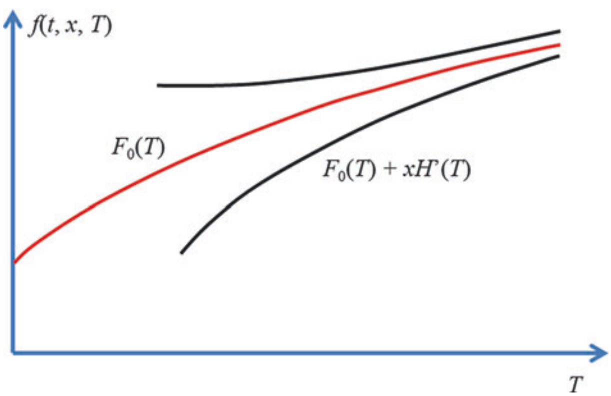

Fig. 26.1 Cash flows for a swaption

$$\left[R_f \sum_{j=1}^{n} \alpha_j p(t_{ex}, X, t_j) + p(t_{ex}, X, t_n) - p(t_{ex}, X, t_0)\right]$$

$$\frac{p(t_{ex}, x, T)}{N(t_{ex}, x)} = D(t_j) e^{-H(t_j)\cdot x - \frac{1}{2}H^2(t_j)\cdot \varsigma(t_{ex})}.$$

If we integrate this for a receiver swaption we get

$$V_{rec} = R_f \sum_{j=1}^{n} \alpha_j D_j N\left(\frac{x - H_j \varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right) + D_n N\left(\frac{x - H_n \varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right) - D_0 N\left(\frac{x - H_0 \varsigma_{ex}}{\sqrt{\varsigma_{ex}}}\right)$$

where x solves

$$\begin{cases} R_f \sum_{j-1}^{n} \alpha_j D_j e^{H_j x - \frac{1}{2}H_j^2 \varsigma_{ex}} + D_n e^{H_n x - \frac{1}{2}H_n^2 \varsigma_{ex}} = D_0 e^{H_0 x - \frac{1}{2}H_0^2 \varsigma_{ex}} \\ \varsigma_{ex} = \varsigma(t_{ex}), \quad H_j = H(t_j), \quad H_0 = H(t_0) \end{cases}$$

As we can see, the price only depends on the accumulated volatility at the expiry date $\zeta(t_{ex})$ and on the mean reversion on pay days $H(t_j), j = 0, 1, \ldots, n$. From a "put/call" parity, we also have the price of a payer swaption

$$V_{pay} = V_{rec} - R_f \sum_{j=1}^{n} \alpha_j D_j - D_n + D_0.$$

26.1.12 Bermudan Swaption

We study a "2 into 4, annual pay, receiver at 8%, 5 days' notice". The fixed leg is given by

$$\begin{aligned} R_f \alpha_j &\quad \text{paid at} \quad t_j = 1, 2, \ldots, n-1 \\ 1 + R_f \alpha_n &\quad \text{paid at} \quad t_n, \end{aligned}$$

where R_f is the fixed rate and with notification dates $\tau_j, j = 0, 1, \ldots, n-1$. If we exercise at τ_j we get a swap starting at t_j with fixed leg payments at $t_{j+1}, t_{j+2}, \ldots, t_n$ and a floating leg at t_j worth 1. The current value is given by

$$V = R_f \sum_{j=1+1}^{n} \alpha_j p(\tau_j, X, t_k) + p(\tau_j, X, t_n) - p(\tau_j, X, t_j).$$

26.1.13 Calibration, Diagonal + Constant κ

In the calibration process we choose the model parameters $\zeta(t)$ and $H(t)$ to match the LGM prices to the market. For a "2 into 4 Bermudan receiver @ 8%" we use for the underlying risk: "2 into 4", "3 into 3", "4 into 2" and "5 into 1", i.e. the diagonal. For the smile risk we calibrate to swaptions @ 8% (Table 26.2).

$$\kappa(t) = -H''(t)/H'(t) \quad \Rightarrow \quad H(t) = A + B \cdot e^{-\kappa t}$$
$$H(t) \to H(t) + C$$
$$H(t) \to H(t)/eC, \qquad \varsigma(t) \to C^2 \cdot \varsigma(t)$$

Set $H(t) = (1 - e^{-\kappa t})/\kappa$ Then calibrate

$$\begin{aligned}
&\text{2 into 4 } \zeta(t_2), H(t_2), H(t_3), \ldots, H(t_6) => \zeta(t_2)\\
&\text{3 into 3 } \zeta(t_3), H(t_3), H(t_4), \ldots, H(t_6) => \zeta(t_3)\\
&\text{4 into 2 } \zeta(t_4), H(t_4), H(t_5), H(t_6) \qquad => \zeta(t_4)\\
&\text{5 into 1 } \zeta(t_5), H(t_5), H(t_6) \qquad\qquad => \zeta(t_5).
\end{aligned}$$

Then use $\zeta(0) = 0$ and the above and use linear interpolation. Choose $\kappa = 2\%$, which gives $H(t)$ and then use the diagonal to get ζ_j. The calibrated LGM prices are shown in Table 26.3 and their predicted implied volatility in Table 26.4. The error in the At-The-Money volatility with diagonal k = 2% are shown in Table 26.5.

Table 26.2 ATM swaption volatilities

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	13.62%	13.64%	13.65%	13.56%	13.59%	13.51%	13.42%	13.42%	13.42%	13.42%
2y	14.36%	14.17%	14.18%	14.11%	13.78%	13.47%	13.71%	13.63%	13.55%	13.47%
3y	14.18%	14.13%	14.02%	14.00%	13.76%	13.67%	13.59%	13.49%	13.40%	13.20%
4y	14.10%	14.10%	13.96%	13.93%	13.70%	13.59%	13.48%	13.37%	13.26%	13.15%
5y	14.09%	13.99%	13.86%	13.77%	13.53%	13.40%	13.27%	13.16%	13.05%	12.94%
6y	13.83%	13.69%	13.53%	13.42%	13.22%	13.11%	13.01%	12.91%	12.81%	12.72%
7y	13.53%	13.38%	13.19%	13.07%	12.90%	12.82%	12.75%	12.66%	12.58%	12.50%
8y	13.34%	13.15%	12.96%	12.83%	12.62%	12.53%	12.45%	12.34%	12.22%	12.11%
9y	13.11%	12.92%	12.72%	12.58%	12.32%	12.24%	12.15%	12.01%	11.87%	11.73%
10y	12.87%	12.69%	12.49%	12.34%	12.01%	11.94%	11.85%	11.68%	11.51%	11.35%

Table 26.3 LGM prices

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	0.58%	1.62%	2.33%	2.97%	3.48%	3.96%	4.39%	4.77%	5.13%	5.93%
2y	0.88%	1.68%	2.42%	3.03%	3.60%	4.11%	4.57%	4.99%	5.33%	5.61%
3y	0.89%	1.71%	2.40%	3.03%	3.61%	4.13%	4.61%	5.01%	5.35%	5.65%
4y	0.89%	1.65%	2.35%	2.99%	3.57%	4.10%	4.54%	4.93%	5.28%	5.59%
5y	0.83%	1.59%	2.28%	2.91%	3.48%	3.97%	4.40%	4.80%	5.14%	5.46%
6y	0.81%	1.55%	2.22%	2.84%	3.37%	3.84%	4.26%	4.64%	4.99%	5.32%
7y	0.78%	1.49%	2.14%	2.70%	3.20%	3.65%	4.05%	4.43%	4.78%	5.11%
8y	0.75%	1.44%	2.03%	2.57%	3.05%	3.49%	3.89%	4.37%	4.63%	4.96%
9y	0.72%	1.35%	1.92%	2.44%	2.90%	3.33%	3.74%	4.12%	4.47%	4.81%
10y	0.66%	1.26%	1.81%	2.30%	2.74%	3.19%	3.59%	3.96%	4.32%	4.66%

Table 26.4 Implied ATM Volatilities (from LGM prices)

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	14.65%	14.46%	14.30%	14.15%	13.99%	13.83%	13.69%	13.55%	13.42%	13.30%
2y	14.67%	14.53%	14.38%	14.21%	14.05%	13.90%	13.76%	13.63%	13.50%	13.39%
3y	14.53%	14.38%	14.19%	14.02%	13.87%	13.72%	13.59%	13.46%	13.35%	13.26%
4y	14.40%	14.20%	14.02%	13.87%	13.72%	13.59%	13.46%	13.35%	13.26%	13.18%
5y	14.12%	13.95%	13.81%	13.66%	13.53%	13.41%	13.30%	13.22%	13.15%	13.08%
6y	13.83%	13.70%	13.55%	13.42%	13.30%	13.20%	13.13%	13.06%	13.00%	12.95%
7y	13.47%	13.32%	13.19%	13.07%	12.98%	12.92%	12.86%	12.81%	2.77%	12.73%
8y	13.27%	13.15%	13.04%	12.96%	12.91%	12.86%	12.81%	12.78%	12.76%	12.74%
9y	13.11%	13.00%	12.94%	12.90%	12.86%	12.82%	12.80%	12.78%	12.76%	12.75%
10y	12.97%	12.94%	12.91%	12.88%	12.85%	12.83%	12.82%	12.81%	12.80%	12.79%

Table 26.5 Error in ATM volatility (diagonal, kappa = 2%)

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	1.03%	0.81%	0.65%	0.59%	0.40%	0.32%	0.26%	0.13%	0.00%	–0.12%
2y	0.31%	0.36%	0.20%	0.10%	0.27%	0.16%	0.05%	0.00%	–0.05%	–0.08%
3y	0.35%	0.24%	0.16%	0.02%	0.11%	0.05%	0.00%	0.03%	–0.05%	–0.04%
4y	0.30%	0.10%	0.06%	–0.06%	0.02%	0.00%	–0.02%	0.01%	0.01%	0.04%
5y	0.03%	–0.04%	–0.05%	–0.11%	0.00%	0.00%	0.03%	0.06%	0.10%	0.15%
6y	0.00%	0.01%	0.03%	0.00%	0.08%	0.09%	0.11%	0.14%	0.18%	0.24%
7y	–0.11%	–0.06%	0.00%	0.00%	0.08%	0.09%	0.11%	0.14%	0.19%	0.24%
8y	–0.07%	0.00%	0.08%	0.14%	0.30%	0.33%	0.36%	0.44%	0.53%	0.62%
9y	0.00%	0.08%	0.22%	0.32%	0.53%	0.59%	0.65%	0.77%	0.90%	1.02%
10y	0.10%	0.25%	0.42%	0.54%	0.81%	0.89%	0.97%	1.13%	1.29%	1.44%

26.1.14 Calibration to the Diagonal with H(T) Specified

Suppose that $H(T)$ is specified a priori. (A possible source of such curves $H(T)$ is indicated next.) Typically $H(T)$ is given at discrete points $H(T_1), H(T_2), \ldots, H(T_N)$. In that case, piecewise linear interpolation is used between nodes. This is equivalent to assuming that all shifts of the forward rate curve are piecewise constant curves.

With $H(T)$ set, we can use the preceding procedure and formulas to calibrate on the diagonal swaptions. This determines the value of $\zeta(t)$ at $\tau_1, \tau_2, \ldots, \tau_n$. As before, one adds the point $\zeta(0) = 0$, ensures that the $\zeta_j = \zeta(\tau_j)$ are increasing, and uses piecewise linear interpolation to obtain $\zeta(t)$ at other values of t.

Origin of the ***H(T)***. Suppose one had the set of Bermudan swaptions 30 NC 20, 30 NC 15, 30 NC 10, 30 NC 5 and 30 NC 1. Wouldn't it be nice if the same curve $H(T)$ were used for each of these Bermudans? The 30 NC 10 Bermudan includes the 30 NC 15 and the 30 NC 20 Bermudans. It would be satisfying if our valuation procedure for the 30 NC 15 and 30 NC 20 assigned the same price to these Bermudans regardless of whether they were individual deals or part of a larger Bermudan.

One could arrange this by first using a constant κ, let's call it κ_4, to calibrate and price the 30 NC 20 Bermudan. Without loss of generality, we could select

$$H'(T) = e^{\kappa_4(T_{30}-T)}$$

$$H(T) = \frac{1 - e^{\kappa_4(T_{30}-T)}}{\kappa_4}$$

for $T_{20} \leqslant T \leqslant T_{30}$. We would calibrate on the diagonal to find $\zeta(t)$ at expiry dates $\tau_m, \tau_{m+1}, \ldots$ beyond 20 years, and then price the 30 NC 20 Bermudan. Selecting the right value of κ_4 would match the Bermudan price to its market value. Neither the swaption prices nor the Bermudan prices depend on $H(T)$ or $\zeta(t)$ for dates before the 20-year point.

To price the 30 NC 15, one could use the $H(T)$ obtained from $\kappa 4$ for years 20 to 30, and choose a different kappa, say κ_3, for years 15 to 20:

$$H'(T) = e^{\kappa_3(T_{20}-T)} e^{\kappa_4(T_{30}-T)}$$

$$H(T) = \frac{1 - e^{\kappa_3(T_{20}-T)}}{\kappa_3} e^{\kappa_4(T_{30}-T_{20})} + \frac{1 - e^{\kappa_4(T_{30}-T_{20})}}{\kappa_4}.$$

Calibrating would produce the same $\zeta(t)$ values for years 20 to 30 as before. In addition, for each κ_3 it would determine $\zeta(t)$ for years 15 to 20. By selecting the right κ_3, one could match the 30 NC 15 Bermudan's market price.

Continuing in this way, one produces the values of $\zeta(t)$ and $H(T)$ for years 10 to 15, for years 5 to 10, and finally for years 1 to 5. These $\zeta(t)$ and $H(T)$ would then yield a model which matches all the diagonal swaptions and happens to correctly price all the liquid, 30y co-terminal Bermudans. These $\kappa(t)$'s turn out to be extremely stable, only varying very rarely, and then by small amounts. Typically a desk would remember the $\kappa(t)$'s as a function of the co-terminal points, relying on the same $\kappa(t)$'s for years.

In general, if T_n is the co-terminal point and $T_0, T_1, \ldots, T_{n-1}$ are the "no call" points, then $H(T)$ is

$$H(T) = \frac{1 - e^{\kappa_j(T_j - T)}}{\kappa_j} \prod_{i=j+1}^{n} e^{\kappa_i(T_i - T_{i-1})} + \sum_{k=j+1}^{n} \frac{1 - e^{\kappa_k(T_k - T_{k-1})}}{\kappa_k} \prod_{i=k+1}^{n} e^{\kappa_i(T_i - T_{i-1})}$$

for $T_{j-1} \leqslant T \leqslant T_j$.

After $H(T)$ and $\zeta(t)$ have been found, one can use the invariants to re-scale them if desired.

26.1.15 Calibration, Diagonal + Linear $\zeta(t)$

This is an idea pioneered by Solomon brothers. Let us use a constant local volatility α. Then

$$\zeta(t) = \int_0^t \alpha^2 du = \alpha^2 t$$

is linear. By using the invariance $\zeta(t) \to \zeta(t)/C^2, H(T) \to CH(T)$ we can choose α to be any arbitrary constant without affecting any prices. So we choose

$$\zeta(t) = \alpha_0^2 t,$$

where t is measured in years, and the dimensionless constant α_0 is typically 10^{-2}. For this calibration, we use the other invariant to set $H_n = H(t_n) = 0$. We now determine the values of H_i for other values of i by calibrating on the diagonal swaptions, starting with the last swaption. Recall that the price of the jth diagonal swaption is

$$\hat{V}_j^{\text{mod}}(0,0) = R^{fix} \sum_{i=j}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i N\left(\frac{y_j^* + [H_i - H_{j-1}]\varsigma_j}{\sqrt{\varsigma_j}}\right)$$
$$+ D_n N\left(\frac{y_j^* + [H_n - H_{j-1}]\varsigma_j}{\sqrt{\varsigma_j}}\right) - D_j N\left(\frac{y_j^*}{\sqrt{\varsigma_j}}\right)$$

and its derivative with respect to H_{j-1} is

$$\frac{\partial}{\partial H_{j-1}} \hat{V}_j^{\text{mod}} = -\sqrt{\varsigma_j} \sum_{i=j}^{n} \alpha_i \left(R^{fix} - S_i\right) D_i \left[H_i - H_{j-1}\right]$$
$$\times G\left(\frac{y_j^* + \left[H_i - H_{j-1}\right]\varsigma_j}{\sqrt{\varsigma_j}}\right)$$
$$- \sqrt{\varsigma_j} D_n \left[H_n - H_{j-1}\right] G\left(\frac{y_j^* + \left[H_n - H_{j-1}\right]\varsigma_j}{\sqrt{\varsigma_j}}\right)$$

Here y_j^* is given implicitly by

$$\sum_{i=j}^{n} \alpha_i (R^{fix} - S_i) D_i e^{-(H_i - H_{j-1}) y_j^* - \frac{1}{2}(H_i - H_{j-1})^2 \varsigma_j} + D_n e^{-(H_n - H_{j-1}) y_j^* - \frac{1}{2}(H_n - H_{j-1})^2 \varsigma_j} = D_{j-1}$$

Consider the last swaption, $j = n$. It depends on $\zeta_n = \zeta(\tau_n)$, on H_n, and H_{n-1}. Of these, ζ_n is known, H_n has been set to zero, so only H_{n-1} is unknown. Since V_n^{mod} is a decreasing function of H_{n-1}, there is a unique value of H_{n-1} which matches the model price to the market price. This can be found easily using a global Newton's method. We can then move onto the $j = n - 1$ swaption. This swaption depends on H_{n-2}, which is unknown, and ζ_{j-1}, H_{n-1}, and H_n, which are known. Working backwards like this, we can calibrate all of the swaptions, and for each calibration there will only be a single unknown parameter, H_{j-1}.

This calibration procedure will yield $H_0, H_1, \ldots, H_n$ on the dates $t_0, t_1, \ldots, t_n$. One uses linear interpolation/extrapolation to get $H(t)$ at other values of t.

Infeasible values. In deriving the swaption formulas, we assumed that $H(T)$ was an increasing function of T. (This assumption was stronger than we needed: inspection of the previous argument shows that one only needs to assume that there is a unique break-even point y^* , with in-the-moneyness on the left.) Since we are calibrating the H_j 's separately, it may happen that H_{n-1} may exceed H_j . (In practice, this has never happened to my knowledge. Still one must be prepared.) After each H_{n-1} is found, one should check to see that $H_{j-1} \leqslant H_j$

If this condition is violated, one should reset $H_{n-1} = H_j$. This means the jth swaption would not match its market price exactly. Instead it would be the closest feasible price.

This kind of calibration is used by Solomon & Brothers. The hedging instruments for the underlying risk are again "2 into 4", "3 into 3", "4 into 2" and "5 into 1", i.e. the diagonal.

We then assume that $\zeta(t)$ is linear

$\varsigma(t) = \alpha_0^2 t$ set $\alpha_0 = 10^{-2}$

$$H(t) \to H(t) + C$$

$$H(t) \to H(t)/C, \qquad \varsigma(t) \to C^2 \cdot \varsigma(t)$$

Set $H(t_6) = 0$ Then calibrate

$$\begin{aligned}
&5 \text{ into } 1\ \zeta(t_5), H(t_5), H(t_6) && => H(t_2)\\
&4 \text{ into } 2\ \zeta(t_4), H(t_4), H(t_5), H(t_6) && => H(t_3)\\
&3 \text{ into } 3\ \zeta(t_3), H(t_3), H(t_4), \ldots, H(t_6) && => H(t_4)\\
&2 \text{ into } 4\ \zeta(t_2), H(t_2), H(t_3), \ldots, H(t_6) && => H(t_5).
\end{aligned}$$

Use a global Newton-Raphson method and linear interpolation to find $H(t_j)$.

The error after calibration are shown in Table 26.6.

Table 26.6 Error in ATM volatility (diagonal, constant alpha)

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	–1.29%	–0.52%	–0.35%	–0.13%	–0.08%	0.06%	0.10%	0.05%	0.00%	–0.06%
2y	–0.39%	–0.33%	–0.32%	–0.24%	0.11%	0.04%	–0.01%	0.00%	0.00%	0.00%
3y	–0.46%	–0.32%	–0.18%	–0.11%	0.01%	0.00%	0.00%	0.01%	0.02%	0.05%
4y	–0.16%	–0.16%	0.01%	–0.13%	–0.01%	0.00%	0.01%	0.03%	0.07%	0.11%
5y	–0.13%	0.02%	–0.08%	–0.14%	0.00%	0.02%	0.06%	0.10%	0.14%	0.20%
6y	0.24%	0.00%	0.00%	0.00%	0.10%	0.11%	0.14%	0.18%	0.22%	0.28%
7y	–0.29%	–0.14%	0.00%	0.04%	0.15%	0.17%	0.19%	0.24%	0.29%	0.34%
8y	–0.14%	0.00%	0.10%	0.17%	0.34%	0.37%	0.41%	0.50%	0.58%	0.68%
9y	0.00%	0.08%	0.22%	0.32%	0.53%	0.59%	0.65%	0.77%	0.90%	1.02%
10y	0.04%	0.19%	0.36%	0.48%	0.75%	0.83%	0.91%	1.07%	1.23%	1.35%

26.1.16 Calibration, Diagonal + Row

The hedging instruments for the underlying risk is again "2 into 4", "3 into 3", "4 into 2" and "5 into 1", i.e. the diagonal. And exercise or wait (forward volatility risk), use "2 into 1", "2 into 2", "2 into 3" and "2 into 4" (i.e. a row).

Set $\varsigma(t_2) = \alpha_0^2 t_2 H(t_2) = 0$ and set $\alpha_0 = 10^{-2}$

$$H(t) \to H(t) + C$$
$$H(t) \to H(t)/C, \qquad \varsigma(t) \to C^2 \cdot \varsigma(t)$$

Then calibrate

$$\begin{aligned}
&\text{2 into 1 } \zeta(t_2), H(t_2), H(t_3) && => H(t_3)\\
&\text{2 into 2 } \zeta(t_2), H(t_2), H(t_3), H(t_4) && => H(t_4)\\
&\text{2 into 3 } \zeta(t_2), H(t_2), H(t_3), \ldots, H(t_5) && => H(t_5)\\
&\text{2 into 4 } \zeta(t_2), H(t_2), H(t_3), \ldots, H(t_6) && => H(t_6)
\end{aligned}$$

$$\begin{aligned}
&\text{3 into 3 } \zeta(t_3), H(t_3), H(t_4), \ldots, H(t_6) && => \zeta(t_3)\\
&\text{4 into 2 } \zeta(t_4), H(t_4), H(t_5), H(t_6) && => \zeta(t_4)\\
&\text{5 into 1 } \zeta(t_5), H(t_5), H(t_6) && => \zeta(t_5).
\end{aligned}$$

Use a global Newton-Raphson method and linear interpolation. As we will see next, over-calibrating gives worse result. Similar result will be found if we use a column instead of a row.

The error after calibration are shown in Table 26.7.

Table 26.7 Error in ATM volatility (diagonal and row)

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	–0.02%
2y	–0.47%	–0.27%	–0.41%	–0.30%	–0.06%	–0.13%	–0.09%	0.00%	0.06%	0.11%
3y	–0.27%	–0.43%	–0.23%	–0.33%	–0.20%	–0.10%	0.00%	0.08%	0.15%	0.22%
4y	–0.56%	–0.31%	–0.33%	–0.41%	–0.14%	0.00%	0.10%	0.19%	0.27%	0.35%
5y	–0.13%	–0.38%	–0.42%	–0.28%	0.00%	0.11%	0.22%	0.31%	0.39%	0.47%
6y	–0.58%	–0.53%	–0.19%	0.00%	0.21%	0.30%	0.39%	0.46%	0.54%	0.62%
7y	–0.81%	–0.30%	0.00%	0.12%	0.27%	0.33%	0.37%	0.43%	0.49%	0.56%
8y	–0.20%	0.00%	0.12%	0.20%	0.37%	0.41%	0.45%	0.54%	0.63%	0.72%
9y	0.00%	0.08%	0.22%	0.32%	0.53%	0.59%	0.65%	0.77%	0.90%	1.02%
10y	–0.01%	0.14%	0.32%	0.43%	0.70%	0.78%	0.86%	1.02%	1.18%	1.34%

26.1.17 Calibration, Caplets + Constant κ

In the calibration process we choose the model parameters $\zeta(t)$ and $H(t)$ to match the LGM prices to the market. For a "2 into 4 Bermudan receiver @ 8%" we use for the underlying risk "2 into 1", "3 into 1", "4 into 1" and "5 into 1", i.e. a column.

$$\begin{aligned}
&\kappa(t) = -H''(t)/H'(t) \quad \Rightarrow \quad H(t) = A + B \cdot e^{-\kappa t}\\
&H(t) \to H(t) + C\\
&H(t) \to H(t)/C, \qquad \varsigma(t) \to C^2 \cdot \varsigma(t)
\end{aligned}$$

Set $H(t) = (1 - e^{-\kappa t})/\kappa$ Then calibrate

$$\begin{aligned}
&2 \text{ into } 1\ \zeta(t_2), H(t_2), H(t_3) => \zeta(t_2)\\
&3 \text{ into } 1\ \zeta(t_3), H(t_3), H(t_4) => \zeta(t_3)\\
&4 \text{ into } 1\ \zeta(t_4), H(t_4), H(t_5) => \zeta(t_4)\\
&5 \text{ into } 1\ \zeta(t_5), H(t_5), H(t_6) => \zeta(t_5).
\end{aligned}$$

Then use $\zeta(0) = 0$ and the above and linear interpolation. You can also use caplets + linear $\zeta(t)$.

26.1.18 Calibration to Diagonals with Prescribed ζ(t)

Suppose $\zeta(t)$ is a known function which is increasing and has $\zeta(0) = 0$. We could carry out the preceding calibration procedure to determine $H(T)$ from the diagonal swaptions; the procedure does not depend on $\zeta(t)$ being linear.

26.1.19 Calibration to Diagonal Swaptions and Caplets

A Bermudan swaption can be viewed as the most expensive of its component European swaptions, plus an option to "switch" to a different swaption should market conditions change. The component swaptions are just the diagonal swaptions, so calibrating to the diagonals accounts for this part of the pricing. On any exercise date, "switch" option is the option to exercise immediately, or to delay the exercise decision until the next exercise date. Since these delays are short, typically 6 months, one may believe that the switch option can best be represented by short underlyings. Accordingly, one could argue that one should calibrate to either a column of caplets or a column of 1 year underlyings, as well as the diagonal swaptions. Here we calibrate on the caplets and swaptions simultaneously; in the next section we calibrate to the diagonal swaptions and the swaptions with 1 year underlyings.

26.1.20 Calibration to Diagonal Swaptions and a Column of Swaptions

One could argue that caplet and swaption markets have distinct identities, and that mixing the 2 markets introduces small, but needless, noise. Instead one could calibrate on the diagonal swaptions and a column of swaptions with 1 year tenors. (In most currencies, these are the swaptions with the shortest underlying available.)

26.1.21 Other Calibration Strategies

There are many other simple calibration strategies; although they are not overly appropriate for pricing a Bermudan, they may well be appropriate for other deal types.

26.1.21.1 Calibrate on Swaptions with Constant κ or Specified H(T)

Suppose we have chosen a constant mean reversion parameter κ, or have otherwise specified $H(T)$. Then the calibration procedure just needs to find $\zeta(t)$. Suppose we have selected an arbitrary set of n swaptions to be our calibration instruments. In LGM valuation of each swaption the only unknown parameter is $\zeta(t)$ at the swaption's exercise date. Using a global Newton's method to calibrate each swaption to its market value thus determines $\zeta(t)$ and the exercise dates $\tau_1, \tau_2, \ldots, \tau_n$ of the n swaptions. After obtaining the $\zeta j = \zeta(\tau j)$, we need to ensure that $\zeta(\tau j)$ are non-decreasing, altering the offending values if necessary. We then include the value $\zeta 0 = \zeta(0) = 0$, and use piecewise linear interpolation to obtain $\zeta(t)$ at other dates.

Note that this method fails if 2 swaptions share the same exercise date τ; calibration would either yield the same ζ, in which case one of the swaptions is redundant, or differing ζ, in which case our data is contradictory. If the exercise dates of any 2 swaptions are too close, say within 1–2 months, the results may be problematic. For this reason one usually ensures that the swaption exercise dates are, say, at least 2½ months apart, excluding instruments from the calibration set to achieve this spacing, if necessary.

26.1.21.2 Calibrate on Swaptions with Specified ζ(t)

Suppose we have chosen a linear $\zeta(t)$, or otherwise specified parameter $\zeta(t)$. The calibration procedure just needs to find $H(T)$. Suppose we have selected an arbitrary set of n swaptions to be our calibration instruments. We can then arrange the swaptions in increasing order of their final pay dates. Let these final pay dates be $T_1, T_2, \ldots, T_n$. Suppose we use our invariance to set $H = H(0) = 0$, and we use piecewise linear interpolation

$$H(T) = \Delta_1 T \qquad \text{for } T < T_1,$$

$$H(T) = \sum_{i=1}^{k-1} \Delta_i(T_i - T_{i-1}) + \Delta_k(T - T_{k-1}) \qquad \text{for } T_{k-1} < T < T_k,$$

$$H(T) = \sum_{i=1}^{n-1} \Delta_i(T_i - T_{i-1}) + \Delta_n(T - T_{n-1}) \qquad \text{for } T_n < T$$

where $T_0 = 0$.

For the first swaption, the slope Δ_1 determines the value of $H(T)$ at all the swaption's pay dates. Since $\zeta(t)$ is known, the LGM value of the swaption depends only on a single unknown quantity, Δ. It is easily seen that the value is an increasing function of $\Delta 1$, so one can use a global Newton scheme to find the unique $\Delta 1$ which matches the swaption's price to its market value. The value of $H(T)$ at the second swaption's pay dates is determined by both Δ_1 and Δ_2, of which only Δ_2 is unknown at this stage. Again a global Newton scheme can be used to find the Δ_2 needed to calibrate the swaption to its market value. (In rare cases it may occur that $\Delta_2 < 0$; in this case we need to set $\Delta_2 = 0$, its minimum feasible value.)

We then continue in this way, calibrating the swaptions and obtaining the Δ_j's in succession. This method will fail only if 2 deals have the same final pay date, and will work poorly if the final pay dates are too near together. For this reason one usually ensures that the final pay dates are, say, at least 2½ months apart, excluding instruments from the calibration set to achieve this spacing, if necessary.

References

[1] Cox, J., Ross, S.A., Rubinstein, M.: Option Pricing: A Simplified Approach. *Journal of Financial Economics*, 7, pp. 145–166 (1979).
[2] Hull, J.: *Option Futures and Other Derivatives*, New Jersey: Prentice-Hall (2014). ISBN-13: 978-0133456318.
[3] Haug, E.: *The Complete Guide to Option Pricing Formulas*, McGraw-Hill (1997).
[4] Clewlow, L., Strickland, C: *Implementing Derivative Models*, Wiley (2000).
[5] Shreve, Steven.: *Stochastic Calculus for Finance I*, Springer (2005).
[6] Shreve, Steven.: *Stochastic Calculus for Finance II*, Springer (2004).
[7] Munk, Claus.: "Fixed Income Analysis: Securities, Pricing and Risk Management" (2003).
[8] Björk, Tomas.: *Arbitrage Theory in Continuous Time*, Oxford University Press (1998).
[9] Wilmott, Paul.: *On Quantitative Finance*, Wiley (2000)
[10] Neftci, Salih N.: *An Introduction ro the Mathematics of Financial Derivatives*, Academic Press (2000).
[11] Neftci, Salih N.: *Principles of Financial Engineering*, Academic Press (2004).
[12] Ross, Sheldon M.: *Stochastic Processes*, Wiley (1983).
[13] Kijima, Masaaki.: *Stochastic Processes with Application to Finance*, Chapman & Hall (2003).
[14] Ingersoll, J. E., *Theory of Financial Decision Making*, Rowman & Littlefield (1987).
[15] Rose-Anne Dana & Monique Jeanblanc: *Financial Markets in Continuous Time*, Springer (2007).
[16] Emilio Barucci: *Financial Markets Theory*, Springer (2003).

J.R.M. Röman, *Analytical Finance: Volume II*,
https://doi.org/10.1007/978-3-319-52584-6

[17] Kerry Back: *A Course in Derivative Securities*, Springer (2005).
[18] Yue-Kuen Kwok: *Mathematical Models of Financial Derivatives*, Springer (2008).
[19] David F. DeRosa: *Options on Foreign Exchange*, John Wiley & Sons (2000).
[20] Tomas, S. Y. Ho & Sang Bin Lee: *Securities Valuations*, Oxford University Press (2005).
[21] Justin London: *Modeling Derivatives in C++*, Wiley Finance (2005).
[22] Hunt, P. J. & J. E. Kennedy: *Financial Derivative in Theory and Practise*, Wiley (2004).
[23] Martin, J. S: *Applied Math For Derivatives*, Wiley (2001).
[24] Masaaki Kijima: *Stochastic Processes with Applications to Finance*, Chapman & Hall (2003).
[25] Paul Glasserman: *Monte Carlo Methods in Financial Engineering*, Springer (2000).
[26] Curran, Michael: Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price. *Management Science*, 40(12): December 1994: pp. 1705–1711.
[27] Francis, A. Longstaff: Pricing Options with Extendible Maturities. *Journal of Finance*, 45(3): (1989).
[28] Drezner, Z.: Computation of the Bivariate Normal Integral. *Mathematics of Computation*, 32: p. 277 (January1978).
[29] Reiner, E. and Rubinstein, M.: Breaking Down the Barriers. *Risk*, 4 pp. 28–35 (1991).
[30] Stoklosa, J.: *Studies of Barrier Options and their Sensitivities*. The University of Melbourne. (2007).
[31] Robert M.Conroy: *Binomial Option Pricing*. University of Virginia. August (2003).
[32] Schweizer, Martin: *On Bermudan Options*. ETH Zurich (2012).
[33] Glasserman, P: *Monte Carlo Methods in Financial Engineering*, Springer (2003).
[34] Black, F. & Scholes, M.: The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy*, 8 (1973)
[35] Albanese, Claudio & Giuseppe Campolieti: *Advanced Derivatives Pricing and Risk Management*, San Diego: Elsevier Academic Press (2006)
[36] Barone-Adesi, et al.: VaR Without Correlations for Nonlinear Portfolios. *Journal of Futures Markets*, 19: pp. 583–602 (1999).
[37] Berkowitz, J., & J.O'Brien: How Accurate are Value-at-Risk Models at Commercial Banks?. *Journal of Finance* (2002)
[38] Campbell, Sean D.: *A Review of Backtesting and Backtesting Procedures. Finance and Economics, Discussion Series Divisions of Research and Statistics and Monetary Affairs*, Federal Reserve BoardWashington, D.C. (2005)

[39] Glasserman, Paul: *Monte Carlo Methods in Financial Engineering*, Springer (2004)

[40] Jorion, Philippe.: *Value at Risk The New Benchmark for Managing Financial Risk*, McGraw-Hill (2001)

[41] Merton C. Robert: Theory of Rational Option Pricing. *The Bell Journal of Economics and Management Science*, 4(1):pp. 141–183, (1973)

[42] Mark Rubinstein: On the Relation Between Binomial and Trinomial Option Pricing Models. *The Journal of Derivatives*, 8(2): pp. 47–50 (2000)

[43] Lenos Trigeorgis: A Log-Transformed Binomial Numerical Analysis Method for Valuing Complex Multi-Option Investments. *The Journal of Financial and Quantitative Analysis*, 26(3):pp. 309–326. (1991)

[44] Black, F.: The Pricing of Commodity Contracts. *Journal of Financial Economics*,3: pp. 1 (1976)

[45] Davis, M. & Etheridge, A.: *Louis Bachelier's Theory of Speculation: The Origins of Modern Finance*. Princeton University Press (2006).

[46] Merton, C. Robert: Option Pricing when Stock Returns are Discontinuous. *Jornal of Financial Economics*, 3 (1976)

[47] Rendleman, R. J., & Bartter, B. J.: The Pricing of Options on Debt Securities. *The Journal of Financial and Quantitative Analysis* (1980)

[48] Patrick, S. Hagan, & Diana, E. Woodward: Equivalent Black Volatilities March 30, 1998

[49] Martin, S. B., & Laursen Meik, Bruhs: Pricing of Interest Rate Swaps in the Aftermath of the Financial Crisis, Aarhus University August 2011

[50] Ametrano, F., & Bianchetti, M.: Bootstrapping the illiquidity: Multiple yield curves construction for market coherent forward rates estimation. In *Modeling Interest Rates: Latest Advances for Derivatives Pricing*, edited by F. Mercurio, Risk Books (2009)

[51] Bianchetti, M.: Two Curves, One Price: Pricing & Hedging Interest Rate Derivatives Decoupling Forwarding and Discounting Yield Curves, *Quantitative Finance Papers* (2010)

[52] Brigo, D., & Mercurio, F.: *Interest Rate Models: Theory and Practice: With Smile, Inflation, and Credit*, Springer Verlag (2006)

[53] Fujii, M., Shimada, Y., & Takahashi, A.: "A Market Model of Interest Rates with Dynamic Basis Spreads in the Presence of Collateral and Multiple Currencies." (2009).

[54] Fujii, M., Shimada, Y., & Takahashi, A.: "A Note on Construction of Multiple Swap Curves with and Without Collateral." (2009)

[55] Fujii, M., Shimada, Y., & Takahashi, A.: "A Survey on Modeling and Analysis of Basis Spreads." (2009)

[56] Fujii, M., Shimada, Y., & Takahashi, A.: "On the Term Structure of Interest Rates with Basis Spreads, Collateral and Multiple Currencies." (2010)

[57] Hagan, P., & West, G.: Interpolation Methods for Curve Construction. *Applied Mathematical Finance*, 13(2): pp. 89–129. (2006)
[58] Hagan, P., & West, G.: Methods for Constructing a Yield Curve. *WILMOTT Magazine*, May: pp. 70–81 (2008)
[59] Henrard, M.: The Irony in the Derivatives Discounting. *Wilmott Magazine*, 92(98): (2007)
[60] Henrard, M.: The Irony in Derivatives Discounting Part II: The Crisis. *Wilmott Journal*, 2(6): pp. 301–316 (2010)
[61] Mercurio, F.: *Interest Rates and the Credit Crunch: New Formulas and Market Models*, Preprint. Bloomberg, QFR (2009)
[62] Morini, M.: "Solving the Puzzle in the Interest Rate Market (Part 1 & 2)". (2009)
[63] Dan Li: Dilemmas in the Multiple Curve Approach, Numerix Suppport Papers, October2013
[64] Labuszewski, J. W.: *Eurodollar Futures: Interest Rate Market Building Blocks*, CME Group.
[65] Svensson, L. E. O.: *Estimating and Interpreting Forward Interest Rates*, Sweden 1992–1994, NBER Working Papers No. 4871, National Bureau of Economic Research, Inc (1994)
[66] Ion Mihai: The OIS and FVA Relationship, Numerix.
[67] Barbashova, A.: Avoiding Collateral Surprises: Managing Multi-Currency CSAs. Numerix (2013)
[68] Brace, A.: Primer: Curve Stripping with Full Collateralization. Numerix (2013).
[69] Alavian, S.: Funding Value Adjustment (FVA) (2011)
[70] Bianchetti, M., & Carlicchi, M.: Interest Rates After The Credit Crunch: Multiple-Curve Vanilla Derivatives and SABR. *Capco Journal of Financial Transformation, Applied Finance*, 32, August (2012)
[71] Bianchetti, M., & Carlicchi, M.: Markets Evolution After the Credit Crunch (2012) https://arxiv.org/ftp/arxiv/papers/1103/1103.2567.pdf
[72] Bianchetti, M.: The Zeeman Effect in Finance Libor Spectroscopy and Basis Risk Management (2011) https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1951578
[73] Morini, M., & Prampolini, A.: Risky Funding: A Unified Framework for Counterparty and Liquidity Charges. August 30, 2010 https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1669930&rec=1&srcabs=1609587&alg=7&pos=1.
[74] Ferdinando, M. A., & Bianchetti, M.: Everything You Always Wanted to Know About Multiple Interest Rate Curve Bootstrapping But Were Afraid To Ask. April 2, 2013 https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2219548
[75] Ferdinando, M. A., & Bianchetti, M.: Smooth yield curves bootstrapping for forward libor rate estimation and pricing interest rate derivatives.

In *Modelling Interest Rates: Latest Advances for Derivatives Pricing*, edited by FabioMercurio, Risk Books (2009)

[76] Sandmann, K., & Sondermann, D.: Log-Normal Interest Rate Models: Stability and Methodology (1997)

[77] Brace, A., Gatarek, D. & Musiela, M.,The Market Model of Interest Rate Dynamics. *Mathematical Finance*, 7: pp. 127–154 (1997).

[78] Jamshidian, F: LIBOR and Swap Market Models and Measures. *Finance and Stochastics*, 1(4): pp. 293–330 (1997).

[79] Andersen, L., Andreasen, J.: Volatility Skews and Extensions of the Libor Market Model. *Applied Mathematical Finance*, 7 (2000): pp. 1–32.

[80] Andersen, L.: A Simple Approach to the Pricing of Bermudan Swaptions in the Multi-Factor LIBOR Market Model. *Journal of Computational Finance*, 3(2): pp. 5–32 Winter 1999/2000.

[81] Ball, C., & Torous, W.: Bond Prices Dynamics and Options. *Journal of Financial and Quantitative Analysis*, 18: pp. 517–532 (1983).

[82] Schaefer, S., & Schartz, E.: Time-Dependent Variance and the Pricing of Bond Options. *Journal of Finance*, 42(5): pp. 1113–1128 (1987).

[83] Rasy, S., & Sandmann, K.: The Direct Approach to Dept Option Pricing. *The Review of Futures Markets*, 13: pp. 461–514 (1994).

[84] Miltersen, K. R., Sandmann, K., & Sondermann, D.: Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates. *The Journal of Finance*, 52(1): pp. 409–430 (1997)

[85] De Jong, F., Driessen, J., & Pelsser, A.A.J.: Libor Market Models Versus Swap Market Models for Pricing Interest Rate Derivatives: An Empirical Analysis. *European Finance Review*, 5(3): pp. 201–237 (2001)

[86] Rebonato, R.: *Modern Pricing of Interest-Rate Derivatives: The LIBOR Market Model and Beyond* (2002) https://www.amazon.co.uk/Modern-Pricing-Interest-Rate-Derivatives-Market/dp/0691089736

[87] William, H. Teukolsky, Saul A. Vetterling, William T., Flannery, Brian P: *Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press (2007)

[88] Mari, C.: Single Factor Models with Markovian Spot Interest Rate: An Analytical Treatment. *Decisions in Economics and Finance* (2003).

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