Asset Analytics
Performance and Safety Management
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Kusum Deep
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Logistics, Supply Chain and Financial Predictive Analytics

Theory and Practices

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Asset Analytics

Performance and Safety Management

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Cooperative/Non-cooperative Supply Chain Models for Imperfect Quality Items with Trade Credit Financing

Rita Yadav, Sarla Pareek and Mandeep Mittal

1 Introduction

Game theory is a powerful mathematical tool, which studies the capacity of the players. This theory plays a significant role in the area of supply chain connected problems whose purpose is to construct supply chain policies under numerous assumptions with different perspectives. These policies demonstrate coordination between several channels in the supply chain to get effectual outcome. Now a days, the maximum supply chain industries are using credit period policy to improve the profit of both the partners of the supply chain. The trade credit policy is generally offered to the buyer by the seller which is authorized settlement between buyer and seller for the late payment. Many researchers explored their study in this area. Hayley and Higgins [1] studied the buyer's lot size problem having a trade credit contract by assuming a fixed demand and showed that lot size is not affected by the length of trade credit period. Kim et al. [2] formulated a mathematical model to find out the optimal trade credit period with the assumption that the seller's price is fixed. Hwang and Shinn [3] showed in his study that the order quantity of the buyer fluctuates with the length of the trade credit period by considering demand, price sensitive.

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Primarily, Schwaller [4] developed *EOQ* models on defective items by counting inspection cost. Some researchers like Jamal et al. [5] and Aggarwal and Jaggi [6] considered under the fixed demand, the deteriorating item problems with permissible delay in payments, the market demand depends on the selling price of the buyer; Jaggi and Mittal [7] established policies for imperfect quality deteriorating items under inflation and permissible delay in payments. Thangam and Uthaya [8] developed an *EPQ* model for perishable items and with trade credit financing policy in which demand depends on both the variable's selling price and the trade credit period. Under permissible delay in payments, Jaggi and Mittal [9] developed an inventory model for imperfect quality items in which shortages are permitted and fully backlogged. Shortages are eliminated during the screening process and with the assumption that demand rate is less than the screening rate. Zhou et al. [10] developed a specific condition in the Supplier-Stackelberg game in which the trade credit policy not only increases the overall supply chain profit but also gives benefits to each partner of the supply chain.

Zhou et al. [11] also found a synergic economic order quantity model, in which the concept of imperfect quality, shortages with trade credit, and inspection errors are considered. Abad and Jaggi [12] deliberated supply chain model, in which the end market demand is price sensitive and with trade credit period offered by the seller to the buyer. They developed non-cooperative (Seller-Stackelberg game) and cooperative (Pareto-efficient solution) relationship between buyer and seller. Esmaeili et al. [13] also explained cooperative and non-cooperative games in which the end demand was price sensitive as well as marketing expenditure sensitive. This work was extended by Esmaeili and Zeephongscul [14] for asymmetric information game. Zhang and Zeephongsekul [15] investigated non-cooperative models with credit option by assuming the same demand function.

None of the researchers considered the effects of imperfect production on the supply chain model in the cooperative and non-cooperative environment with trade credit financing. In this paper, we develop a supply chain model for imperfect items with credit option which is offered by the seller. Seller delivers a lot to the buyer. In a delivered lot, some items may not be of perfect quality. Thus, buyer applies inspection process to separate defective items on the supplied lot. The perfect quality items are sold at selling price, and imperfect items are sold at a discounted price. In this paper cooperative and non-cooperative relationship are derived between the two players (buyer and seller) of the game. Effect of credit period is also considered when the demand is price sensitive.

This paper consists of seven sections. The first section consists of an introduction and literature review. The second section introduces notation and assumptions used in the paper. The third section formulates the noncooperative Seller-Stackelberg mathematical model, in which seller is the leader and buyer is the follower. In the fourth section, we discuss the cooperative game model with Pareto-efficient approach. In the fifth and sixth sections, we provide numerical examples with sensitivity analysis. The last section consists of conclusion part with suggestions for future research work.

2 Notations and Assumptions

2.1 Notations

The notations that are used in the proposed models are given below:

Decision variables	
c_b	Buyer's unit purchasing cost ($/unit)
Q	Order quantity of the buyer (units)
M	Credit period offered to the buyer by the seller (year)
Parameters	
A_b	Ordering cost of the buyer ($/order)
H_b	Inventory carrying cost ($/unit/unit time)
p_b	Buyer's retail price ($/unit)
T	Cycle time in years
I_s	Seller's opportunity cost of capital ($/year)
A_s	Ordering cost of the seller ($/order)
I	Inventory carrying cost ($/year)
I_e	Interest earned rate for the buyer ($/year)
I_p	Interest paid rate to the buyer ($/year)
D	Annual demand rate (unit/year)
α	Percentage of defective items delivered by the seller to the buyer
λ	Buyer's screening rate (unit/year)
c_s	Cost of defective items per unit ($/year)
C	Seller's unit purchasing cost ($/unit)
T	Required time to screen the defective items, $t = Q/\lambda$ (years)

2.2 Assumption

(1) The annual demand is price sensitive; i.e., $D = Kp^{-e}$ Abad and Jaggi [12].
(2) Shortages are not taken into consideration, and planning horizon is infinite.
(3) In each lot, there is α percentage defective items with known probability density function. In order to avoid shortage, the demand has to be less than the screened lot, Jaggi et al. [16].

(4) A specific credit period is offered by the seller to the buyer. In this paper, interest earned by the buyer, I_e, and interest paid, I_p, are considered equal, Zhang and Zeephongsekul [15].
(5) $I_s = a_1 + a_2 M$, $a_1 > 0$, $a_2 > 0$, Seller's opportunity cost of capital is assumed as linear and function of credit period [1, 2].
(6) There is no carrying cost that links with lot size as the seller considers a lot to lot strategy.

3 Mathematical Models

In this section, mathematical formulation of buyer's model, seller's model, and Seller-Stackelberg game model are explained and optimal solutions are found.

3.1 The Buyer's Model

In this model, the objective of the buyer is to determine the lot size/order quantity, Q, and selling price which maximizes his expected profit. In the lot size, Q, α Q, items are defective and sold at a discounted price c_s and $(1-\alpha)Q$ are non-defective items and are sold at a price p_b. Therefore, the total revenue of the buyer is $p_b(1-\alpha)Q + c_s\alpha Q$.

There are three possible cases (Fig. 1):

Case I: $M \le t \le T$

The interest earned by the buyer for the inventory in the time period 0 to M is $D\frac{M^2 I_e c_b}{2}$, and the interest paid by the player buyer for the inventory not sold after the credit period is $\frac{D(T-M)^2 I_p c_b}{2} + c_s I_p \alpha Q(t-M)$. Hence, buyer's total profit is expressed as $TP_{b1}(p_b, Q)$.

$TP_{b1}(p_b, Q)$ = Sales Revenue − Purchasing cost − Ordering cost − Inventory carrying cost + Interest gain − Interest paid

$$= p_b(1-\alpha)Q + c_s\alpha Q - c_b Q - A_b - \left(\frac{Q(1-\alpha)T}{2} + \frac{\alpha Q^2}{\lambda}\right)H_b + D\frac{M^2 I_e c_b}{2} - \frac{D(T-M)^2 I_p c_b}{2} - c_s I_p \alpha Q(t-M)$$

Put $T = \frac{(1-\alpha)Q}{D}$, $t = \frac{Q}{\lambda}$, $H_b = \mathrm{I}c_b$, then buyer's profit is

$$TP_{b1}(p_b, Q) = p_b(1-\alpha)Q + c_s\alpha Q - c_b Q - A_b - \left(\frac{Q^2(1-\alpha)^2}{2D} + \frac{\alpha Q^2}{\lambda}\right)\mathrm{I}c_b + D\frac{M^2 I_e c_b}{2} - \frac{D\left(\frac{(1-\alpha)Q}{D} - M\right)^2 I_p c_b}{2} - c_s I_p \alpha Q\left(\frac{Q}{\lambda} - M\right)$$

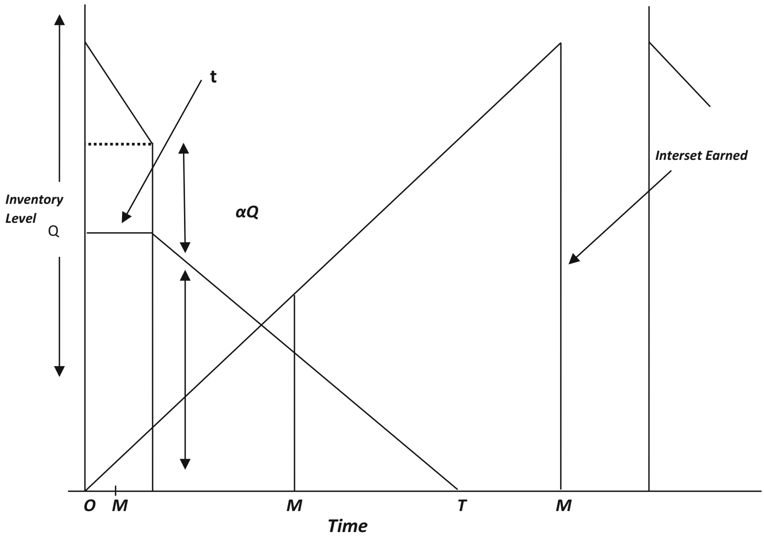

Fig. 1 Inventory system with inspection and trade credit for all the three cases $(i)\, M \leq t \leq T \quad (ii)\, t \leq M \leq T \quad (iii) M \geq T$

$$= p_b(1-\alpha)Q + c_s\alpha Q - c_bQ - A_b - \left(\frac{Q^2(1-\alpha)^2}{2D} + \frac{\alpha Q^2}{\lambda}\right)\mathrm{I}c_b + D\frac{M^2 I_e c_b}{2}$$
$$-\frac{D}{2}\left(\frac{Q^2(1-\alpha)^2}{D^2} + M^2 - 2QM(1-\alpha)I_p c_b\right) - c_s I_p \alpha\frac{Q^2}{\lambda} + c_s I_p QM\alpha$$

Thus, the buyer's total expected profit is given by

$$E\left[TP_{b1}(p_b, Q)\right] = p_b E[1-\alpha]Q + c_s E[\alpha]Q - c_bQ - A_b - \left(\frac{Q^2 E\left[(1-\alpha)^2\right]}{2D} + \frac{E[\alpha]Q^2}{\lambda}\right)Ic_b$$
$$+ D\frac{M^2 I_e c_b}{2} - \left(\frac{Q^2 E\left[(1-\alpha)^2\right]}{2D} + \frac{DM^2}{2} - QME[1-\alpha]\right)I_p c_b - c_s I_p E[\alpha]\frac{Q^2}{\lambda}$$
$$+ c_s I_p QME[\alpha]$$
$$= -\left[\left(\frac{E\left[(1-\alpha)^2\right]}{2D} + \frac{E[\alpha]}{\lambda}\right)Ic_b + \left(\frac{E\left[(1-\alpha)^2\right]}{2D}\right)I_p c_b + c_s I_p\frac{E[\alpha]}{\lambda}\right]Q^2$$
$$+ \left[p_b(1 - E[\alpha] + c_s E[\alpha] - c_b + M(1 - E[\alpha]I_p c_b + c_s I_p ME[\alpha]\right]Q - A_b$$
$$+ D\frac{M^2 I_e c_b}{2} - D\frac{M^2 I_p c_b}{2}$$

By using the concept of renewal theory used in the paper Maddah and Jaber [17], the buyer's expected total profit per cycle is

$$E\left[TP_{b1}^{c}(p_b, Q)\right] = \frac{E\left[TP_{b1}(p_b, Q)\right]}{E(T)}$$

$$= \frac{D}{Q(1 - E[\alpha])} E\big[TP_{b1}(p_b, Q)\big]$$

$$E\big[TP^c_{b1}(p_b, Q)\big] = \frac{D}{Q(1 - E[\alpha])}\Bigg[-\Bigg[\left(\frac{E\big[(1-\alpha)^2\big]}{2D} + \frac{E[\alpha]}{\lambda}\right)Ic_b + \left(\frac{E\big[(1-\alpha)^2\big]}{2D}\right)I_p c_b + c_s I_p \frac{E[\alpha]}{\lambda}\Bigg]Q^2 + \big[p_b(1 - E[\alpha] + c_s E[\alpha] - c_b + M(1 - E[\alpha]I_p c_b + c_s I_p M E[\alpha]\big]Q - A_b + D\frac{M^2 I_e c_b}{2} - D\frac{M^2 I_p c_b}{2}\Bigg] \quad (1)$$

Case II: $t \leq M \leq T$

In this case, interest gain due to credit balance in the time period 0 to M is $D\frac{M^2 I_e p_b}{2} + c_s I_e \alpha Q(M - t)$ and interest paid by the buyer for the period M *to* T is $\frac{D(T-M)^2 I_p c_b}{2}$. The total profit for the player buyer is expressed as $TP_{b2}(p_b, Q)$.

$TP_{b2}(p_b, Q)$ = Sales Revenue − Purchasing cost − Ordering cost − Inventory carrying cost + Interest gain − Interest paid

$$TP_{b2}(p_b, Q) = p_b(1-\alpha)Q + c_s\alpha Q - c_b Q - A_b - \left(\frac{Q(1-\alpha)T}{2} + \frac{\alpha Q^2}{\lambda}\right)\mathrm{I}c_b + D\frac{M^2 I_e p_b}{2} + c_s I_e \alpha Q(M - t) - \frac{D(T-M)^2 I_p c_b}{2}$$

Put $T = \frac{(1-\alpha)Q}{D}, t = \frac{Q}{\lambda}$

Thus, the total expected buyer's profit is given by

$$TP_{b2}(p_b, Q) = p_b(1 - E[\alpha])Q + c_s E[\alpha]Q - c_b Q - A_b - \left(\frac{Q^2 E[(1-\alpha)^2]}{2D} + \frac{E[\alpha]Q^2}{\lambda}\right)\mathrm{I}c_b + D\frac{M^2 I_e p_b}{2} + c_s I_e E[\alpha]Q\left(M - \frac{Q}{\lambda}\right) - \frac{1}{2}D\left(\frac{(1-E[\alpha])Q}{D} - M\right)^2 I_p c_b$$

Using renewal theory used in the paper Maddah and Jaber [17], the buyer's expected total profit per cycle will be

$$\begin{aligned} E\big[TP^c_{b2}(p_b, Q)\big] &= \frac{E\big[TP_{b2}(p_b, Q)\big]}{E(T)} \\ &= \frac{D}{Q(1 - E[\alpha])}\big[p_b(1 - E[\alpha])Q + c_s E[\alpha]Q - c_b Q - A_b \\ &\quad - \left(\frac{Q^2\big[E(1-\alpha)^2\big]}{2D} + \frac{E[\alpha]Q^2}{\lambda}\right)\mathrm{I}c_b + D\frac{M^2 I_e p_b}{2} + c_s I_e E[\alpha]Q\left(M - \frac{Q}{\lambda}\right) \\ &\quad - \frac{1}{2}D\left(\frac{(1-E[\alpha])Q}{D} - M\right)^2 I_p c_b\Bigg] \end{aligned} \quad (2)$$

Case III: $t \le T \le M$

For this case, the buyer's total earned interest is $D\frac{T^2 I_e p_b}{2} + p_b I_e DT(M - T) + c_s I_e \alpha Q(M - t)$ and there is no interest which is payable to the seller by the buyer. The buyer's total profit is expressed as $TP_{b3}(p_b, Q)$.

$TP_{b3}(p_b, Q)$ = Sales Revenue − Purchasing cost − Ordering cost − Inventory carrying cost + Interest gain due to credit

$$= p_b(1-\alpha)Q + c_s\alpha Q - c_b Q - A_b - \left(\frac{Q(1-\alpha)T}{2} + \frac{\alpha Q^2}{\lambda}\right)\mathrm{I}c_b + D\frac{T^2 I_e p_b}{2} + p_b I_e DT(M - T) + c_s I_e \alpha Q(M - t)$$

Put $T = \frac{(1-\alpha)Q}{D}$, $t = \frac{Q}{\lambda}$, thus the total expected buyer's profit is given by

$$\mathrm{T}P_{b3}(p_b, Q) = p_b(1 - E[\alpha])Q + c_s E[\alpha]Q - c_b Q - A_b - \left(\frac{Q^2\left[E(1-\alpha)^2\right]}{2D} + \frac{E[\alpha]Q^2}{\lambda}\right)\mathrm{I}c_b + D\frac{\frac{(E[(1-\alpha)^2])Q^2}{D^2} I_e p_b}{2} + p_b I_e D\left[\frac{(1-E[\alpha])Q}{D}M + \frac{\left(E\left[(1-\alpha)^2\right]\right)Q^2}{D^2}\right] + c_s I_e E[\alpha]Q\left(M - \frac{Q}{\lambda}\right)$$

By using the concept of renewal theory as used in the paper Maddah and Jaber [17], the buyer's expected total profit per cycle is

$$E\left[TP^c_{b3}(p_b, Q)\right] = \frac{E\left[TP_{b3}(p_b, Q)\right]}{E(T)} = \frac{D}{Q(1 - E[\alpha])}\left[p_b(1 - E[\alpha])Q + c_s E[\alpha]Q - c_b Q - A_b - \left(\frac{Q^2 E\left[(1-\alpha)^2\right]}{2D} + \frac{E[\alpha]Q^2}{\lambda}\right)\mathrm{I}c_b + \frac{1}{2}\frac{(E[(1-\alpha)^2])Q^2}{D} I_e p_b + p_b I_e D\left[\frac{(1-E[\alpha])Q}{D}M + \frac{\left(E\left[(1-\alpha)^2\right]\right)Q^2}{D^2}\right] + c_s I_e E[\alpha]Q\left(M - \frac{Q}{\lambda}\right)\right] \tag{3}$$

Under the assumption $I_e = I_p$, it is found that the mathematical expression for buyer's expected profit per cycle for case I, case II, and case III is same [15] and is denoted by $E\left[TP^c_b(p_b, Q)\right]$

Demand is assumed to be price sensitive, $D = Kp_b^{-e}$

Thus, the buyer's expected total profit can be written as

$$E\left[TP^c_b(p_b, Q)\right] = Kp_b^{-e}\left[p_b - A_1 Q - A_2 - \frac{A_3}{Q}\right] - c_b A_4 Q \tag{4}$$

Let, $A_1 = \frac{E[\alpha]Ic_b}{(1-E[\alpha])\lambda} + \frac{c_s I_e E[\alpha]}{(1-E[\alpha])\lambda}$

$$A_2 = \frac{c_b}{(1 - E[\alpha])} - \frac{c_s E[\alpha]}{(1 - E[\alpha])} - \frac{c_s I_e E[\alpha]M}{(1 - E[\alpha])} - c_b I_p M$$

$$A_3 = \frac{A_b}{(1 - E[\alpha])} \quad , \quad A_4 = \frac{E\left[(1-\alpha)^2\right]}{2(1 - E[\alpha])}\left(\mathrm{I} + I_p\right)$$

Differentiating Eq. (4) with respect to p_b for a fixed Q and equating to zero. The resultant equation gives the unique value of p_b that maximize $E\left[TP_b^c(p_b, Q)\right]$. The value of p_b is given below:

$$p_b = \frac{e}{e-1}(A_1 Q + A_2 + \frac{A_3}{Q}) \tag{5}$$

where A_1, A_2, and A_3 are defined by Eq. (4)

Putting the value of p_b into Eq. (4),

$$E\left[TP_b^c\left(p_b^*(Q), Q\right)\right] = \frac{K}{e}\left(\frac{e}{e-1}\left[A_1 Q + A_2 + \frac{A_3}{Q}\right]\right)^{-e+1} - c_b A_4 Q \tag{6}$$

The first-order condition for maximization with respect to Q yields,

$$Q = \frac{kA_3 p_b^{-e}}{c_b A_4 + kA_1 p_b^{-e}} \tag{7}$$

And second-order differentiation of $E\left[TP_b^c(p^*(Q), Q)\right]$ with respect to Q

$$\frac{\partial^2}{\partial Q^2} E\left[TP_b^c\left(p_b^*(Q), Q\right)\right] = \frac{e^2}{(e-1)p_b}\left(A_1 - \frac{A_3}{Q^2}\right)^2 - 2\frac{A_3}{Q^3} \tag{8}$$

$$\text{where} \quad p_b = \frac{e}{e-1}\left(A_1 Q + A_2 + \frac{A_3}{Q}\right)$$

$\frac{\partial^2 E\left[TP_b^c(p,Q)\right]}{\partial p_b^2} > 0$, $\frac{\partial^2 E\left[TP_b^c(p,Q)\right]}{\partial Q^2} > 0$ and

$$\left[\frac{\partial^2 E\left[TP_b^c(p,Q)\right]}{\partial p_b^2}\right]\left[\frac{\partial^2 E\left[TP_b^c(p,Q)\right]}{\partial Q^2}\right] - \left[\frac{\partial^2 E\left[TP_b^c(p,Q)\right]}{\partial p_b \partial Q}\right]^2 < 0$$

It is very difficult to prove the concavity of the expected total profit function analytically.

Thus, expected total profit $E\left[TP_b^c(p_b, Q)\right]$ in Eq. (4) is concave function with respect to p_b and Q for $e \geq 1$ is shown with the help of graph (Fig. 2). Further, first and second derivatives are defined in Appendix A in the end of the paper.

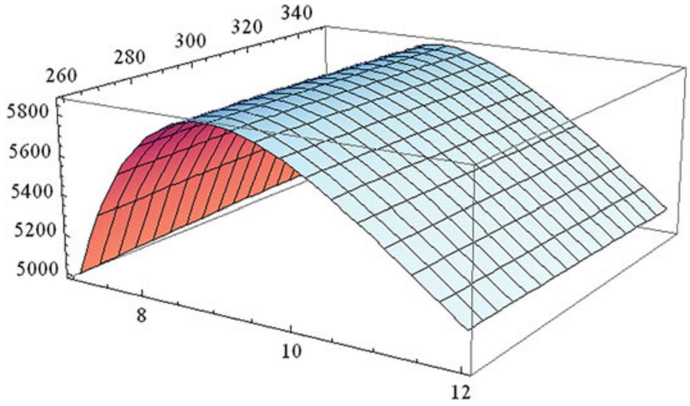

Fig. 2 Optimal buyer's total expected profit with respect to p_b and Q

3.2 Seller's Model

The seller transports products to the buyer by offering a fixed credit period, M to the buyer. The motive of the seller is to find his optimal policies which are selling price, c_b, and length of credit period, M, to maximize his expected profit. Supply chain's cycle length is $\mathrm{T} = (1-\alpha)Q/D$. The player seller considers a lot to lot strategy; thus, there is no inventory cost.

The seller's total annual profit function is

Seller profit = Sales Revenue − Purchasing cost − Ordering cost − Opportunity cost

$$\mathrm{Tp}_s(c_b, M) = c_b Q - CQ - A_s - (a_1 + a_2 M)((1-\alpha)c_b M\, Q)$$

The seller's expected profit per cycle is

$$\begin{aligned} E\left[\mathrm{TP}_s^c(c_b, M)\right] &= \frac{D}{Q(1-E[\alpha])}\left[c_b Q - CQ - A_s - (a_1 + a_2 M)((1-E[\alpha])c_b M Q)\right] \\ &= D\left[\frac{1}{(1-E[\alpha])}\left(c_b - C - \frac{A_s}{Q}\right) - (a_1 + a_2 M)c_b M\right] \end{aligned} \tag{9}$$

For fixed c_b, the first-order condition with respect to M results in

$$\frac{\partial}{\partial M} E\left[\mathrm{TP}_s^c(c_b, M)\right] = -D[a_1 c_b + 2a_2 c_b M]$$

and second-order differentiation yields the result

$$\frac{\partial^2}{\partial M^2} E[\mathrm{TP}_s^c(c_b, M)] = -2a_2 c_b D < 0.$$

Given the expected profit function, $E[\mathrm{TP}_s^c(c_b, M)]$ is concave for a fixed c_b

By Eq. (9), it can be easily seen that $E[\mathrm{TP}_s^c(c_b, M)]$ is linear in c_b,; therefore, the selling price of the seller is unbounded and is denoted by c_b^*, and the seller has to set the selling price by setting zero profit; i.e., $E(\mathrm{TP}_s^c(c_b, M)) = 0$, we have

$$c_{b_0} = \frac{4a_2\left(c + \frac{A_s}{Q}\right)}{4a_2 + a_1^2(1 - E[\alpha])}$$

$c_b^* = T\, c_{b_0} = T\frac{4a_2\left(C+\frac{A_s}{Q}\right)}{4a_2+a_1^2(1-E[\alpha])}$, for some $T > 1$ can be obtained through the mediation with the buyer.

3.3 The Non-cooperative Seller-Stackelberg Model

The Stackelberg model is a non-cooperative strategic game model in which both the players (seller and buyer) have interaction with each other. Among them, one player performs the leader and second player behaves as the follower. The leader moves first, and the follower moves sequentially. In this Seller-Stackelberg model, the seller player behaves as a leader, whereas buyer player behaves as follower. The seller (leader) moves first and offers credit period, M, and selling price,c_b to the player buyer. The buyer chooses his best response based on the policies given by the seller with an aim to increase his expected profit. Further, the seller chooses his best strategy for finding the optimum credit period and his selling price based on the buyer's best response.

$$\mathrm{Max}\, E(\mathrm{TP}_s^c(c_b, M))$$

Subject to the conditions

$$p_b = \frac{e}{e-1}(A_1 Q + A_2 + \frac{A_3}{Q}) \tag{10}$$

$$Q = \frac{kA_3 p_b^{-e}}{c_b A_4 + kA_1 p_b^{-e}} \tag{11}$$

$$\text{where} \quad A_1 = \frac{E[\alpha]Ic_b}{(1 - E(\alpha))\lambda} + \frac{c_s I_e E[\alpha]}{(1 - E(\alpha))\lambda}$$

$$A_2 = \frac{c_b}{(1 - E[\alpha])} - \frac{c_s E[\alpha]}{(1 - E[\alpha])} - \frac{c_s I_e E[\alpha] M}{(1 - E[\alpha])} - c_b I_p M$$

$$A_3 = \frac{A_b}{(1 - E[\alpha])}, A_4 = \frac{E\left[(1-\alpha)^2\right]}{2(1 - E[\alpha])}\left(\mathrm{I} + I_p\right)$$

From Eq. (11), we get

$$p_b = \left(\frac{k\left(A_3 - A_1 Q^2\right)}{c_b A_4 Q^2}\right)^{1/e} \tag{12}$$

From Eqs. (10) and (12), we get

$$M = \left(\frac{c_b}{(1-E[\alpha])} - \frac{c_s E[\alpha]}{(1-E[\alpha])} - \frac{e-1}{e} p_b + A_1 Q + \frac{A_3}{Q}\right) / \left(\frac{c_s I_e E[\alpha]}{(1-E[\alpha])} + c_b I_p\right) \tag{13}$$

Putting the values of p_b and M into Eq. (9), this problem becomes

$$\mathrm{Max}\, E(\mathrm{TP}_s^c(c_b, Q)) = \mathrm{K}\, p_b^{-e}\left[\frac{1}{(1 - E[\alpha]))}(c_b - C - \frac{A_s}{Q}) - (a_1 + a_2 M) c_b M\right] \tag{14}$$

where $p_b = \left(\frac{k\left(A_3 - A_1 Q^2\right)}{c_b A_4 Q^2}\right)^{1/e}$

$$M = \left(\frac{c_b}{(1-E[\alpha])} - \frac{c_s E[\alpha]}{(1-E[\alpha])} - \frac{e-1}{e} p_b + A_1 Q + \frac{A_3}{Q}\right) / \left(\frac{c_s I_e E[\alpha]}{(1-E[\alpha])} + c_b I_p\right)$$

$E[\mathrm{TP}_s^c(c_b, Q)]$ is a nonlinear objective function. Solution of the above problem can be obtained for different optimal values of Q and c_b.

4 The Cooperative Game

In the cooperative games, the interaction between different players is established. The Pareto-efficient solution is one approach for cooperative game for solving seller–buyer chain problem in which both the players make effort together with an aim to maximize their profit. This is an economic state or situation in which one player cannot make better off without making another player's worse off. In this approach, the objective is to optimize the profits of both the members by finding retailer price, p_b, selling price of the seller, c_b, trade credit, M, offered by seller and lot size, Q.

The Pareto-efficient solution concept can be obtained by maximizing the combined sum (weighted) of both the players, buyer and seller's expected profit [13].

$$E[JTP_{sb}] = \mu E[\mathrm{TP}_s^c] + (1-\mu) E[\mathrm{TP}_b^c], \quad 0 \le \mu \le 1 \tag{15}$$

$E[JTP_{sb}] = \mu D[\frac{1}{(1-E[\alpha])}(c_b - C - \frac{A_s}{Q}) - (a_1 + a_2 M) c_b M] + (1-\mu) K p_b^{-e}\left[p_b - A_1 Q - A_2 - \frac{A_3}{Q}\right] - c_b A_4 Q$, where A_1, A_2, A_3, and A_4 are defined by Eq. (4)

The first-order condition for maximizing $E[JTP_{sb}]$ with respect to c_b gives the following result

$$\mu = \frac{w_1}{w_1 - w}, \tag{16}$$

$$w = \left(\frac{D}{1 - E[\alpha])} - (a_1 + a_2 M)MD\right)$$

$$w_1 = I_p MD - D\frac{E[\alpha]IQ}{(1 - E[\alpha])\lambda} - \frac{E[(1-\alpha)^{2]}Q}{2(1 - E[\alpha])}\left(\mathrm{I} + I_p\right) - D\frac{1}{(1 - E[\alpha])}$$

And the first-order condition with respect to $Q, p_b\ and\ M$ yields the following results

$$Q = \sqrt{\left(\frac{\mu A_s D}{(1 - E[\alpha])} + (1-\mu)A_3 D\right)/(1-\mu)(A_1 D + c_b A_4)} \tag{17}$$

$$p_b = \frac{e}{e-1}\frac{(w_3(1-\mu) - \mu w_2)}{(1-\mu)}, \tag{18}$$

where

$$w_2 = \frac{1}{(1 - E[\alpha])}[c_b - C - \frac{A_s}{Q}] - (a_1 + a_2 M)c_b M, \quad w_3 = A_1 Q + A_2 + \frac{A_3}{Q}$$

$$M = \frac{1}{2\mu a_2 c_b}\left((1-\mu)\left(\frac{c_s I_e E[\alpha]M}{(1 - E[\alpha])} + c_b I_p M\right) - \mu a_1 c_b\right) \tag{19}$$

5 Numerical Examples

Example 1 Suppose $C = \$3$ units, $A_b = \$40$, $A_s = \$300$, $a_1 = 0.08, a_2 = 0.06, I_e = 0.15, I_p = 0.15$, $c_s = 5$, $\mathrm{I} = 0.12, e = 2.5$, $\mathrm{K} = 400000, \lambda = 125280$ unit/year. The fraction of imperfect quality item, α, uniformly distributed on $(a, b), 0 < a < b < 1\ i.e., \alpha \sim \mathrm{U(a, b)}$ with $\alpha \sim \mathrm{U(a, b)}$, $\mathrm{E}[\alpha] = \frac{\mathrm{a+b}}{2}$ and can be determined with the formula $E[(1-\alpha)^2] = \int_a^b (1-\alpha)^2 f(\alpha) = \frac{a^2+ab+b^2}{3} + 1 - a - b$; the expected value of α is $E[\alpha] = 0.01$, $E[(1-\alpha)^2] = 0.980$, where $a = 0$ and $b = 0.02$. Then from Eq. (14), the results are $Q = 292$ units and $c_b = \$5.72$.Eqs. (12) and (13) yield $p_b = \$9.070$ and $M = 0.485$ years. With these results, the end demand $D = 1613$ units. The buyer's expected profit, $E\left[(\mathrm{TP}_b^c\right] = \$5631.330$, and the seller's profit $E\left[(\mathrm{TP}_s^c\right] = \$2266.440$. The cycle length $T = 0.179$ years.

Example 2 We consider that all parameters assume the same values of as defined in Example 1 except $c_s = 4$. Suppose the seller and buyer agree at $c_b = \$4.4/$ unit through negotiation under the cooperative approach. Then using Eqs. (16)–(19), we obtain $\mu = 0.505$, $p_b = \$5.080$/unit, $Q = 2022$/unit, $M = 0.570$ years. The end demand D = 6877 units. The cycle length is 0.294. The seller's profit $E\left[(\mathrm{TP}_s^c\right] = \$6725$, and the buyer's profit $E\left[(\mathrm{TP}_b^c\right] = \$\ 5932$. It can be easily be seen from the numerical example that although selling price is less, but profits of the seller and buyer are high as compare to non-cooperative Seller-Stackelberg game because of high demand in this case. Through this approach, both players, seller and buyer, get benefited.

6 Sensitivity Analysis

Sensitivity analysis is performed to explore the impact of three parameters: fraction of defective items, α, price elasticity, e, the interest earned, I_e, on decision variable

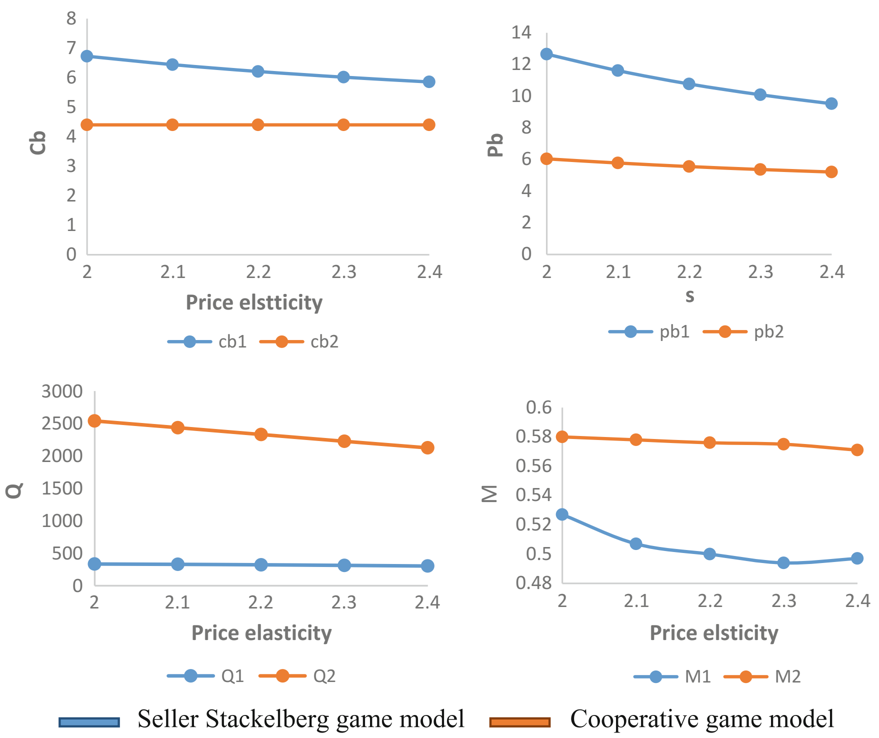

Fig. 3 Effect of e parameter on p_b, M, Q, C_b

of the players, c_b, p_b, M, Q in both the models in the Seller-Stackelberg game and cooperative game. Sensitivity analysis's results are shown in the graph.
Observations:

1. It is evident from Fig. 3 that as the value of ***e*** increases, then there are considerable decreases in selling price, less demand which results in decrease in buyer's profit. Findings suggest, there is decrease in the value of credit period, but cycle length increases which results in decrease in the demand and seller's expected profit (Seller-Stackelberg model).
2. It is depicted from Fig. 4 that when fraction of imperfect quality items increases, then order quantity increases. The buyer places orders more frequently as the imperfect rate increases. Demand depends upon the selling price, as the selling price decreases, then the demand increases (Seller-Stackelberg model).
3. It is also observed from the Fig. 5 that increase in I_e, increases credit period which implies that when interest earned by the buyer is high which results in

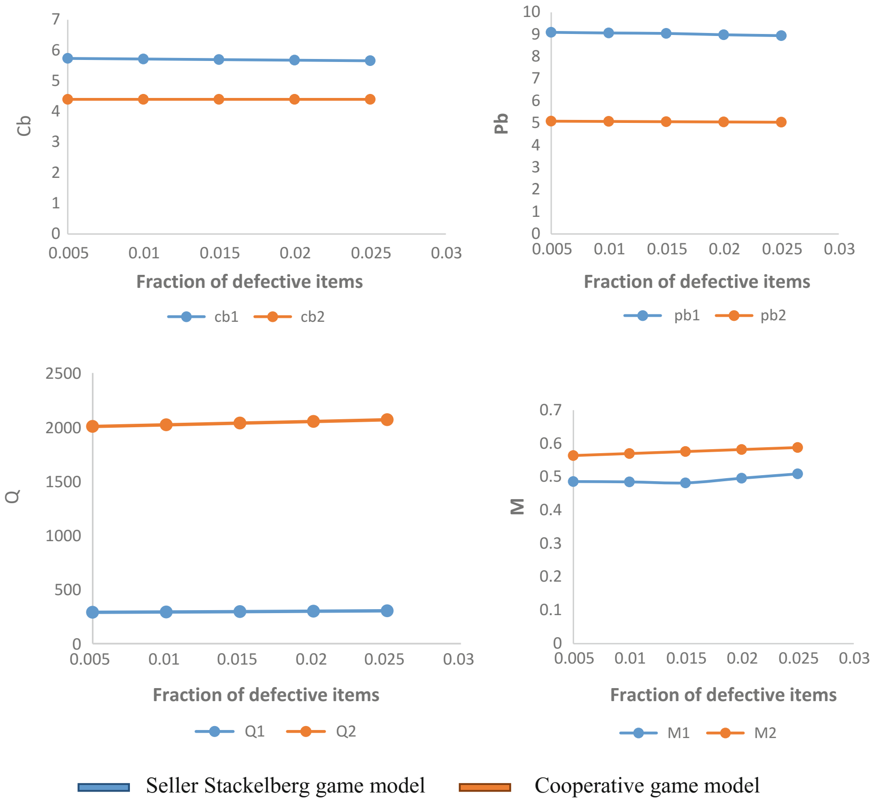

Fig. 4 Effect of α parameter on p_b, M, Q, C_b

higher expected profit. When I_e increase, order quantity decreases which results in less seller's profit (Seller-Stackelberg model).

4. It can be also seen from the Fig. 3 that as price of elasticity increase, the selling price decreases significantly and optimal order quantity decreases, which results decrement in seller's and buyer's profit (cooperative game). Selling price in the non-cooperative game is more than in the cooperative game, and optimal order quantity in the non-cooperative game is less than in the cooperative game. Findings suggest that the seller and the buyer get more profit in the cooperative game.
5. Further, Fig. 4 shows that as the expected number of imperfect quality items increases, the optimal order quantity increases, and the cycle length decreases marginally, whereas the retailer's expected profit decreases significantly (cooperative game). If we compare results of Seller-Stackelberg model with cooperative game model in this Fig. 4, we find that order quantity in the cooperative game is more than in the non-cooperative game where selling price is less in cooperative

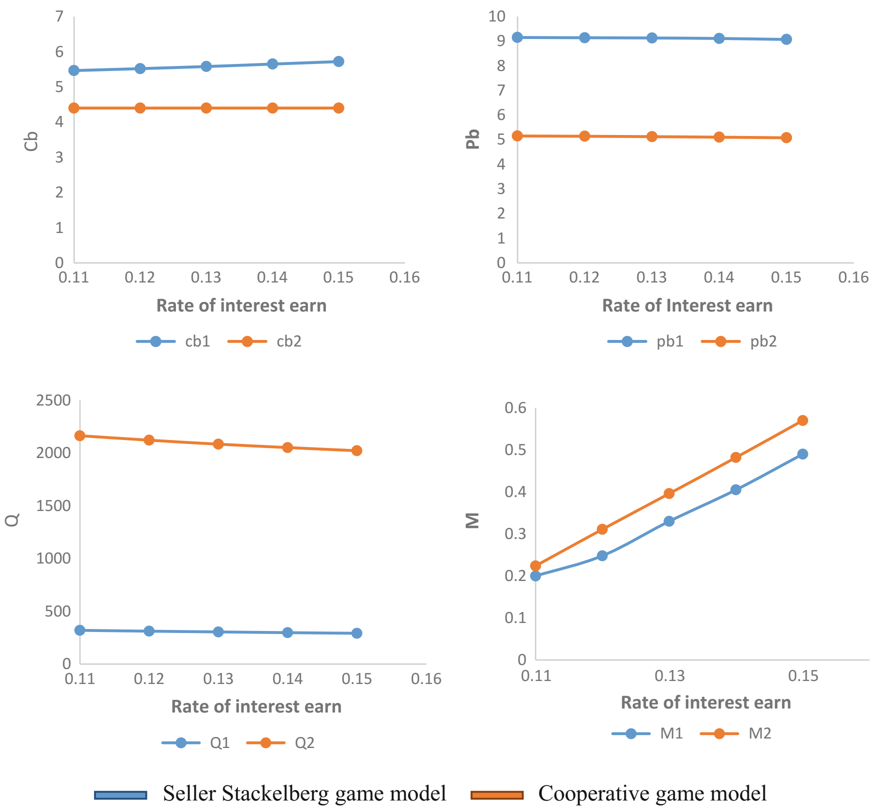

Fig. 5 Effect of I_e parameter on p_b, M, Q, C_b

game. When the number of imperfect quality items increases, the profit of both the players in the cooperative game is more than in the non-cooperative game.

6. Figure 5 investigates that order quantity and market demand in the non-cooperative game is more than in the cooperative game, whereas selling price of the buyer is less in the cooperative game.

7 Conclusions

In this study, non-cooperative and cooperative game theoretic approaches are discussed in which end demand depends on the selling price of the buyer with trade credit period offered by the seller to the buyer. Seller-Stackelberg game is discussed under non-cooperative approach, whereas in cooperative approach, Pareto-efficient solution is obtained. Sensitivity analysis is conducted to show the comparison between Seller-Stackelberg and Pareto-efficient solution concept. This study shows the effect of trade credit financing on the expected total profit of the players of the supply chain. Both the members of supply chain improve their expected profit by using credit financing policy. The study also shows that the expected profit function is impacted by the production of imperfect quality items. Further, it is also clearly observable from the numerical examples that order quantity is increasing due to effect of imperfect production of the items. The result shows that both the players would prefer cooperative game, because they got more profit in cooperative game approach than to non-cooperative game approach. Further, the presented research can be extended to supply chain under asymmetric information game environment.

Appendix 1

Expected profit function for the buyer is given by
$E\left[TP_b^c(p,Q)\right] = Kp_b^{-e}\left[p_b - A_1Q - A_2 - \frac{A_3}{Q}\right] - c_bA_4Q$, where A_1, A_2, A_3 and A_4 are defined by Eq. (4).

First and second derivative with respect to p_b, Q are given below

$$\begin{aligned}\frac{\partial E\left[TP_b^c(p,Q)\right]}{\partial p_b} &= kp_b^{-e} - kep_b^{-e-1}\left(p_b - A_1Q - A_2 - \frac{A_3}{Q}\right)\\ &= (1-e)kp_b^{-e} + kep_b^{-e-1}(A_1Q + A_2 + \frac{A_3}{Q})\end{aligned}$$

$$\frac{\partial^2 E\left[TP_b^c(p,Q)\right]}{\partial p_b^2} = e(e-1)kp_b^{-e-1} - ke(e+1)p_b^{-e-2}\left(A_1Q + A_2 + \frac{A_3}{Q}\right)$$

$$\frac{\partial E\left[TP_b^c(p,Q)\right]}{\partial Q} = kp_b^{-e}\left(-A_1 + \frac{A_3}{Q^2}\right) - c_bA_4$$

$$\frac{\partial^2 E\left[TP_b^c(p, Q)\right]}{\partial Q^2} = kp_b^{-e}\left(-\frac{2A_3}{Q^3}\right)$$

$$\frac{\partial^2 E\left[TP_b^c(p, Q)\right]}{\partial p_b \partial Q} = ekp_b^{-e-1}\left(A_1 - \frac{A_3}{Q^2}\right)$$

where A_1, A_2, A_3 and A_4 are defined by Eq. 4.

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Determination of Initial Basic Feasible Solution for Transportation Problems by: “Supply–Demand Reparation Method” and “Continuous Allocation Method”

Balasubramani Prajwal, J. Manasa and Radha Gupta

1 Introduction

Transportation problems play major role in logistics and supply chain management for decreasing cost and optimizing service. These problems are basically concerned with the optimal distribution of single commodity from several sources (supply origins/plants) to different localities (demand destinations/warehouses). The primary objective in a transportation problem is to find the shipping schedule that minimizes the total cost of shipping goods so that destination requirements are satisfied within the production capacity constraints [1].

In general, transportation problem can be formulated as linear programming problem as follows:

Let there be “m” plants where a commodity is manufactured and stocked (supply point) and “n” markets (destination points). Let “x_{ij}” be number of units shipped from plant i to market j and “c_{ij}” represents the cost of transporting one unit of commodity supply area i to demand point j [1].

$$\text{Minimize } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

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Subject to the constraints,

$$\sum_{j=1}^{n} x_{ij} = a_{i,} i = 1, 2, \ldots, m \quad \text{(Supply constraints)}$$

$$\sum_{i=1}^{m} x_{ij} = b_{j,} j = 1, 2, \ldots, n \quad \text{(Demand constraints)}$$

$$x_{ij} \geq 0$$

The internal dynamics of a transport problem are built on balanced transport problems.

Some of the conceptual assumptions of transportation model are

- Balanced model, i.e. total demand = total supply.
- Unbalanced transport problem can be converted to a balanced transport problem by adding a dummy production centre or consumption centre.
- Commodities are homogeneous.

Transportation modelling using linear programming technique helps in finding an optimal solution to companies and saves the costs in transportation activities. It can be applied to other areas of operations such as inventory control, employment scheduling and personnel assignment [2]. There are several types of transportation models, and Hitchock presented the simplest transportation model in 1941 and in 1949; Koopman developed it.

Numerous extensions of transportation model and methods have been subsequently developed to obtain an optimal solution. There are many efficient methods, and some of them include modified distribution method (Dantzig [3]), stepping-stone method (Charnes and Cooper [4]), the modified stepping-stone method, the simplex-type algorithm and the dual-matrix approach [3–6].

All the transportation algorithms for obtaining optimal solution start with IBFS. There are several effective techniques to find initial solution, such as “LCM—least cost method”, “NWCR—North-West Corner Rule”, “VAM—Vogel’s Approximation Method”, “CMM—column minima method”, “RMM—row minima method” [7].

Though there are many methods existing in the literature to find the IBFS, there is always zeal among the researchers to find a method which can give an IBFS that converges faster to optimal solution. This study presents two new algorithms to find an initial solution for a TP which requires less computational procedures and appropriately gives the same solution as other existing methods.

In Sect. 2, algorithms of proposed procedures CAM and SDRM are discussed. VAM along with both proposed methods is illustrated by considering five problems in Sect. 3. Comparisons and other key observations are discussed in Sects. 4 and 5.

2 Materials and Methods

The methodology in this paper is to find initial basic feasible solution using VAM followed by the proposed methods. It is observed that IBFS obtained by both techniques are either better or same or slightly less than as that of VAM.

VAM is an iterative procedure for calculating a basic feasible solution of a transportation problem. In VAM, penalty cost for each row or column is computed if the smallest cost cell is not selected from that row or column. The allocation is made on the basis of penalty costs that have been incurred if the allocation in certain cells with minimum transportation cost was missed. IBFS obtained using this method is closer to an ideal solution or is the optimal solution itself [8].

2.1 Algorithm of Continuous Allocation Method (CAM)

Continuous allocation method (CAM) is sequential approach to obtain IBFS for TP. Since the aim is to minimize the total transportation cost, start allocation in the least cost cell. The steps involved are summarized below:

1. Select the cell that has smallest cost in the transportation matrix, and allocate the maximum value to the least cost cell to initiate allocation process.
2. Meet the demand or supply requirements in the adjoining row or column.
3. From there on move row-/column-wise to least possible cost in the row/column of the currently allocated cell.
4. Navigate allocating through all rows and columns to complete the allocation.
5. Continue to follow the above steps until the allocations satisfy the demand and supply constraints. In cases of discrepancy/termination, move to the next least cost cell and continue the above steps.

2.2 Algorithm of Supply–Demand Reparation Method (SDRM)

Supply–demand reparation method (SDRM) is an effective procedure to obtain IBFS that takes into account the supply–demand entities to make allocations. As there is not much resource available in the literature where allocations are done by considering these two parameters, this method can be considered as a unique approach to obtain IBFS. The basic steps involved are discussed as follows:

1. From the given data of supply and demand requirements, choosing the supply–demand parameter with the highest value allocates resources in the element with least cost of the concerned row/column.

Table 1 Cost matrix

Origin	Destination				Supply
	B1	B2	B3	B4	
A1	4	19	22	11	100
A2	1	9	14	14	30
A3	6	6	16	14	70
Demand	40	20	60	80	

Table 2 Initial basic feasible solution by CAM

Origin	**Destination**				**Supply**
	B1	B2	B3	B4	
A1	10	10		80	**100**
A2	30				**30**
A3		10	60		**70**
Demand	**40**	**20**	**60**	**80**	

2. Select the next highest supply or demand requirement, and allocate in the least cost element of that concerned row/column.
3. Continue the above steps until all supplies and demands are met.
4. Choose the row or column with the lower cost element that is available to allocate if there is tie or any other discrepancy.

3 Numerical Illustrations

Illustration 1: Three manufacturers (A) of bottled water have supplies of 100, 30 and 70 units, respectively, to be distributed to four markets (B) with demands 40, 20, 60 and 80 units. Each transportation process has an associated cost represented in Table 1. Obtain the initial basic feasible solution [9].

Solution:

The problem is balanced. Supply = Demand = 200

IBFS obtained by VAM: **Rs. 2170** [9] (Table 2).

$$\begin{aligned}\text{Transportation cost} &= (30 \times 1) + (10 \times 4) + (80 \times 11) + (10 \times 19) + (10 \times 6) + (60 \times 16) \\ &= \textbf{Rs. 2160}\end{aligned}$$

Table 3 Initial basic feasible solution by SDRM

Origin	Destination				Supply
	B1	B2	B3	B4	
A1	40			60	100
A2			30		30
A3		20	30	20	70
Demand	40	20	60	80	

Steps

1. Select the cell (2, 1) which is having least cost in the transportation problem, and allot as much as possible (30) to this cell. Since the supply is met at A1, second row is eliminated.
2. Then continue the allocation in the current row/column. Here, in the same column (Column 1) as the row is completely allocated.
3. Choose the next least cost cell, i.e. cell (1, 1), and allocate the maximum possible, i.e. 10. Eliminate column 1 as demand at destination is met. Allocate maximum allocation, i.e. 80 to least cost cell (1, 3) (excluding elements in eliminated rows and columns) in first row.
4. Use the same logic to move from rows to columns and then columns to rows, each time fulfilling the supply–demand requirements.
5. Continue these steps to finally complete all allocations and satisfy the supply and demand. If during the process of making allocation at a particular cell, the supply equals demand, then the next allocation (Table 3).

$$\begin{aligned}\text{Transportation cost(SDRM)} &= (40 \times 4) + (60 \times 11) + (20 \times 6) + (30 \times 14) + (20 \times 14) + (30 \times 16)\\ &= \textbf{Rs. 2120}\end{aligned}$$

Steps

1. Identify the maximum value in the supply and demand parameters. Here, supply of 100 is the highest and hence allocation will take place in first row to begin. The lowest cost in this adjoining row is allocated with maximum resources, cell (1, 1).
2. Similarly, find the next highest parameter, i.e. 80, and allocate in its adjoining lowest cost cell (1, 4).
3. Continue the process of allocation in a similar way till supply–demand constraints are satisfied.

Table 4 Cost matrix

Origin	Destination			Supply
	F1	F2	F3	
C1	6	4	1	50
C2	3	8	7	40
C3	4	4	2	60
Demand	20	95	35	

Table 5 Initial basic feasible solution by CAM

Origin	**Destination**			**Supply**
	F1	**F2**	**F3**	
C1		15	35	**50**
C2	20	20		**40**
C3		60		**60**
Demand	**20**	**95**	**35**	

Table 6 Initial basic feasible solution by SDRM

Origin	**Destination**			**Supply**
	D1	**D2**	**D3**	
C1		15	35	**50**
C2	20	20		**40**
C3		60		**60**
Demand	**20**	**95**	**35**	

Illustration 2: Consider the TP whose cost matrix is (Table 4).
Solution:
The given problem is balanced. IBFS by VAM: **Rs. 555** (Tables 5 and 6).

$$\text{Transportation cost} = (35 \times 1) + (15 \times 4) + (60 \times 4) + (20 \times 8) + (20 \times 3) = \mathbf{Rs.\ 555}$$

$$\text{Transportation cost} = (60 \times 4) + (35 \times 1) + (20 \times 3) + (15 \times 4) + (20 \times 8) = \mathbf{Rs.\ 555}$$

Table 7 Cost matrix [11]

Factory	Destinations							Supply
	P1	P2	P3	P4	P5	P6	P7	
M1	12	7	3	8	10	6	6	60
M2	6	9	7	12	8	12	4	80
M3	10	12	8	4	9	9	3	70
M4	8	5	11	6	7	9	3	100
M5	7	6	8	11	9	5	6	90
Demand	20	30	40	70	60	80	100	

Table 8 Initial solution by CAM

Factory	Destinations							Supply
	P1	P2	P3	P4	P5	P6	P7	
M1			40			20		60
M2	20				60			80
M3				70				70
M4							10	100
M5		30				60		90
Demand	20	30	40	70	60	80	100	

Illustration 3: Consider the transportation matrix:
Solution:
The problem is balanced transportation problem. Supply = Demand = 400, IBFS by VAM = **Rs. 1930** (Table 7).

IBFS by CAM: The allocations done using continuous allocation method are shown in the table. It is observed that transportation cost obtained is less than that of VAM and closer to optimal solution (Tables 8 and 9).

$$\begin{aligned}\text{Transportation cost} &= (20 \times 6) + (30 \times 6) + (40 \times 3) + (70 \times 4) + (60 \times 8) \\ &\quad + (20 \times 6) + (60 \times 5) + (100 \times 3) \\ &= \textbf{Rs. 1900}\end{aligned}$$

Table 9 Initial basic feasible solution by SDRM

Factory	Destinations							Supply
	P1	P2	P3	P4	P5	P6	P7	
M1		20	40					60
M2	20				60			80
M3				70				70
M4							10	100
M5		10				80		90
Demand	20	30	40	70	60	80	100	

$$\text{Transportation cost} = (20 \times 6) + (20 \times 7) + (10 \times 6) + (40 \times 3) + (70 \times 4) + (60 \times 8) + (80 \times 5) + (100 \times 3) = \textbf{Rs. 1900}$$

Illustration 4: The Bombay Transport Company has trucks available at four different sites in the following numbers:

Site	A	B	C	D
No of trucks	5	10	7	3

Customers W, X and Y require trucks as shown:

Customer	W	X	Y
No of trucks	5	8	10

Variable costs of getting trucks to the customers are given in the matrix [12] (Table 10).

Solution:

The problem is unbalanced since the availability of trucks is greater than the requirement of trucks.

Table 10 Cost matrix

Origin	Destination				Supply
	W	X	Z	Dummy	
A	7	3	6	0	5
B	4	6	8	0	10
C	5	8	4	0	7
D	8	4	3	0	3
Demand	5	8	10	2	

Table 11 Initial basic feasible solution by CAM

Origin	**Destination**				**Supply**
	B1	B2	B3	Dummy	
A1		5			5
A2	5	3		2	10
A3			7		7
A4			3		3
Demand	5	8	10	2	

Table 12 Initial basic feasible solution by SDRM

Origin	**Destination**				**Supply**
	B1	B2	B3	Dummy	
A1		5			5
A2	5	3		2	10
A3			7		7
A4			3		3
Demand	5	8	10	2	

Hence, dummy column is created. IBFS obtained by VAM: **Rs. 98** (Tables 11 and 12).

$$\text{Transportation cost} = (2 \times 0) + (5 \times 4) + (3 \times 6) + (5 \times 3) + (7 \times 4) + (3 \times 3) = \textbf{Rs. 90}$$

Table 13 Cost matrix

Origin	Destination					Supply
	M	N	O	P	Q	
X1	46	74	9	28	99	461
X2	12	75	6	36	48	277
X3	35	199	4	5	71	356
X4	61	81	44	88	9	488
X5	85	60	14	25	79	393
Demand	278	60	461	116	1060	

Table 14 Initial basic feasible solution by CAM

Origin	**Destination**					**Supply**
	M	N	O	P	Q	
X1	106	60		116	177	461
X2	172		105			277
X3			356			356
X4					488	488
X5					393	393
Demand	278	60	461	116	1060	

$$\text{Transportation cost} = (2 \times 0) + (5 \times 4) + (3 \times 6) + (5 \times 3) + (7 \times 4) + (3 \times 3) = \mathbf{Rs.\ 90}$$

Illustration 5: Consider the transportation matrix: (Table 13).
Solution:
The problem is balanced.

IBFS obtained by VAM: **Rs. 68804** (Tables 14 and 15).

$$\text{Transportation cost} = (356 \times 4) + (105 \times 6) + (172 \times 12) + (106 \times 46) + (116 \times 28) + (60 \times 74) + (177 \times 99) + (488 \times 9) + (79 \times 393) = \mathbf{Rs.\ 69842}$$

$$\text{Transportation cost} = (488 \times 9) + (277 \times 48) + (356 \times 4) + (105 \times 9) + (116 \times 25) + (278 \times 46) + (277 \times 79) + (60 \times 74) + (18 \times 99) = \mathbf{Rs.\ 63850}$$

Table 15 Cost matrix

Origin	Destination					Supply
	M	N	O	P	Q	
X1	278	60	105		18	461
X2					277	277
X3			356			356
X4					488	488
X5				116	277	393
Demand	278	60	461	116	1060	

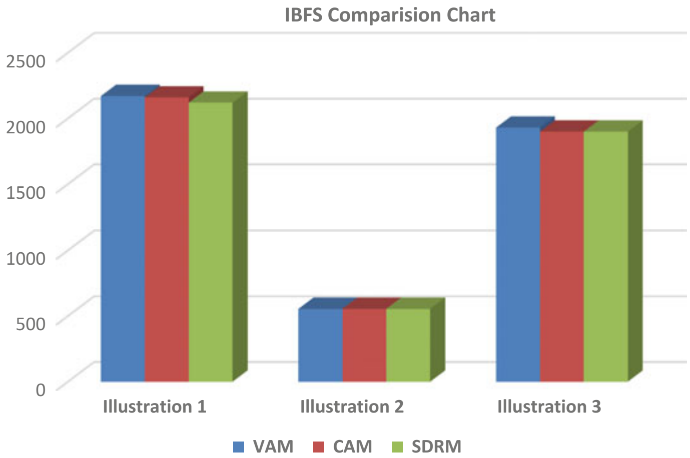

Fig. 1 Comparison of results

4 Results and Discussions

We have considered five problems from different research papers and applied proposed methods to obtain IBFS and also compared with most effective method VAM. Cost matrices considered were of different orders, i.e. 3×4, 3×3, 5×7, 4×3, 5×5. Results obtained for all illustrations and comparison chart for three illustrations are summarized as follows (Fig. 1).

As seen from the results, CAM and SDRM yielded better initial basic feasible solution or tied for best with another procedure (Table 16).

Table 16 Summarization of results

Illustration	Size of matrix	IBFS obtained		
		VAM	CAM	SDRM
1	3×4	2170	2160	2120
2	3×3	555	555	555
3	5×7	1930	1900	1900
4	4×3	98	90	90
5	5×5	68804	69842	63850

Table 17 ANOVA analysis

Source of variation	SS	DOF	Variance	F-ratio	Critical F-value
Between sample	4185792.5333	2	2092896.2667	0.00237	F(12,2) = 19.41
Within sample	10582043022.4	12	881836918.5333		
Total	10586228814.9	14			

ANOVA Test

Null hypothesis: There is no significant difference between the initial basic feasible solutions obtained by all three methods.

We applied one-way analysis of variance (ANOVA) to check the null hypothesis that has been made. The underlying basic principle of ANOVA is to test for difference among the means of the populations by investigating the amount of variation within each of the samples, relative to the amount of variation between the samples [10].

Through ANOVA technique, we can explain whether all three ways of obtaining IBFS for a TP differ significantly or not (Table 17).

The ANOVA table for the problem is summarized as follows.

We see that calculated value of F-ratio is 0.00245 which is below the critical value of 19.41 at 5% level. This analysis supports the null hypothesis. We may, therefore, conclude that difference in solutions obtained by three methods is insignificant and is just a matter of chance.

5 Conclusions

The above-discussed methods on initial solution give another paradigm on solutions with more efficient way. CAM and SDRM give results exactly or even lesser or slightly more than that of VAM method. All necessary qualities of being time efficient, easy applicability, etc., form the core of being implemented successfully. Both the procedures presented in paper involve extremely simple operations. This research paper in a conclusive way unravels to pinpoint on the methodical approach on providing initial basic feasible solution directly in fewer steps.

In a truthful attempt on providing a new method for obtaining IBFS of transportation problem, we may propose SDRM which is unique from all known previous methods as allocations are done considering supply–demand entities.

Future Scope

Having proposed a better methodology for obtaining IBFS, it is required to check its convergence to optimal solution in a faster way. The authors have succeeded in writing code for finding IBFS using SDRM, and they are in the process of writing an algorithm in Python for obtaining the optimal solution using SDRM to find the IBFS.

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Comparison Study on Exponential Smoothing and ARIMA Model for the Fuel Price

Sheik Abdullah Abdul Azees and Ramraj Sasikumar

1 Introduction

Financial analysis and forecasting have an essential role in the fields of finance and the stock market. Most important of the analysis of a time series is the selection of a suitable probability model for the data. Time series analysis has become more significant in various fields of research, such as finance, insurance, business, economics, engineering, medicine, social sciences, and politics which can be used to carry out different goals such as descriptive analysis, comparative analysis, spectral analysis, forecasting, intervention analysis, and explanative analysis.

Exponential smoothing methods are one of the most efficacious methods to use forecasting the time series in seasonal patterns; moreover, it is also easy to adjust for past errors and several different practices are used depending on the presence of trend or cyclical variations. Meanwhile, it is an averaging technique that uses unequal weights where weights are applied to past observations decline in an exponential manner.

The construction of forecast model based on discounted past observations is most commonly carried out using exponential smoothing methods. These methods are very attractive and vital in time series forecasting techniques. This forecast function can be updated very easily in an efficient manner and every time new observations are considered for forecasting and it becomes more popular. They are very easy to implement, and it can be quite effective for forecasting, which is implemented without respect to properly defined statistical models.

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2 Literature Survey

We discuss here various forecasting methods based on SES and ARIMA. Landsman and Damodaran [7] have estimated the James-Stein ARIMA parameter estimator which improves the forecast accuracy relative to other methods, under an MSE loss criterion. Kim [6] estimated the model parameter and the forecasting of AR models in small samples. He was found that (bootstrap) bias-corrected parameter estimators produce more accurate forecasts than the least square estimator.

During the year 1950, Brown extended single exponential smoothing and developed the methods for trends and seasonality in discrete data. The developed model was applied in forecasting the demand for spare parts in Navy inventory systems. Gardner [4] revealed that many believed that exponential smoothing should be disregarded because it was either a special case of ARIMA modeling. Hyndman et al. [5] have discussed a new approach for automatic forecasting based on an extended range of exponential smoothing methods. Taylor [8] introduced a new damped exponential smoothing method which follows the multiplication trend formulation. He suggest that the assumption of a multiplicative trend is not difficult to forecast the values.

3 Materials and Methods

3.1 Data Source

The monthly maximum prices (Rupees) for petrol price are obtained during the period October 2006 to September 2016 by collecting through www.yahoofinance.com.

3.2 Exponential Smoothing Method

Time series data occur frequently in many real-world applications. One of the major important steps in analyzing a time series data is the selection of appropriate statistical model, which will help to prediction, hypothesis testing, and rule discovery. Exponential smoothing is the most efficient method for forecasting the seasonal time series data. An exponential smoothing over an already smoothed time series is called double exponential smoothing. In some other cases, it might be necessary to extend it even to a triple exponential smoothing.

3.2.1 Single Exponential Smoothing

SES calculates the smoothed series as a damping coefficient period for the actual values. The extrapolated smoothed series is a constant, equal to the last value of the smoothed series during the period when actual data on the underlying series are available. The simple moving average method is a special case of the exponential smoothing; the exponential smoothing is more in its data usage. In exponential smoothing, a new estimate is the addition of the estimate for the present time period and a portion of the error $(x_t - \hat{x}_t)$ generated in the present time period, that is

$$\hat{x}_{t+1} = \hat{x}_t + \alpha(e_t) \tag{1}$$

This equation is usually written as

$$S_t = S_{t-1} + \alpha(x_t - S_{t-1}) \tag{2}$$

where

S_t = the new estimated value or forecasted value for the next time period (made in the present time period);

S_{t-1} = the estimated or forecasted value for the present time period (made in the last time period);

x_t= the actual data value in the present time period; $x_t - S_{t-1}$ = estimating or forecasting error for the present time period;

α = a weight value or discount ranges between 0.01 and 1.00.

The smoothing equation can be written as

$$\hat{x}_{t+1} = S_t + \alpha x_t + (1 - \alpha)S_{t-1} \tag{3}$$

or in another way of smoothing equation can be written as follows:
$Next\ period\ forecast = weight\ (present\ period\ observations) + (1 - weight)$ $(present\ period\ forecast)$.

The smoothing equation is constructed based on averaging (smoothing) of past values of a series in decreasing (exponential) manner. The observations are weighted, with the more recent observations are given more weight. The weights are α used for the most recent observation, α(1-α) for the next most recent observation, α(1-α)^2 for the next, and so on. At each time for producing a new forecasted value, the weighted observation along with the weighted estimate for the present period is combined.

$$\hat{x}_{t+1} = \hat{S}_t = \alpha x_t + \alpha(1 - \alpha)x_{t-1} + \alpha(1 - \alpha)^2 x_{t-2} + \cdots + \alpha(1 - \alpha)^{t-1} x_1 \tag{4}$$

or

$$\hat{x}_{t+1} = \hat{S}_t = \alpha \sum_{j=0}^{t} (1 - \alpha)^j x_{t-j} \tag{5}$$

Since, the double exponential smoothing can evaluate in linear trends with no seasonal pattern. The Triple exponential smoothing can handle both trend and seasonal pattern in time series data. Figure 1 shows the selection procedure of different exponential smoothing methods.

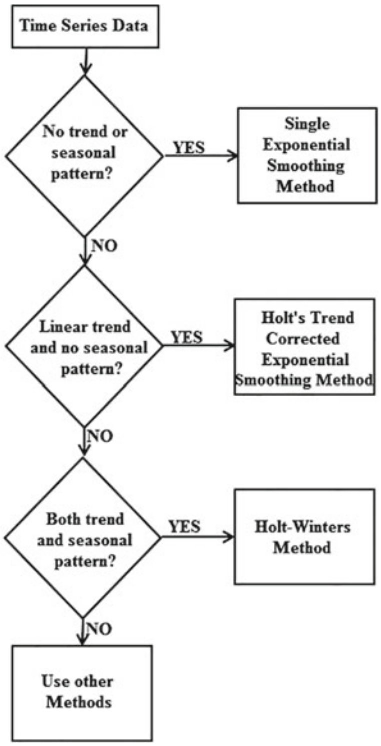

Fig. 1 Selection procedure for exponential smoothing method

3.3 ARIMA Model

ARIMA model was introduced by Box and Jenkins [1], and they recommend differencing non-stationary series one or more times to obtain stationary. The term integrated is used because the differencing process can be reversed to obtain the original series. When the explanatory variables in a regression model are time-lagged values of the forecast variable, then the model is called an autoregressive (AR) model. The general form of an autoregressive model of order p denoted as AR (p) is

$$Y_t = c + b_0 + b_1 Y_{t-1} + b_2 Y_{t-2} + \cdots b_p Y_{t-p} + e_t \tag{6}$$

where e_t is the error or residual term and p is an integer denoting the order in which the observations in the time series are correlated.

When a time series is analyzed using its dependence relation with the past error terms, a moving average (MA) model is applied. The general form of the MA(q) model of order q is

$$Y_t = c + e_t - \varphi_1 e_{t-1} - \varphi_2 e_{t-2} - \cdots - \varphi_q e_{t-q} \tag{7}$$

Autoregressive (AR) model can be efficiently coupled with moving average (MA) model to form a general and useful class of time series models called autoregressive moving average ARMA (p, q) models. However, it can be used only when the time series is stationary. When a time series is studied based on the dependence relationship among the time-lagged values of the forecast variable and the past error terms, an autoregressive integrated moving average (ARIMA) model is more appropriate. It can be used when the time series is non-stationary. The general form of the ARIMA (p, d, q) model is

$$Y_t = c + b_1 Y_{t-1} + b_2 Y_{t-2} + \cdots b_p Y_{t-p} + e_t - \varphi_1 e_{t-1} - \varphi_2 e_{t-2} - \cdots - \varphi_q e_{t-q} \tag{8}$$

where p, d, and q represent, respectively, the order of an autoregressive part, the degree of difference involved in the stationary time series which is usually 0,1 or at most 2 and the order of the moving average part.

An ARIMA model can be obtained by estimating its parameters. The values of p and q can be determined from the patterns in the plotting of the values of ACF and PACF. The spikes falling above the time axis are used to estimate the value of p. The spikes falling below the time axis are used to estimate the value of q. For an AR (p) model, the spikes of ACF decay exponentially or there is a sine wave pattern and the spikes of PACF are close to zero beyond the time lag q whereas the spikes of PACF decay exponentially or there is a sine wave pattern.

4 Results and Discussions

The selected data have no trend and seasonality so we should have to use SES method. Forecasted values were obtained for different α values (0–1). When îś = 0.9, forecasted values are much closer to the actual values compared with the other îś values. The maximum difference of actual and forecasted value is Rs.0.79 for May 2012, and percentage of error is 0.010.

The selected data values have possessed the ARIMA (0, 2, 3) model. Actual values do not possess the stationary conditions so second-order difference is calculated, and it satisfies the stationary, so we have used d=2. Then, the combination of p, d, and q values (without changing the d=2) has found the different BIC values. Among the BIC values, ARIMA (0, 2, 3) has the minimum value (2.16). Based on ARIMA (0, 2, 3) values are forecasted and are presented in Table 1. The maximum difference of actual and forecasted value is Rs.7.90 for May 2012 and Percentage of error is 10.16.

Table 1 Forecasted values using SES and ARIMA

Month	Actual	Forecasted		Month	Actual	Forecasted	
		SES	ARIMA			SES	ARIMA
6-Oct	51.83	51.83	–	11-Oct	70.82	70.79	71.57
6-Nov	49.67	49.89	–	11-Nov	72.73	72.54	71.13
6-Dec	49.67	49.69	47.51	11-Dec	69.55	69.85	73.16
7-Jan	49.67	49.67	48.72	12-Jan	69.55	69.58	69.20
7-Feb	47.51	47.73	49.11	12-Feb	69.55	69.55	69.47
7-Mar	47.48	47.50	46.17	12-Mar	69.55	69.55	69.73
7-Apr	47.48	47.48	46.64	12-Apr	69.55	69.55	69.66
7-May	47.44	47.44	46.92	12-May	77.53	76.73	69.65
7-Jun	47.44	47.44	46.90	12-Jun	75.4	75.53	79.14
7-Jul	47.44	47.44	46.96	12-Jul	72.67	72.96	75.58
7-Aug	47.44	47.44	47.01	12-Aug	72.19	72.27	72.03
7-Sep	47.44	47.44	47.05	12-Sep	72.19	72.20	72.13
7-Oct	47.44	47.44	47.08	12-Oct	71.77	71.81	72.35
7-Nov	47.44	47.44	47.10	12-Nov	70.57	70.69	71.80
7-Dec	47.44	47.44	47.13	12-Dec	70.57	70.58	70.42
8-Jan	47.44	47.44	47.14	13-Jan	70.42	70.44	70.60
8-Feb	49.61	49.39	47.16	13-Feb	72.17	72.00	70.49
8-Mar	49.61	49.59	49.85	13-Mar	72.68	72.61	72.56
8-Apr	49.61	49.61	49.56	13-Apr	69.71	70.00	72.95
8-May	49.64	49.64	49.41	13-May	65.9	66.31	69.22
8-Jun	55.07	54.53	49.50	13-Jun	69.31	69.01	65.08
8-Jul	55.07	55.02	56.15	13-Jul	73.6	73.14	69.84
8-Aug	55.07	55.06	55.41	13-Aug	74.49	74.36	74.70
8-Sep	55.07	55.07	55.00	13-Sep	78.51	78.09	74.86
8-Oct	55.07	55.07	55.12	13-Oct	75.68	75.92	79.28
8-Nov	55.07	55.07	55.13	13-Nov	74.22	74.39	75.45
8-Dec	49.66	50.20	55.11	13-Dec	74.71	74.68	73.81
9-Jan	44.24	44.84	48.55	14-Jan	75.71	75.61	74.86
9-Feb	44.24	44.30	42.70	14-Feb	75.71	75.70	76.03
9-Mar	44.24	44.25	43.82	14-Mar	76.48	76.40	75.82
9-Apr	44.24	44.24	44.10	14-Apr	75.49	75.58	76.67
9-May	44.24	44.24	43.97	14-May	74.6	74.70	75.42
9-Jun	44.24	44.24	43.98	14-Jun	74.73	74.73	74.43
9-Jul	48.58	48.15	43.99	14-Jul	76.93	76.71	74.78
9-Aug	48.58	48.54	49.22	14-Aug	73.6	73.91	77.41
9-Sep	48.58	48.58	48.65	14-Sep	73.47	73.51	73.15
9-Oct	48.58	48.58	48.34	14-Oct	70.23	70.56	73.26

(continued)

Table 1 (continued)

Month	Actual	Forecasted		Month	Actual	Forecasted	
		SES	ARIMA			SES	ARIMA
9-Nov	48.58	48.58	48.44	14-Nov	67.01	67.36	69.72
9-Dec	48.58	48.58	48.45	14-Dec	64.99	65.23	66.25
10-Jan	48.58	48.58	48.44	15-Jan	61.38	61.76	64.49
10-Feb	51.59	51.29	48.44	15-Feb	59.36	59.60	60.63
10-Mar	51.59	51.56	52.05	15-Mar	63.31	62.94	58.76
10-Apr	52.13	52.07	51.66	15-Apr	62.32	62.38	63.91
10-May	52.13	52.12	52.09	15-May	68.76	68.12	62.30
10-Jun	55.92	55.54	52.08	15-Jun	70.12	69.92	69.72
10-Jul	55.92	55.88	56.59	15-Jul	68.99	69.08	70.67
10-Aug	55.92	55.92	56.10	15-Aug	64.13	64.63	68.68
10-Sep	56.16	56.14	55.82	15-Sep	61.46	61.78	63.10
10-Oct	57.09	56.99	56.19	15-Oct	61.02	61.10	60.68
10-Nov	57.26	57.23	57.28	15-Nov	61.38	61.35	60.81
10-Dec	60.65	60.31	57.33	15-Dec	60.28	60.39	61.37
11-Jan	60.65	60.62	61.29	16-Jan	59.77	59.83	59.99
11-Feb	60.65	60.65	60.86	16-Feb	59.42	59.46	59.49
11-Mar	61.93	61.80	60.62	16-Mar	57.09	57.33	59.23
11-Apr	61.93	61.92	62.22	16-Apr	60.95	60.59	56.53
11-May	67.22	66.69	62.05	16-May	62.05	61.90	61.40
11-Jun	67.22	67.17	68.26	16-Jun	65.09	64.77	62.37
11-Jul	67.5	67.47	67.60	16-Jul	63.08	63.25	65.50
11-Aug	67.5	67.50	67.55	16-Aug	60.5	60.77	62.73
11-Sep	70.82	70.49	67.63	16-Sep	59.5	59.63	59.75

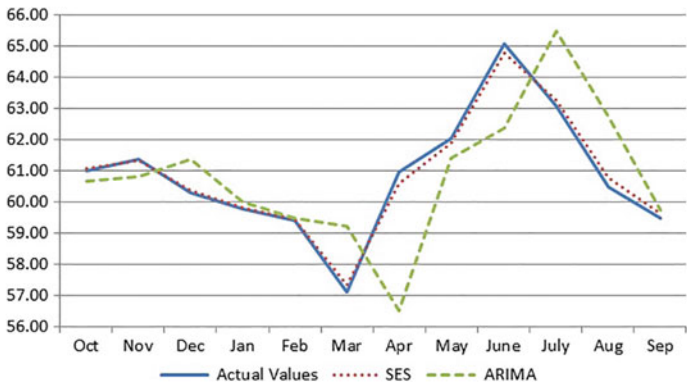

Fig. 2 Comparison of observed and estimated value of petrol price

Table 2 Forecasted error values

Errors	Single exponential smoothing	ARIMA(0, 2, 3)
MSE	0.049	5.058
RMSE	0.221	2.249
MAD	0.140	2.293
MAPE	0.224%	2.381%

From the Table 1, we compare the forecasted values of SES and ARIMA (0, 2, 3) by the SES values are very closer to actual values.

Figure 2 shows the line chart of actual and forecasted values, which shows that actual values are closely associated with the SES model. Table 2 shows MSE, RMSE, MAD, and MAPE.

5 Conclusions

Forecasting of fuel price prediction is a very tedious job because its price values are nonlinear. Over the years, a lot of forecasting techniques have tried and used for forecasting. In this paper, petrol price is forecasted using SES and ARIMA (0, 2, 3). SES is very flexible to use the nonlinear model because we can adjust the values of îś, from which we can reduce the error values. The ARIMA models present some difficulties in estimating and validating the model. The error values such as MSE, RMSE, MAD, and MAPE are compared for two models. In this case, SES model error values are less than the ARIMA (0, 2, 3) model. It concludes that SES model is more appropriate for petrol price prediction.

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An Ordering Policy with Generalized Deterioration, Ramp-Type Demand Under Complete Backlogging

Trailokyanath Singh, Nirakar Niranjan Sethy and Ameeya Kumar Nayak

1 Introduction

In the real-world problems, several factors play important roles to obtain the optimum policies for the deteriorating items. Therefore, various deteriorating inventory models are developed for controlling and maintaining the inventories. From the beginning of the research on inventory, Whitin [1] observed the decaying of fashion items after their expired date. Later, Ghare and Schrader [2] developed the exponentially deteriorating inventory model using the differential terms as: $\frac{dI(t)}{dt} = -(\theta I(t) + R(t)), \quad 0 \leq t \leq T$ where $I(t)$, θ, $R(t)$, and T represent the level of inventory at any time t, rate of deterioration, demand rate, and cycle time, respectively. They also classified the decaying inventory into main categories as direct spoilage, physical depletion, and deterioration. The term, "deterioration" means decay, the physical change or damage that cannot be considered for the use as original it is. Fashion items, vegetables, fruits, automobiles, mobile phones, electronic equipment, chemicals, and radioactive substances are the examples of such items. Most of the deteriorating inventory models are established for the purpose of minimizing inventory costs. In this context, Donaldson [3] suggested an analytical solution procedure for the classical inventory model with linear trend in demand

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and without shortage. Deb and Chaudhuri [4] developed the heuristic for trended inventory model by incorporating shortages.

The deterioration acts as one of the key factors of the deteriorating inventory model that describes the deteriorating nature of items which cannot be neglected while developing the models. Several researchers developed the deteriorating inventory models by considering the constant deterioration rate. However, it is seen in real-life situation that the rate of deterioration rate increases with age. Hence, many deteriorating inventory models are developed by assuming the deterioration rate for two-parameter or three-parameter Weibull distribution rate. Covert and Philip [5] extended Ghare and Schrader's model by assuming two-parameter Weibull distribution deterioration rate to study the variable deterioration rate. Later, Philip [6] generalized the work of Covert and Philip by assuming the time to deterioration as the three-parameter Weibull distribution deterioration. Singh and Pattnayak [7] established the EOQ model with trapezoidal-type demand rate, the two-parameter Weibull distribution deterioration rate, and shortages. Raafat [8], Goyal and Giri [9], and Li et al. [10] gave a detailed survey of literature reviews on deteriorating inventory items in 1991, 2001, and 2010, respectively. Goyal and Giri stated that the assumption of the constant demand rate is not always applicable to inventory items like fashion items, electronics equipment because they experience fluctuations in the demand rate. The age of the inventory has a negative impact on demand due to loss of consumer confidence in the quality of the items and the physical loss of materials. Many items experience a period of raising demand and a period declining demand during the growth phase and the introduction of more attractive substitute items, respectively. These effects have encouraged many researchers for developing models with linearly increasing/decreasing, exponentially increasing/decreasing, quadratic demand patterns, etc. An optimal policy with linear demand rate, generalized deterioration rate, and permissible delay in payment was studied by Singh and Pattanayak [11]. Ghosh and Chaudhuri [12] studied the optimal policy for deteriorating items with consideration of the two-parameter Weibull distribution deterioration and shortages. Ouyang et al. [13] studied an optimal ordering policy with exponential declining demand and partial backlogging. But the ramp-type demand is one in which demand rate increases up to a certain time after which it stabilizes the demand and becomes constant. It depicts the demand rate of the items such as newly launched fashion items, cosmetics, garments, automobiles, and mobile phones for which the demand rate increases as they are launched into the market and after some time it becomes constant. Mandal and Pal [14] studied an order-level inventory model for deteriorating items with ramp-type demand rate. Wu and Ouyang [15] extended Mandal and Pal's model by considering two types of shortages, viz. complete and partial shortages, in their model. Wu [16] studied an EOQ model with Weibull distribution deterioration rate, ramp-type demand rate, shortages, and partial backlogging. Jalan et al. [17] developed the inventory model with shortages by assuming ramp-type demand rate and two-parameter Weibull distribution deterioration rate. Samanta and Bhowmick [18] generalized Wu and Ouang's model by considering the two-parameter Weibull distribution deterioration rate as time to the deterioration and by allowing shortages in the inventory. Skouri et al. [19] established the partial backlogging model considering two-parameter of Weibull deterioration and ramp-type demand to deal with deteriorating items. An inventory model with general ramp-type demand rate, partial backlogging of unsatisfied demand, and time-varying holding cost was studied by Karmakar and Choudhury [20]. The ordering policy with the ramp-type demand rate, time-dependent deterioration rate, unit production cost, and shortage is developed by Manna and Chaudhuri [21]. Jain and Kumar [22] suggested an optimal policy with ramp-type demand, three-parameter Weibull distribution deterioration, and starting with shortage. Recently, Sanni and Chukwu [23] developed an EOQ model with shortages by assuming ramp-type demand rate and Weibull distribution deterioration rate.

The objectives of the present paper are to determine the optimal order quantity for which the average total cost will be minimum taking into account the factors of ramp-type demand rate, two-parameter Weibull distribution deterioration rate, and shortages with complete backlogging. In real-life situations, the occurrence of shortages in inventory is a natural phenomenon and the deterioration over time is a natural feature for items. Demand is completely backlogged for a certain period and for the rest of the period as a function of ramp-type demand rate. This model is much useful for analyzing situations for newly launched high-tech products like android mobiles, 4G SIM cards, and automobiles, and seasonal items, etc. Using the differential equation, the instantaneous state of on-hand inventory is derived. Actually, the deterioration rate is not always constant. In this study, an optimal policy for deteriorating items is developed in which the deterioration rate is taken to have a two-parameter Weibull distribution deterioration rate and shortages are permitted to make the model relevant and more realistic. With suitable cost consideration, the average total cost function is derived. By minimizing the average total cost function, the optimal order quantity is obtained. Now with the help of numerical examples, the model has been illustrated and sensitivity analysis of the model with respect to parameters and costs is also discussed.

2 Assumptions and Notations

In this section, we briefly present the assumptions and notations used for developing the inventory model.

(i) The inventory deals with single item only.
(ii) The inventory system is considered over infinite time horizon.
(iii) Zero time level is considered at the beginning.
(iv) A ramp-type function of time is considered for demand rate since it varies linearly with time up to a fixed time and then stays constant.
(v) The deterioration rate is taken a two-parameter Weibull distribution deterioration rate because the items in the inventory start deteriorating the instant they are received into the inventory.
(vi) The ordering cost is fixed regardless of the lot size.

(vii) The inventory holding cost, shortage cost, and deterioration cost remain constant over time.
(viii) The model allows shortages, which are completely backlogged.
(ix) The repair or replacement of the deteriorated units during the cycle time are not allowed.

Notations

$I(t)$ The inventory level at any instant of time t during the period $[0, T]$.
$\theta(t)$ The deterioration rate at any time t follows the two-parameter Weibull distribution, i.e., $\theta(t) = \alpha\beta t^{\beta-1}$ where $\alpha(0 < \alpha << 1)$ and $\beta(> 0)$ indicate the scale parameter and the shape parameter, respectively. Here, the deterioration rate is decreasing with time, the case of exponential decay, and increasing with time at $\beta < 1$, $\beta = 1$, and $\beta > 1$, respectively.
$D(t)$ The ramp-type demand rate at any time t, i.e., $D(t) = a_0[t - (t - \mu)H(t - \mu)]$. Here, $a_0(> 0)$, $\mu(> 0)$, and $H(t - \mu) = \begin{cases} 0, & t < \mu \\ 1, & t \geq \mu \end{cases}$ stand for the initial demand rate, a fixed point in time, and Heaviside unit step function, respectively.
T The cycle length of each ordering cycle.
t_b The procurement time.
c_d The deterioration cost per unit.
h The inventory holding cost per unit per unit time.
c_b The shortage cost per unit per unit time.
c_o The ordering cost per order.
$I_{\max}$ The maximum inventory level for each ordering cycle.
$\mathbb{Z}(t_b)$ The average total cost per unit per unit time.
t_b^* The optimal procurement time.
I_0^* The optimal value of I_0.
$\mathbb{Z}^*(t_b)$ yThe optimal average total cost per unit per unit time

In the next section, we describe the model and present the average total cost per unit time.

3 Model Development

In this section, the deteriorating inventory model with ramp-type demand is developed starting with shortage and ending without shortage. At the beginning of the model, the inventory level is zero and continues up to time $t = t_b$ and then shortage is allowed to occur during the interval $[0, t_b]$. Replenishment occurs at time $t = t_b$ when the inventory level attains its maximum $I_{\max}$. The quantity received partly fulfills the shortages which occurred in the interval $[0, t_b]$, and the rest of the quantity is allowed to meet the demand and the deterioration for the interval $[t_b, T]$. Finally,

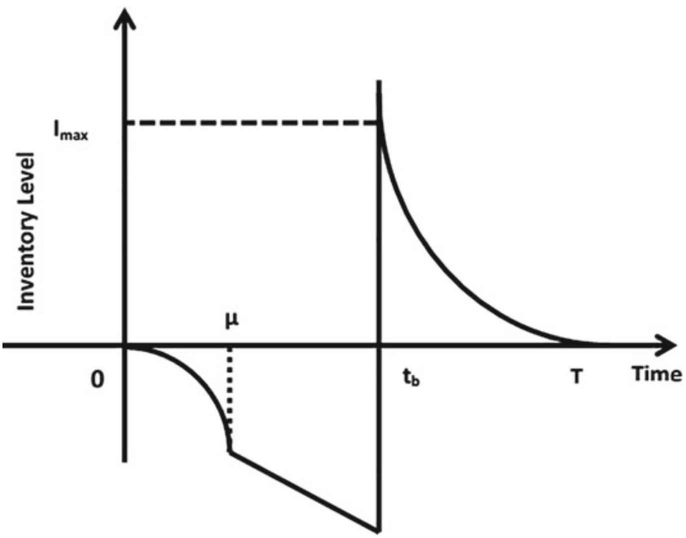

Fig. 1 Relation between inventory level and time

the inventory level diminishes with T (time). The characteristics of the system are presented in Fig. 1.

The instantaneous state of the inventory level $I(t)$ at any time t is governed by the differential equation

$$\frac{dI(t)}{dt} = -a_0 t, \qquad 0 \le t \le \mu, \tag{1}$$

and

$$\frac{dI(t)}{dt} = -a_0 \mu, \qquad \mu \le t \le t_b, \tag{2}$$

with boundary condition $I(0) = 0$.

The solution of Eq. (1) and Eq. (2) are, respectively, given by

$$I(t) = -\frac{a_0 t^2}{2}, \qquad 0 \le t \le \mu, \tag{3}$$

and

$$I(t) = a_0 \mu \left(\frac{\mu}{2} - t \right), \qquad \mu \le t \le t_b. \tag{4}$$

The total demand backlogged ($\mathbb{C}_b$) during the interval $[0, t_b]$ can be computed

$$\mathbb{C}_b = \int_0^{\mu} (a_0 t)dt + \int_{\mu}^{t_b} (a_0\mu)dt = a_0\mu\left(t_b - \frac{\mu}{2}\right). \tag{5}$$

Furthermore, at time $t = t_b$, inventory is procured and part of received quantity is used to meet the demand and deterioration in the interval $[t_b, T]$. Thus, the inventory level $I(t)$ at time t is governed by the differential equation

$$\frac{dI(t)}{dt} = -(\theta(t)I(t) + a_0\mu), \qquad t_b \le t \le T, \tag{6}$$

with boundary condition $I(T) = 0$ and $\theta(t) = \alpha\beta t^{\beta-1}$ where $0 < \alpha << 1$ & $\beta > 0$.

The solution of Eq. (6) is given by

$$I(t) = a_0\mu\left(T + \frac{\alpha T^{\beta+1}}{\beta+1} - t - \frac{\alpha t^{\beta+1}}{\beta+1}\right)e^{-\alpha t^{\beta}}, \qquad t_b \le t \le T, \tag{7}$$

by neglecting the terms containing higher power of α as $0 < \alpha << 1$.

By letting $t = t_b$, the maximum inventory level ($I_{\max}$) is given by

$$I_{\max} = a_0\mu\left(T + \frac{\alpha T^{\beta+1}}{\beta+1} - t_b - \frac{\alpha t_b^{\beta+1}}{\beta+1}\right)e^{-\alpha t_b^{\beta}}. \tag{8}$$

Thus, the initial order quantity (I_0) becomes

$$I_0 = I_{\max} + \mathbb{C}_b = a_0\mu\left[\left(T + \frac{\alpha T^{\beta+1}}{\beta+1} - t_b - \frac{\alpha t_b^{\beta+1}}{\beta+1}\right)e^{-\alpha t_b^{\beta}} + t_b - \frac{\mu}{2}\right]. \tag{9}$$

The following costs are calculated for the determination of the average total cost ($\mathbb{Z}(t_b)$):

(i) Deterioration cost ($\mathbb{k}_d$):

$$\mathbb{k}_d = I_{\max} - \int_{t_b}^{T} (a_0\mu)dt = \alpha a_0\mu\left[\frac{T^{\beta+1} - t_b^{\beta+1}}{\beta+1} - t_b^{\beta}(T - t_b)\right], \tag{10}$$

by neglecting the terms containing higher power of α as $0 < \alpha << 1$.

(ii) Inventory holding cost ($\mathbb{k}_h$):

$$\mathbb{k}_h = a_0\mu \int_{t_b}^{T} \left(T + \frac{\alpha T^{\beta+1}}{\beta+1} - t - \frac{\alpha t^{\beta+1}}{\beta+1}\right)e^{-\alpha t^{\beta}} dt = a_0\mu\left(T + \frac{\alpha T^{\beta+1}}{\beta+1}\right)(T - t_b)$$
$$- a_0\mu\left[\frac{T^2 - t_b^2}{2} + \frac{\alpha T\left(T^{\beta+1} - t_b^{\beta+1}\right)}{\beta+1} - \frac{\alpha\beta\left(T^{\beta+2} - t_b^{\beta+2}\right)}{(\beta+1)(\beta+2)}\right], \tag{11}$$

by neglecting the terms containing higher power of α as $0 < \alpha << 1$.

(iii) The shortage cost ($\mathbb{k}_b$):

$$\mathbb{k}_b = -\left[\int_0^{\mu} \left(-\frac{a_0 t^2}{2}\right)dt + \int_{\mu}^{t_b} \left[a_0\mu\left(\frac{\mu}{2} - t\right)\right]dt\right] = \frac{a_0\mu}{2}\left[\frac{\mu^2}{3} + t_b(t_b - \mu)\right]. \tag{12}$$

(iv) Ordering cost ($\mathbb{k}_s$)

$$\mathbb{k}_s = c_o. \tag{13}$$

The average total cost per unit time during $[0, T]$ can be expressed by establishing the relationship as with the cost of deterioration and inventory holding cost, the shortage, and the ordering cost as:

$$\mathbb{Z}(t_b) = \frac{1}{T}[c_d\mathbb{k}_d + h\mathbb{k}_h + c_b\mathbb{k}_b + \mathbb{k}_s]$$
$$= \frac{a_0\mu h}{T}\left[\left(T + \frac{\alpha T^{\beta+1}}{\beta+1}\right)(T - t_b) - \frac{T^2 - t_b^2}{2} - \frac{\alpha T\left(T^{\beta+1} - t_b^{\beta+1}\right)}{\beta+1} + \frac{\alpha\beta\left(T^{\beta+2} - t_b^{\beta+2}\right)}{(\beta+1)(\beta+2)}\right]$$
$$+ \frac{a_0\mu}{T}\left[\alpha c_d\left(\frac{T^{\beta+1} - t_b^{\beta+1}}{\beta+1} - t_b^{\beta}(T - t_b)\right) + c_b\left(\frac{\mu^2}{6} + \frac{t_b(t_b - \mu)}{2}\right)\right] + \frac{c_o}{T}. \tag{14}$$

The primary aim of this problem is to find the optimal value of t_b, i.e., t_b^* such that $\mathbb{Z}(t_b)$ is minimum.

For the optimum value of $\mathbb{Z}(t_b)$, the respective necessary and sufficient conditions are:

$$\frac{\partial\mathbb{Z}(t_b)}{\partial t_b} = 0 \tag{15}$$

and

$$\frac{\partial^2\mathbb{Z}(t_b)}{\partial t_b^2} > 0. \tag{16}$$

From Eq. (14), we have

$$\frac{\partial \mathbb{Z}(t_b)}{\partial t_b} = \frac{a_0\mu}{T}\left[\alpha\beta c_d t_b^{\beta-1}(t_b - T) + h\left(t_b + \alpha T t_b^{\beta} - \frac{\alpha\beta t_b^{\beta+1}}{\beta+1} - T - \frac{\alpha T^{\beta+1}}{\beta+1}\right)\right] + \frac{a_0\mu c_b(2t_b-\mu)}{2T} = 0. \tag{17}$$

provided the sufficient condition is

$$\frac{\partial^2 \mathbb{Z}(t_b)}{\partial t_b^2} > 0.$$

(See Appendix).

The solution procedure for the above-described model is given below.

4 Solution Procedure: Algorithms

To obtain the optimal value of $\mathbb{Z}^*(t_b)$ and I_0^*, the following steps are adopted.

Step I. Put the appropriate value of the parameters.
Step II. Determine the value of t_b^* from Eq. (17) by Newton–Raphson method.
Step III. Compare t_b^* with μ.

(i) If $\mu < t_b^*$, then t_b^* is a feasible solution. Then, go to Step IV.
(ii) If $\mu > t_b^*$, then t_b^* is infeasible.

Step IV. Substitute t_b^* in Eqs. (14) and (9) to get $\mathbb{Z}^*(t_b)$ and I_0^*, respectively.

5 Numerical Examples

To illustrate the inventory problem with ramp-type demand rate, two-parameter Weibull distribution deterioration rate, and shortages, the following numerical examples are considered.

Example 1 Let us take the parametric values of the inventory model of deteriorating items in their proper units as follows:

Considering $a_0 = 120$, $c_d = 5$, $h = 3$, $c_b = 15$, $c_o = 80$, $T = 1$, $\mu = 0.05$, $\alpha = 0.001$, and $\beta = 1$. Solving Eq. (17), the optimal cycle time is $t_b^* = 0.187781$ year which satisfies the sufficient condition, i.e., $\frac{\partial^2 \mathbb{Z}^*(t_b)}{\partial t_1^2} = 108.045$. Substituting the value of $t_b^* = 0.187781$ in Eqs. (14) and (9), the optimal values of the average total cost and the optimal order quantity are $\mathbb{Z}^*(t_b) = \$87.1506$ and $I_0^* = 5.85198$ units, respectively.

Example 2 Let us take the parametric values of the inventory model of deteriorating items in their proper units as follows:

Considering $a_0 = 150$, $c_d = 5$, $h = 3$, $c_b = 12$, $c_o = 120$, $T = 1$, $\mu = 0.08$, $\alpha = 0.002$, and $\beta = 2$. Solving Eq. (17), the optimal cycle time is $t_b^* = 0.232353$ year which satisfies the sufficient condition, i.e., $\frac{\partial^2 \mathbb{Z}^*(t_b)}{\partial t_1^2} = 179.897$. Substituting the value of $t_b^* = 0.232353$ in Eqs. (14) and (9), the optimal values of the average total cost and the optimal order quantity are $\mathbb{Z}^*(t_b) = \$133.351$ and $I_0^* = 11.5269$ units, respectively.

6 Sensitivity Effects

The variation and changing effects due different parametric values a_0, c_d, h, c_b, c_o, T, μ, α, and β are presented to study the optimum cost and order quantity. The sensitivity analysis is performed by changing the each of the parameters by +50%, +25%, +5%, −5%, −25%, and −50% taking one parameter at a time and keeping remaining parameters unchanged. The analysis is based on Example 1, and the results are shown in Table 1. The following points are observed.

(i) t_b^* remains constant for increment of $\mathbb{Z}^*(t_b)$ and I_0^* but increases with a_0. t_b^* performs no sensitiveness, but $\mathbb{Z}^*(t_b)$ and I_0^* perform average sensitiveness with a_0 variation.

(ii) t_b^* and $\mathbb{Z}^*(t_b)$ increase, while I_0^* remains constant with improvement in c_d. t_b^* and $\mathbb{Z}^*(t_b)$ show weak sensitivity, and variation in c_d shows no sensitiveness in I_0^*.

(iii) t_b^* and $\mathbb{Z}^*(t_b)$ increase with decrement in the values of I_0^* with the increment in h, where t_b^*, $\mathbb{Z}^*(t_b)$, and I_0^* are average sensitive for change of h.

(iv) t_b^* diminishes with the incremented values of $\mathbb{Z}^*(t_b)$ and I_0^* and follows an in c_b. t_b^*, $\mathbb{Z}^*(t_b)$, and I_0^* are found to be moderately sensitive with the variation in c_b.

(v) t_b^* and I_0^* are remained constant, but $\mathbb{Z}^*(t_b)$ shows the improvement with the parameter c_o. Here, t_b^* and I_0^* are insensitive and $\mathbb{Z}^*(t_b)$ are highly sensitive to changes in c_o.

(vi) t_b^* and I_0^* increase, while $\mathbb{Z}^*(t_b)$ increases with the increase in the value of the parameter T. Here, t_b^*, $\mathbb{Z}^*(t_b)$, and I_0^* are highly sensitive to changes in T.

(vii) t_b^*, $\mathbb{Z}^*(t_b)$, and I_0^* increase with the increase in the value of the parameter μ and α. Here, t_b^*, $\mathbb{Z}^*(t_b)$, and I_0^* are moderately sensitive to changes in μ and α.

(viii) t_b^* decreases $\mathbb{Z}^*(t_b)$, and I_0^* finds an increment with the increment in β. In this case, t_b^*, $\mathbb{Z}^*(t_b)$, and I_0^* find weak sensitivity for change of β.

From Table 2, it is observed that, with the decrement of parameter μ by 80%, the value of the optimal procurement time, the optimal average total cost, and the optimal order quantity decrease by 8.87204%, 6.49772%, and 79.5896%, respectively.

Table 1 Sensitivity analysis

Parameters	% Change in parameters	t_b^*	% Change in t_b^*	$\mathbb{Z}^*(t_b)$	% Change in $\mathbb{Z}^*(t_b)$	I_o^*	% Change in I_o^*
a_0	50	0.187781	0	90.7259	3.102	8.77797	50
	25	0.187781	0	88.9382	2.051	7.31497	24.999
	5	0.187781	0	87.5081	0.41	6.14458	5
	−5	0.187781	0	86.793	−0.41	5.55938	−5
	−25	0.187781	0	85.3629	−2.051	4.38898	−25
	−50	0.187781	0	83.5753	−4.102	2.92599	−50
c_d	50	0.187893	0.059	87.1555	0.005	5.85198	0
	25	0.187837	0.029	87.153	0.002	5.85198	0
	5	0.187792	0.005	87.1511	0.001	5.85198	0
	−5	0.187769	−0.006	87.1501	−0.001	5.85198	0
	−25	0.187724	−0.03	87.1481	−0.002	5.85198	0
	−50	0.187668	−0.06	87.1456	−0.005	5.85198	0
h	50	0.250257	33.27	89.8916	3.145	5.85169	−0.004
	25	0.220269	17.301	88.5759	1.635	5.85182	−0.002
	5	0.194493	3.574	87.4451	0.337	5.85195	−0.001
	−5	0.180955	−3.635	86.8511	−0.343	5.85201	0.001
	−25	0.152466	−18.806	85.6013	−1.777	5.85215	0.002
	−50	0.113941	−39.322	83.9111	−3.717	5.85235	0.006
c_b	50	0.139918	−25.488	87.5762	0.488	5.85222	0.004
	25	0.159725	−14.94	87.3996	0.285	5.85212	0.002
	5	0.181272	−3.466	87.2083	0.066	5.85201	0.002
	−5	0.194855	3.767	87.0879	−0.071	5.85194	−0.001
	−25	0.230595	22.8	86.7717	−0.434	5.85178	−0.003
	−50	0.303972	61.875	86.1241	−1.177	5.85145	−0.009
c_o	50	0.187781	0	127.151	45.898	5.85198	0
	25	0.187781	0	107.151	22.949	5.85198	0
	5	0.187781	0	91.1506	4.589	5.85198	0
	−5	0.187781	0	83.1506	−4.589	5.85198	0
	−25	0.187781	0	67.1506	−22.948	5.85198	0
	−50	0.187781	0	47.1506	−45.897	5.85198	0
T	50	0.2713	44.476	64.2365	−26.292	8.85453	51.308
	25	0.229537	22.236	73.0263	−16.206	7.35312	25.651
	5	0.196131	4.446	83.7161	−3.94	6.15219	5.13
	−5	0.17943	−4.447	90.9862	4.401	5.55178	−5.129
	−25	0.146032	−22.232	111.944	28.448	4.35109	−25.647
	−50	0.10429	−44.461	163.408	87.5	2.85047	−51.29

(continued)

Table 1 (continued)

Parameters	% Change in parameters	t_b^*	% Change in t_b^*	$\mathbb{Z}^*(t_b)$	% Change in $\mathbb{Z}^*(t_b)$	I_o^*	% Change in I_o^*
μ	50	0.198193	5.544	90.4705	3.809	8.66539	48.076
	25	0.192987	2.772	88.8307	1.927	7.26807	24.198
	5	0.188822	0.554	87.4899	0.389	6.1367	4.865
	−5	0.186739	−0.554	86.8096	−0.391	5.56651	−4.878
	−25	0.182574	−2.772	85.4287	−1.975	4.41713	−24.519
	−50	0.177368	−5.545	83.6639	−4.001	2.96351	−49.358
α	50	0.187921	0.074	87.1563	0.006	5.85297	0.016
	25	0.187851	0.037	87.1534	0.003	5.85247	0.008
	5	0.187795	0.007	87.1511	0.001	5.85208	0.001
	−5	0.187767	−0.007	87.15	−0.001	5.85188	−0.001
	−25	0.18771	−0.037	87.1477	−0.003	5.85148	−0.008
	−50	0.18764	−0.075	87.1448	−0.006	5.85099	−0.016
β	50	0.187701	−0.042	87.1507	0.001	5.85197	−0.001
	25	0.187741	−0.021	87.1508	0.001	5.852	0.001
	5	0.187773	−0.004	87.1507	0.001	5.85199	0.001
	−5	0.187788	0.003	87.1505	−0.001	5.85196	−0.001
	−25	0.187809	0.014	87.1497	−0.001	5.85185	−0.002
	−50	0.187804	0.012	87.148	−0.002	5.85156	−0.007

Table 2 Parametric effect of μ on optimal factors

Parameters	Change in parameters	Change (%) in μ	t_b^*	Change (%) in t_b^*	$\mathbb{Z}^*(t_b)$	Change (%) in $\mathbb{Z}^*(t_b)$	I_0^*	Change (%) in I_0^*
μ	0.01	−80	0.171121	−8.872	81.4878	−6.497	1.19441	−79.589
	0.1	100	0.208605	11.089	93.6343	7.439	11.4038	94.87
	0.2	300	0.250255	33.269	104.939	20.411	21.6067	269.22
	0.26	420	0.275245	46.577	110.884	27.232	27.1522	363.983
	0.27	440	0.27941	48.795	111.827	28.314	28.0344	379.058
	0.28	460	0.283575	51.013	112.759	29.384	28.9046	393.929
	0.29	480	0.28774	…	…	…	…	…
	0.3	500	0.291905	…	…	…	…	…
	0.4	700	0.333555	…	…	…	…	…
	0.5	900	0.375206	…	…	…	…	…

Here, ". . ." indicates the infeasible solution

However, with the increment of the parameter by 460%, were raised the value of the optimal procurement time, the optimal average total cost and the optimal order to 51.0137%, 29.3841%, and 393.929%, respectively.

From the calculations, it is observed that with the increase of the value μ above $\mu = 0.28$ provides an infeasible solution as $\mu > t_b^*$.

7 Conclusions

To conclude, a model has been illustrated to determine the optimal cost and order quantity with the ramp-type demand rate, a two-parameter Weibull distribution deterioration rate, and shortages. Shortages are allowed and completely backlogged. The inventory model is developed to start with shortages and valid only when the fixed point in time is less than the optimal procurement time. The reason behind it for taking starting with shortages is valid for newly launched automobiles, booking of gas cylinders, high-tech products like android mobiles, 4G SIM cards, and automobiles, and seasonal items, etc. The demand for such items starts with shortages which develops the concept of advance booking of items is the best suitable example for the demand starting with shortages. The average total cost is made up of the inventory holding cost, the shortage cost, the deterioration cost, and the ordering cost. We present some results which lead to the determination of the optimal inventory policy and the minimum total cost per unit of time. The assumptions are very realistic since it will help to determine the optimal order quantity and the optimal average total cost.

There are a number of directions in which this research can be extended. One possible extension stems from the situation when the fixed point in time is greater than the optimal procurement time. For instance, we may extend the ramp-type demand rate to a more realistic time-varying demand rate that is a function of time, trapezoidal-type demand rate, stock-dependent demand, and others. Also, the deterministic inventory model into a stochastic nature model could be extended.

Appendix

$$\frac{\partial^2 \mathbb{Z}(t_b)}{\partial t_b^2} = \frac{a_0 \mu}{T}\left[\alpha\beta c_d t_b^{\beta-2}[(\beta-1)(t_b-T)+t_b] + h\left[1+\alpha\beta t_b^{\beta-1}(T-t_b)\right] + c_b\right].$$

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Inventory Model with Shortages and Deterioration for Three Different Demand Rates

Shalini Singh and G. C. Sharma

1 Introduction

Inventory is the backbone for every organization; along with proper inventory management, it will give new heights and success to the business world. All the activities run smooth if organization have sufficient inventory; it is just like a connection between manufacture and supply procedures. The capital involved in inventory plays an important and critical role in the business world; the profit percentage is also dependent upon the investments. If some organization deals with deteriorating goods, then they have to keep their eye on the level of inventories. Also, they need to know the optimal amount of inventories. Inventory management provides the availability of goods in enough quantity for smooth running of any business enterprise and also helpful in minimizing the cost and maximizing profit. Nowadays, many organizations are manufacturing multi-products at a time and also share the same infrastructure that reduces some cost. The main component in the operational investment of many business organizations is used by inventories as such it is the most important key factor of existing capital of business enterprises in the world. The inventory modelling gives the way to find the optimal solution to firms for their internal operations.

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2 Literature Review

In the field of inventory, Goyal [1] initially developed the perception of delay payments for economic order quantity model. Shah [2] proposed probabilistic inventory model with shortages wherein the late payments are allowed. Aggarwal and Jaggi [3] extended the Goyal's [1] work by introducing the concept of the perishable goods in inventory. Jamal et al. [4] modified Aggarwal and Jaggi's [3] work by adding the concept of shortages to the model. Hwang and Shinn [5] introduced new pricing policy for ELS model and obtained the optimal price for the model by allowing late payments. Liao et al. [6] studied the concept of inflation for perishable items to an inventory model. Chang and Dye [7] applied Weibull distribution to their model and allowed partial backlogging as they considered items to be of deteriorating in nature. The pattern of deteriorating items as linear trend in inventories and delay in payments for the system was considered by Chang et al. [8].

Goyal and Giri [9] studied the deteriorating inventory model with several realistic situations. Teng et al. [10] presented the price-sensitive demand and the concept of delay in payments for his model. Shah [11] considered an inventory model with time value of money for deteriorating items in which payments during a finite planning horizon were permissible. Soni et al. [12] discussed the discounted cash flow (DCF) using the progressive payment scheme for an EOQ model. Ouyang et al. [13] studied EOQ model in which deteriorating items are of non-instantaneous in nature because of those shortages occur, and they also allowed delay payments. An excellent literature review given by Chang et al. [14] was based on trade credit system for inventory models. Rong et al. [15] presented an EOQ model with two warehouses for the perishable goods with fuzzy lead time and partially/fully backlogged shortage. Again, Madhavilata et al. [16] introduced two levels of storage for inventory of single item in their research work. Min et al. [17] developed inventory model in which items are deteriorating exponentially and shortages are allowed. Agrawal et al. [18] also considered an inventory system with two warehouses where demand rate is ramp type and deterioration rate is constant. Sicilia et al. [19] analysed shortages in inventory model where demand is constant and varies with time and follows power pattern. Guchhait et al. [20] developed a model for inventory system with time-dependent deteriorating items to determine the profit maximization. Xu et al. [21] proposed an inventory system periodic review base stock with partial backorders.

In this paper, we propose an inventory model for the production system which consists of deteriorating items, and deterioration rate is time-varying. We also assume three different demand rates, i.e. constant, dependent upon selling price and time-dependent, to deal with three different realistic situations.

3 Notations and Assumptions

Notations used to describe inventory model are given as follows:

- D_1 Constant demand rate; $D_1 = k$.
- D_2 Demand rate is dependent upon selling price; $D_2 = (a - p) > 0$, a is parameter used in demand function, $a > p$, where p denotes selling price/unit item.
- D_3 Demand rate is dependent upon time t; $D_3 = u + vt$, $u > 0$, $0 < v < 1$.
- I(t) Level of inventory at time t, $t \geq 0$.
- θ Deterioration rate function.
- T Length of the interval after which inventories start decline.
- H Entire stretch of time in a system.
- n Number of production cycle during the entire period H.
- p Selling price per unit item.
- I_i Overall quantity of inventory in the *i*th ($i = 1, 2, \ldots, n$) cycle.
- D_i Overall quantity of deteriorated units during the *i*th ($i = 1, 2, \ldots, n$) cycle.
- A Set-up cost of the system.
- C_1 Inventory shortage cost per unit time.
- C_2 Holding cost of inventory per unit time.
- C_3 Unit purchase cost.

Assumptions: For mathematical formulation of inventory model with time-varying deterioration, the following assumptions are made:

(i) Natures of items are deteriorating and dependent upon time.
(ii) Shortages of items are taken into account.
(iii) Three different demand rates are considered.
(iv) Delays in payments are not allowed.
(v) Lead time is considered to be zero.
(vi) Unbounded time horizon is considered.

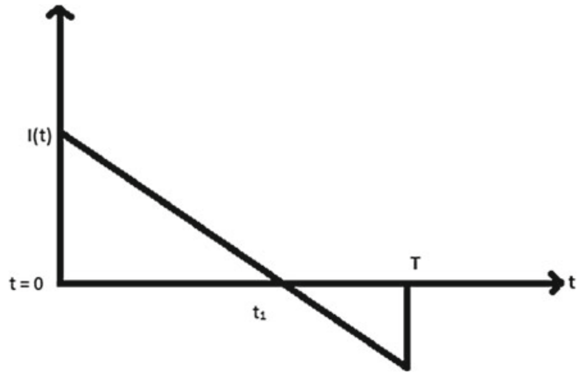
4 Mathematical Modelling

The depiction of inventory model is shown in Fig. 1. Initially, the level of inventory is I(t) at time $t = 0$. Slowly inventory level becomes down as demand increases; it is assumed that the goods in the system start deteriorating with time. At time t_1, the inventory level becomes zero and the shortages start occurring in the system. Formulation of this whole system is given by the following differential equations:

$$\frac{dI(t)}{dt} + \theta t I(t) = -D_i, \qquad 0 \leq t \leq t_1 \tag{1}$$

$$\frac{dI(t)}{dt} = -D_i, \qquad t_1 \leq t \leq T \tag{2}$$

Fig. 1 Inventory level at time t



Here we are taking three different demand rates, as such the solution of Eqs. (1) and (2) are dependent upon demand rates.

4.1 When Demand Is Constant

By applying boundary condition $I(t_1) = 0$ and $I(0) = q$, the solution of Eqs. (1) and (2) is obtained as:

$$I(t) = k\left[(t_1 - t) + a\theta\left(\frac{t_1^3}{6} - \frac{t^3}{3} - \frac{t_1 t^2}{2}\right) + \theta^2\left(\frac{t_1^5}{40} - \frac{t^5}{15} - \frac{t_1^3 t^2}{12} + \frac{t_1 t^4}{8}\right)\right];$$
$$0 \le t \le t_1 \tag{3}$$

$$I(t) = k(t_1 - t); \qquad t_1 \le t \le T \tag{4}$$

Deterioration cost is given by

$$DC_1 = C_3\left(q - \int_0^{t_1} D_i dt\right) = -C_3\left(q - \int_0^{t_1} kdt\right)$$
$$= kC_3\left(\frac{\theta t_1^3}{6} + \frac{\theta^2 t_1^5}{40}\right) \tag{5}$$

Shortage cost is determined by using

$$SC_1 = \int_{t_1}^{t} I(t)dt = -C_1 \int_{t_1}^{t} k(t_1 - t)dt$$
$$= \frac{kC_1}{2}(t - t_1)^2 \tag{6}$$

Holding cost is obtained as

$$HC_1 = C_2 \int_0^{t_1} I(t)dt$$
$$= kC_2\left[\frac{t_1^2}{2} + \frac{\theta t_1^4}{12} + \frac{\theta^2 t_1^6}{90}\right] \tag{7}$$

4.2 When Demand Rate Is Dependent Upon Selling Price

By applying boundary condition $I(t_1) = 0$ and $I(0) = q$, solution of Eqs. (1) and (2) is given by

$$I(t) = (a-p)\left[(t_1 - t) + a\theta\left(\frac{t_1^3}{6} - \frac{t^3}{3} - \frac{t_1 t^2}{2}\right) + \theta^2\left(\frac{t_1^5}{40} - \frac{t^5}{15} - \frac{t_1^3 t^2}{12} + \frac{t_1 t^4}{8}\right)\right];$$
$$0 \le t \le t_1 \tag{8}$$

$$I(t) = (a-P)(t_1 - t); \qquad t_1 \le t \le T \tag{9}$$

Deterioration cost is given by:

$$DC_2 = C_3(a-p)\left[q - \int_0^{t_1} dt\right]$$
$$= C_3(a-p)\left[\frac{\theta t_1^3}{6} + \frac{\theta^2 t_1^5}{40}\right] \tag{10}$$

Shortage cost is

$$SC_2 = \int_{t_1}^{t} I(t)dt = -C_1 \int_{t_1}^{t} k(t_1 - t)dt$$
$$= \frac{(a-p)C_1}{2}(t - t_1)^2 \tag{11}$$

Holding cost is

$$HC_2 = C_2 \int_0^{t_1} I(t)dt$$
$$= (a-p)C_2\left[\frac{t_1^2}{2} + \frac{\theta t_1^4}{12} + \frac{\theta^2 t_1^6}{90}\right] \tag{12}$$

4.3 When Demand Rate Is Depending Upon Time T: We Consider $D_i = U + Vt$

By applying boundary condition $I(t_1) = 0$ and $I(0) = q$, Eqs. (1) and (2) yield

$$I(t) = (u)\left[(t_1 - t) + \frac{\theta}{6}\left(t_1^3 - t^3\right) - \frac{\theta}{2}t^2(t_1 - t)\right] + \frac{v}{2}\left[\left(t_1^2 - t^2\right) + \frac{\theta}{4}\left(t_1^4 - t^4\right) - \frac{\theta}{2}t^2\left(t_1^2 - t^2\right)\right]; \quad 0 \le t \le t_1 \tag{13}$$

$$I(t) = u(t_1 - t) + \frac{v}{2}\left(t_1^2 - t^2\right); \quad t_1 \le t \le T \tag{14}$$

Deterioration cost is

$$DC_3 = C_3\left[q - \int_0^{t_1}(u + vt)dt\right] = C_3\left[\frac{u\theta t_1^3}{6} + \frac{v\theta t_1^4}{8}\right] \tag{15}$$

Shortage cost is

$$SC_3 = \int_{t_1}^{t} I(t)dt = -C_1\int_{t_1}^{t}\left(u(t_1 - t) + \frac{v}{2}\left(t_1^2 - t^2\right)\right)dt = \frac{uC_1}{2}(t - t_1)^2 + \frac{vC_1}{6}\left(t^3 + 2t_1^3 - 3t_1^2 t\right) \tag{16}$$

Holding cost is

$$HC_3 = C_2\int_0^{t_1} I(t)dt = C_2\left[u\left(\frac{t_1^2}{2} + \frac{\theta t_1^4}{12}\right) + \frac{v}{2}\left(\frac{2t_1^3}{2} + \frac{2\theta t_1^5}{15}\right)\right] \tag{17}$$

Total Cost Function

Total average cost per unit for the system is dependent upon their demand rates as we have three different demands for three different situations. Thus,

$$\mathrm{TC_i(t, t_1)} = \frac{1}{\mathrm{t}}(A_i + DC_i + HC_i + SC_i), \ \mathrm{i} = 1, 2, 3 \tag{18}$$

***Case I*: *When demand is constant*.**

$$TC_1 = A + kC_3\left(\frac{\theta t_1^3}{6} + \frac{\theta^2 t_1^5}{40}\right) + \frac{kC_1}{2}(t - t_1)^2 + kC_2\left[\frac{t_1^2}{2} + \frac{\theta t_1^4}{12} + \frac{\theta^2 t_1^6}{90}\right] \tag{19}$$

Case II: ***When demand rate is depending upon selling price***.

$$TC_2 = A + C_3(a-p)\left[\frac{\theta t_1^3}{6} + \frac{\theta^2 t_1^5}{40}\right] + \frac{(a-p)C_1}{2}(t-t_1)^2 + (a-p)C_2\left[\frac{t_1^2}{2} + \frac{\theta t_1^4}{12} + \frac{\theta^2 t_1^6}{90}\right] \tag{20}$$

Case III: ***When demand rate is depending upon time t***.

$$TC_3 = A + C_3\left[\frac{u\theta t_1^3}{6} + \frac{v\theta t_1^4}{8}\right] + \frac{uC_1}{2}(t-t_1)^2 + \frac{vC_1}{6}\left(t^3 + 2t_1^3 - 3t_1^2 t\right) + C_2\left[u\left(\frac{t_1^2}{2} + \frac{\theta t_1^4}{12}\right) + \frac{v}{2}\left(\frac{2t_1^3}{2} + \frac{2\theta t_1^5}{15}\right)\right] \tag{21}$$

To get the maximum profit, we minimize the cost function. We partially differentiate the total cost function with respect to t and t_1 separately and equate them to zero.

$$\frac{\partial TC(t, t_1)}{\partial t} = 0 \quad and \quad \frac{\partial TC(t, t_1)}{\partial t_1} = 0$$

5 Numerical Results

Here we find optimal value of t_1 and calculate total cost TC for all three demand rates:

Case I: ***When demand is constant***.

For fixed parameter values $A = 500$, $k = 300$, $h = 5$, $C_3 = 20$ and $C_1 = 0.8$ and varying value of θ, we obtain the total cost of the system, which is given in Table 1.

Case II: ***When demand rate is depending upon selling price***.

For fixed parameter values $A = 500$, $a = 90$, $p = 20$, $h = 5$, $C_3 = 20$ and $C_1 = 0.8$ and different values of θ, the total cost of the system is given in Table 2.

Case III: ***When demand rate is depending upon time t***.

Here, we set parameters as $A = 500$, $u = 80$, $v = 25$, $h = 5$, $C_3 = 20$ and $C_1 = 0.8$ and vary θ to find total cost of the system. The numerical result for TC is displayed in Table 3.

For numerical results, we have fixed some parameters' values and find optimal value of t_1 and then total cost for the system for all three demand rates, separately. The result obtained shows the effect of parameter θ on total cost. It is noticed that as we increase θ, the total cost also increases which is obvious and can be noticed in the real-time system on t_1.

Table 1 Effect of θ on t_1 and TC for case I

θ	t_1	TC
0.6	0.85	1562.42
0.55	0.85	1524.27
0.5	0.85	1486.47
0.45	0.85	1449.05
0.4	0.85	1411.98

Table 2 Effect of θ on t_1 and TC for case II

θ	t_1	TC
0.60	0.77	735.76
0.55	0.77	726.86
0.50	0.77	718.04
0.45	0.77	709.31
0.40	0.77	700.66

Table 3 Effect of θ on t_1 and TC for case III

θ	t_1	TC
0.60	0.64	589.20
0.55	0.64	586.82
0.50	0.64	584.43
0.45	0.64	582.05
0.40	0.64	579.67

6 Conclusion

In this paper, we have established a mathematical model by including realistic concepts of shortages and deterioration under three different demand rates. This study done is useful for entrepreneurs to run their business smoothly and to minimize the cost function and maximize the profit function. Nowadays, organizations are producing different products which have different patterns of demand which need this type of hybrid model to fulfil customers' demands in minimum time.

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Outlier Labeling Methods for Medical Data

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1 Introduction

It was usual that most of the real-world data sets were affected by outlying observations. Hence, the detection of outlier was considered as preliminary step in any data analysis process. Ferguson [1] defined in a sample of moderate size taken from a certain population, it appears that one or two values which are surprisingly far away from the main group are called outliers. Some applications of outlier identification can be found in credit card fraud detection, criminal behaviors detection, computer intrusion detection, weather prediction, pharmaceutical research, etc. The identification of outlier can be classified into two types: formal methods (test of discordancy) and informal methods (labeling methods).

In this paper, we deal with only the case of outlier labeling methods. The major causes of outlier were categorized by Anscombe [2] as inherent variability, measurement error, and execution error. In case the outlier value is justified as correct, it denotes a rare event. The justification was solely based on the experience of experimenter. The two main reasons for detecting outliers are outliers influencing assumptions of statistical tests and to check for unusual or important values in a data set. The outlier labeling methods create an interval or criterion for detecting outliers. In each labeling method, different location and scale parameters are used to fix a reasonable interval for outlier detection. Gumbel [3] suggest that the rejection of outliers on a purely statistical basis remains a dangerous procedure. Kruskal [4] discussed that no approach may ever be assumed to be the right one in the management of outliers and suggested to carry out an analysis both with and without

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the suspect observation. If the results are quite different, we should be very cautious in data analysis and interpreting results. Hampel [5] suggested that the median and the median absolute deviation to be the robust estimates for the location and spread of the data. Barnett [6] discussed accommodation, identification, and rejection of outliers. Bain and Engelhardt [7] discussed that if the distribution of the data set was normal, the observations two or three standard deviations away from the mean is an outlier. Rousseeuw and Croux [8] proposed an alternative approach of mean absolute deviation (MAD). Olewuezi [9] suggested that the standard deviation was inappropriate to use because it is highly sensitive to extreme values. Barbato et al. [10] used a simple modification of a popular, broadly used method based upon Box plot in order to overcome a major limitation concerning sample size. Leys et al. [11] suggested that standard deviation method was not suitable for outlier detection and used MADe method for robust estimator for the median absolute deviation about the median and discussed the justification of the threshold values. Khaleelur Rahman et al. [12] infer that isolation of outliers improves the quality of the data and many researchers argued that univariate outlier detection methods are useless but favored because outlying data can be hidden in one- or two-dimensional view of the data. Obikee et al. [13] had compared several outlier identification techniques based on the simulation study. Senthamarai Kannan et al. [14] suggested that in univariate case, one of the most robust dispersion scales in the presence of outliers is median absolute deviation and recommended the MADe method for outlier detection.

In this study, the blood pressure data was used for the purpose of comparing the performance of outlier labeling methods. Blood pressure (BP) is the pressure exerted by circulating blood upon the walls of blood vessels. Blood pressure is denoted in systolic (maximum) pressure over diastolic (minimum) pressure in millimeters of mercury (mm Hg). Blood pressure that is low is called hypotension and pressure that is continuously high is called hypertension. Both have many causes and can range from mild to severe. Long-term hypertension is a risk factor for many diseases, including kidney failure, heart disease and stroke. Table 1 describes the classification of blood pressure considered by the American Heart Association for adults of age 18 years and older.

Table 1 Classification of blood pressure for adults

Category	Systolic (mm Hg)	Diastolic (mm Hg)
Hypotension	<90	<60
Desired	$90-119$	$60-79$
Prehypertension	$120-139$	$80-89$
Stage1 hypertension	$140-159$	$90-99$
Stage2 hypertension	$160-179$	$100-109$
Hypertensive urgency	≥ 180	≥ 110

In our study, based on table 1 the blood pressure value less than 90 (< 90) and greater than 160 (> 160) may be considered as outliers. Hence, the method which is close to the interval (90, 160) may be chosen as better method for outlier labeling.

2 Materials and Methods

There are several labeling methods already proposed for the purpose of detection of outliers in a data set. Here we have used six outlier labeling methods, namely the method of Standard Deviation (SD), Median method, Median Absolute Deviation (MADe) method, Z-Score, Modified Z-Score, and Tukey's method.

2.1 Standard Deviation Method (SD Method)

SD method is the simplest classical approach to spot outliers in a given data set. This method uses the measure's mean and standard deviation. It uses less robust measures and also very much affected by extreme values. This method is defined as

$$2SD \ \ Method : \bar{x} \pm 2SD$$

$$3SD \ \ Method : \bar{x} \pm 3SD$$

where
$\bar{x}$ = Sample mean and
SD = Sample standard deviation
The values lying away from this interval are taken as outliers.

2.2 Median Method

Median is the value that lies accurately in the middle of the data when arranged in order Olewuezi [9]. For a skewed distribution, median is one of the robust estimators which was unaffected by outliers. This method is defined as

$$[K_1, K_2] = Q_2 \pm 2.3 \times IQR$$

where
Q_2 = Sample median and
IQR = Interquartile range
The values lying away from this interval are taken as outliers.

2.3 The MADe Method

MADe method is somehow same as the SD method; however, we use median and MADe instead of mean and standard deviation. This method was the robust method which is largely unaltered by the existence of extreme values. It is defined as

$$2MADe \quad Method : Median \pm 2MADe$$
$$3MADe \quad Method : Median \pm 3MADe$$

where MADe =1.483 MAD (for large normal data)

$$MAD = Median|x_i - Median(x)| \qquad\qquad i = 1, 2, \ldots, n$$

The values lying away from this interval are taken as outliers.

2.4 Z-Score

As the SD method, Z-Score method is also affected by extreme values. By using the mean and standard deviation, the Z-Score value is calculated.

$$Z_i = \frac{x_i - \overline{x}}{SD}$$

where
$X_i \sim N(\mu, \sigma^2)$ and
SD = Standard deviation of the data
Z-Scores that go above 3 in absolute value are taken as outliers.

2.5 Modified Z-Scores

The Z-Score method uses sample mean and sample standard deviation which was affected by a few extreme values or by even a single extreme value. To overcome this crisis, the median and the median absolute deviation (MAD) are used in the Modified Z-Scores (Iglewicz and Hoaglin [15]).

$$MAD = Median|x_i - \tilde{x}|$$

where $\tilde{x}$ is the sample median.

This method is defined as

$$M_i = \frac{0.6745(x_i - \tilde{x})}{MAD}$$

where
E(MAD) = 0.675σ for large normal data.
when $|M_i| > 3.5$, the values are outliers.

2.6 Tukey's Method

Tukey's method is a familiar simple graphical method to show some idea about the lower quartile (Q_1), median (Q_2), upper quartile (Q_3), lower extreme and upper extreme of a data set. This method is less sensitive and has no distributional assumptions and also does not rely on a mean or standard deviation.

The formula for the inner fence of the Tukey's method is given by

$$\begin{aligned} Low \quad outliers &= Q_1 - 1.5 \times IQR = Q_1 - 1.5 \times (Q_3 - Q_1) \\ High \quad outliers &= Q_3 + 1.5 \times IQR = Q_3 + 1.5 \times (Q_3 - Q_1) \end{aligned}$$

The formula for the outer fence of the Tukey's method is given by

$$\begin{aligned} Low \quad outliers &= Q_1 - 3 \times IQR = Q_1 - 3 \times (Q_3 - Q_1) \\ High \quad outliers &= Q_3 + 3 \times IQR = Q_3 + 3 \times (Q_3 - Q_1) \end{aligned}$$

where Q_1 = First quartile, Q_3 = Third quartile, IQR = Interquartile range.

The values between inner fence and outer fence are called as outside values, and values away from outer fences are called as far out values.

3 Computational Results

For this study, the blood pressure data was collected from University Health Centre, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, India. The data set consists of 100 observations of systolic blood pressure (in mm Hg) for validating the performance of outlier labeling methods. First the normality of the data set was checked by histogram and shown in Fig. 1, which shows that the data was normally distributed. In Figure 2, the scatter plot shows that there are no remarkable outliers present in the taken data set.

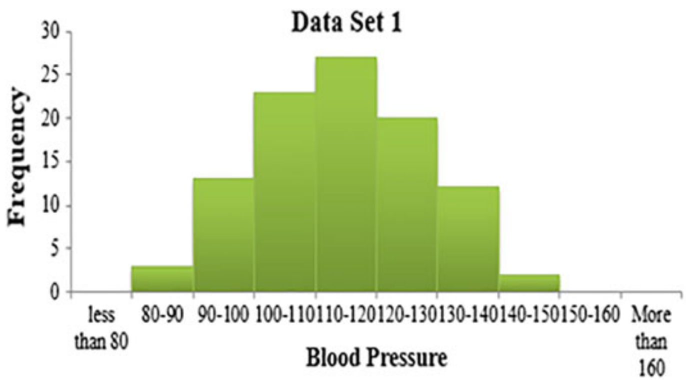

Fig. 1 Histogram of systolic blood pressure data

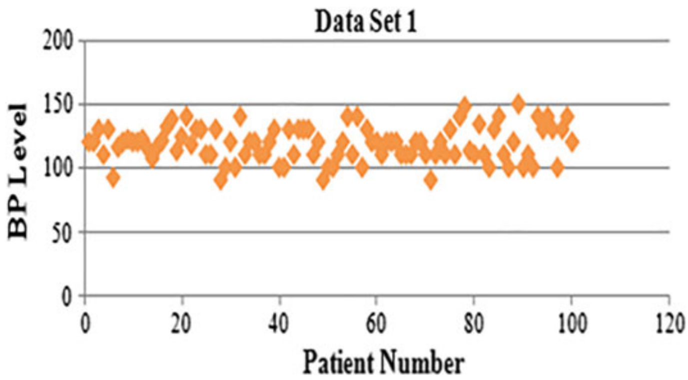

Fig. 2 Scatter plot of systolic blood pressure data

Six labeling methods were compared in this study; they were computed by using MS Excel, and the results are shown in Tables 2 and 3. Table 2 shows the result of original data set (data set 1), in which all the observations are somehow within the normal range. To validate the performance of outlier detection method, we have changed first four values as 70, 50, 180, and 200 (data set 2), and the scatter plot for this data set was shown in Fig. 3.

From Table 2, 2SD method identified 5 outliers, 3SD method identified no outliers, Median method identified no outliers, 2MADe method identified 4 outliers, 3MADe method identified no outliers, Z-Score identified no outliers, Modified Z-Score identified no outliers, and Tukey's method inner fence identified 2 outliers and outer fence identified no outliers.

Table 2 Number of outliers detected before changing values

Methods	Cases	Cutoff value	Outliers	Count
SD method	2SD	[91.27, 145.35]	90, 90, 90, 148, and 150	5
SD method	3SD	[77.75, 158.87]	-NIL-	0
Median method	Median	[74, 166]	-NIL-	0
MADe	2MADe	[90.34, 149.66]	90, 90, 90, and 150	4
MADe	3MADe	[75.51, 164.49]	-NIL-	0
Z-Score	Z-Score	$Z_i > 3$	-NIL-	0
Modified Z-Score	Modified Z-Score	$\lvert M_i \rvert > 3.5$	-NIL-	0
Tukey's method	Inner fence	[90, 170]	148 and 150	2
Tukey's method	Outer fence	[60, 200]	-NIL-	0

Table 3 Number of outliers detected after changing values

Methods	Cases	Cutoff value	Outliers	Count
SD method	2SD	[80.71, 156.31]	70, 50, 180 and 200	4
SD method	3SD	[61.81, 175.21]	50, 180 and 200	3
Median method	Median	[74, 166]	70, 50, 180 and 200	4
MADe	2MADe	[90.34, 149.66]	70, 50, 180, 200, 90, 90, 90 and 150	8
MADe	3MADe	[75.51, 164.49]	70, 50, 180 and 200	4
Z-Score	Z-Score	$Z_i > 3$	50, 180 and 200	3
Modified Z-Score	Modified Z-Score	$\lvert M_i \rvert > 3.5$	50, 180 and 200	3
Tukey's method	Inner fence	[80, 140]	70, 50, 180, 200, 148 and 150	6
Tukey's method	Outer fence	[50, 170]	180 and 200	2

From Table 3, 2SD method identified 4 outliers, 3SD method identified 3 outliers, Median method identified 4 outliers, 2MADe method identified 8 outliers, 3MADe method identified 4 outliers, Z-Score identified 3 outliers, Modified Z-Score identified 3 outliers, and Tukey's method inner fence identified 6 outliers and outer fence identified 2 outliers.

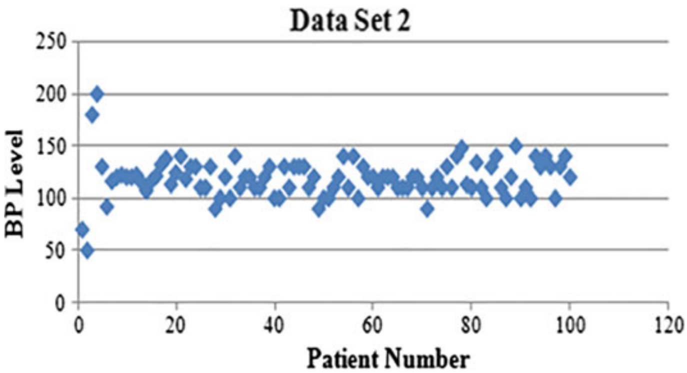

Fig. 3 Scatter plot of systolic blood pressure data after changing values

4 Discussion

From Table 2 based on count, 5 outliers are detected by 2SD method; 4 outliers are detected by 2MADe method; no outliers are detected by 3SD method, Median method, 3MADe method, Z-Score method, and Modified Z-Score method; Box plot inner fence identified 2 outliers and no outliers are detected by outer fence. From Table 3 based on count, 8 outliers are detected by 2MADe method; 4 outliers are detected by 2SD method, Median method, and 3MADe method; 3 outliers are detected by 3SD method, Z-Score, and Modified Z-Score method; Box plot inner fence identified 6 outliers and 2 outliers are detected by outer fence.

If the reason of the outlier detection is to find far out values away from the majority of the data in spite of the distribution, the outlier labeling methods can be used. SD method and Box plot are mostly used and can be appropriate when the data distribution is normal. From the above results, Fig. 1 shows that the original data set was found to be normal and there were no serious outliers present in the data. Table 2 shows that only 2SD method, 2MADe method, and Box plot inner fence have detected some outliers. After replacing first four values as 70, 50, 180, and 200 in the original data set, Table 3 shows that each method has detected some outliers. Here arises the confusion of finding which method is more appropriate for detecting outliers in data set. Hence, we have proposed the approach of fixing appropriate method based on cutoff values. By comparing the cutoff values of Tables 2 and 3, the intervals of Median method, 2MADe method and 3MADe method are unchanged. This is due to the fact that these methods use median as their measure. From these three methods based on the American Heart Association standard blood pressure level Table 1, interval of 2MADe method is found to be appropriate for detecting outliers.

5 Conclusion

The performance of the six outlier labeling methods SD method, 2MADe method, 3MADe method, median rule, Z-Score, Modified Z-Scores and Tukey's method has been compared using blood pressure data set. Also, we have evaluated to find which method has more powerful way for detecting and handling outliers. These methods are applicable only if data set is normally distributed. For choosing the appropriate method for detecting outliers in the data, we have used cutoff values of each. By comparing the cutoff value with the standard blood pressure level value, it was found that 2MADe method with the interval [90.34, 149.66] is appropriate than the other methods. This measure is basically problematic and indicator is itself changed by the occurrence of outlying values. The two main issues are that to choose which method is most suitable and the threshold going to be used. The outcome of outlier detection procedure can vary based on the outlier detection methods or by the type of distribution. Also by using the prior knowledge of the process, outlier detection procedure can be improved. It is tedious to solve the issue of detecting outliers, since in many cases it will depend greatly on the particular case in hand.

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Goal Programming Model to Budgetary Allocation in Garbage Disposal Plant

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1 Introduction

Budgetary allocation has important and also complex task which is directly related to the performance of the company. Budgetary allocation requires cooperation among multiple functional units in any industry. Efficient and effective budgetary allocation helps profitwise growth of industry. To carry out efficient and effective budgetary allocation, sound knowledge of the organizational budgeting process must be known. Inadequate allocation or competent allocation or over allocation of the budget can be identified by the goal programming approach.

Goal programming is an extension of linear programming which is mathematical optimization tool to handle multiple and conflicting objectives. Multiple goals are taken into consideration, and deviation toward the goals is minimized. Deviation in the goals of organization can be found by using goal programming technique. Goal programming technique is applied in many diverse areas like accounting, quality control, human resources, production, agriculture, telecommunication, financial aspect of stock management marketing, forestry, solid waste.

Due to increase in generation of waste, management of wastes became a challenge in many countries. Developing countries undertake the establishment of disposal units which collects the wastes, segregates them, and sells to different vendors of

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different management companies and which in return gives the profit to the unit. In garbage disposal unit, collected wastes are segregated into wet waste and dry wastes. Dry wastes are further divided into plastics, papers, tetra packs, metals, wood, glass, etc. Vendors collect the wastes from disposal units and convert these wastes into useful by-products by recycling process. Many useful products are formed from these wastes. Wet waste recycling leads into the generation of biofuel, organic fertilizers, etc. Plastic wastes are categorized, recycled, and converted into plastic bowls, mugs, buckets, toys, chairs, etc. Categorized plastics are PET bottles, milk covers, high-density plastic, and low-density covers, etc. Papers can be reused, and 100% utilization of wastes can be done by recycling process. If we recycle newspaper, it can be reused as wrapping paper and reprocessed into newsprint. Glass wastes can be recycled, and 100% utilization is possible from that. Glass wastes can be reused and remolded by melting. Recycling the glass materials will yield us new glass containers, bowls, glasses, etc.

If the disposal method is not appropriate, then it causes health hazard, unpleasant odor and uncontrolled air pollution and unhygienic conditions and adverse environmental effect. Thus, establishments of disposal units not only give the profit to the unit; it also helps to keep the environment clean, conservation of resources, protection of environment, social, as well as economic development of the country.

Thus, garbage disposal unit runs with the involvement of many expenditures like personnel cost, infrastructure cost, scientific sanitary landfill cost, liabilities, general expenses, maintenance charges. Budget is allocated to all the above and also assets, revenue generated are taken into consideration in our study. Personnel cost includes pay of officers, pay of establishments, dearness allowance, and allowances. Scientific sanitary landfill includes approach roads, construction of compound walls, development of new sanitary landfill, establishment of primary collection center, improvement to existing landfills, initiatives for small and medium composting and landfill, provided name boards to MSW sites, solid waste management(transfer station) core and in new areas. Maintenance charges include maintenance of landfill/ waste dumping yards, tipping fees, Bruhat Bangalore Mahanagara Palike garbage vehicles. General expenses include consultancy charges and hire charges of SWM vehicles.

2 Review of Literature

Goal programming model is developed for rubber wood door manufacturing factory in Tripura is explained in [1]. In [2], detailed application of linear programming technique in the determination of optimum production capacity, which helped in framing the goal programming model. Goal programming application in agricultural management is shown in [3]. An application to budgetary allocation of an institution of higher learning is explained in [4] which gave proper idea about the development of goal programming model. Goal programming approach for food product distribution of small and medium enterprises is [5] helped in developing the model and arriving the solutions. "Goal programming and its application in management sectors and special attention into plantation management: a review", is given in [6]. Development of goal programming model in rubber tea intercropping management in Tripura is [7] encouraged us to develop a goal programming model. [8] Explains the work on developing optimal solution for distribution of segregated wastes of the garbage disposal unit. Optimal solution for planning of disposal of wastes using mathematical model is explained in [9]. Detailed review work on goal programming model is given in [10]. In [11], discussion of optimization of university resource management through goal programming model is given.

3 Goal Programming Model Formulation

Constraints are framed for personnel cost, general expenses, assets, liabilities, infrastructure cost, scientific sanitary landfill, maintenance charges, and also revenue generated. Data is taken for 3 years, 2009–2011. Statistical data is collected from Bruhat Bangalore Mahanagara Palike.

3.1 Goal 1: Maximize the Personnel Cost

Personnel cost includes pay of officers, pay of establishments, dearness allowance, and allowances.

$$\sum_{i=1}^{3} P_i x_i \geq P_t$$

$$\sum_{i=1}^{3} P_i x_i + d_1^- - d_1^+ = P_t$$

x_i *is amount budgeted in the fiscalyear*

P_i *is the personnelcost in the fiscal year*

P_t *Target Personnel cost*

d_1^-, d_1^+ *Underachievement and overachievement*

3.2 Goal 2: Minimize the General Expenses

$$\sum_{i=1}^{3} G_i x_i \leq G_t$$

$$\sum_{i=1}^{3} G_i x_i + d_2^- - d_2^+ = G_t$$

x_i is the amount budgeted in the fiscal year

G_i is the General expenses in the fiscalyear

G_t is target General expenses

d_2^-, d_2^+ are underachievement and overachievement of the goal

3.3 Goal 3: Maximizing the Assets

$$\sum_{i=1}^{3} A_i x_i \geq A_t$$

$$\sum_{i=1}^{3} A_i x_i + d_3^- - d_3^+ = A_t$$

x_i is amount budgeted in the fiscalyear

A_i is the Assets in *the fiscal year*

A_t Target Assets

d_3^-, d_3^+ Underachievement and overachievement

3.4 Goal 4: Minimizing the Liabilities

$$\sum_{i=1}^{3} L_i x_i \leq L_t$$

$$\sum_{i=1}^{3} L_i x_i + d_4^- - d_4^+ = L_t$$

x_i is the amount budgeted in the fiscal year

L_i is the Liabilities in the fiscalyear

L_t is target Liabilities

d_4^-, d_4^+ are underachievement and overachievement of the goal

3.5 Goal 5: Minimizing Infrastructure Expenses

$$\sum_{i=1}^{3} I_i x_i \leq I_t$$

$$\sum_{i=1}^{3} I_i x_i + d_5^- - d_5^+ = I_t$$

x_i *is the amount budgeted in the fiscal year*

I_i *is the infrastructre charges in the fiscalyear*

I_t*is target maintenance charge*

d_5^-, d_5^+ *are underachievement and overachievement of the goal*

3.6 Goal 6: Minimizing the Scientific Sanitary Landfill Cost

Scientific sanitary landfill includes approach roads, construction of compound walls, development of new sanitary landfill, establishment of primary collection center, improvement to existing landfills, initiatives for small and medium composting landfill, provided name boards to MSW sites, solid waste management (transfer station) core and in new areas.

$$\sum_{i=1}^{3} S_i x_i \leq S_t$$

$$\sum_{i=1}^{3} S_i x_i + d_6^- - d_6^+ = S_t$$

x_i *is the amount budgeted in the fiscal year*

S_i *is the Scientific sanitary land cost in the fiscalyear*

S_t*is target maintenance charge*

d_6^-, d_6^+ *are underachievement and overachievement of the goal*

3.7 Minimizing Maintenance Charges

Maintenance charges include maintenance of landfill/ waste dumping yards, tipping fees, Bruhat Bangalore Mahanagara Palike garbage vehicles.

$$\sum_{i=1}^{3} M_i x_i \leq M_t$$

$$\sum_{i=1}^{3} M_i x_i + d_7^- - d_7^+ = M_t$$

x_i *is the amount budgeted in the fiscal year*

M_i *is the maintenance charges in the fiscalyear*

M_t*is target maintenance charge*

d_7^-, d_7^+ *are underachievement and overachievement of the goal*

3.8 Goal 8: Maximizing the Revenue Generated

$$\sum_{i=1}^{3} R_i x_i \geq R_t$$

$$\sum_{i=1}^{3} R_i x_i + d_8^- - d_8^+ = R_t$$

x_i*is amount budgeted in the fiscalyear*

R_i*is the Revenue generated* in *the fiscal year*

S_t *Target of Revenue generation*

d_8^-, d_8^+ *Underachievement and overachievement*

3.9 Priorities for the Goals

Revenue, personnel cost, assets, infrastructure cost, scientific sanitary land cost, maintenance cost, liabilities, general expenses goals are taken as P1, P2, P3, P4, P5, P6, P7, and P8

3.10 Achievement Function/Objective Function

$$Minimize\quad Z = P_1 d_8^- + P_2 d_1^- + P_3 d_3^- + P_4 d_5^+ + P_5 d_6^+ + P_6 d_7^+ + P_7 d_4^+ + P_8 d_2^+$$

These P's are simply a convenient way of indicating that one goal is more important than the other goal. Convenient weights can be assigned to each P's according to the priorities of the goal, and it can be changed according to the present situation. Z is

Table 1 Budget allocation and target cost

	Goal constraints	Allocation per year (in lakhs)			Target costs (in lakhs)
		2009–10	2010–11	2011–12	
1	Personnel cost	44.45	52.22	61	75
2	General expenses	29.34	25	50	60
3	Assets	1.07	65.93	91	110
4	Liabilities	42.48	80.29	113	125
5	Infrastructure cost	0	2437.3	5200	5000
6	Scientific sanitary land cost	177	1510	2895	3000
7	Maintenance cost	929	248	1425	2000
8	Revenue	1500	7500	10500	13000

the sum of the deviations of different goals. Minimization of deviations is done in this process.

4 Statistical Data Collected from BBMP

In all, 19 variables and 8 constraints are involved in this study. Target is fixed according to the decision maker's requirement. Equations are framed using the above statistical data. Computer software called TORA software which is linear programming solver is used to optimize which is not so easy manually. In goal programming problems, deviational values indicate the underachievement or overachievement while performing the goals. TORA software solves the problem and gives the solution with required iterations, and number of iterations changes when target value changes or weights of priorities changes (Table 1).

5 Solution and Interpretation of the Result

5.1 Case Study 1

Assigning the weights to P_1(*goal* 8 : *revenue*), P_2(*goal*1 : *personnel cost*),
P_3(*goal* 3 : *assets*), P_4(*goal*5 : *Infrastructure cost*), P_5(*Goal*6 : *Scienctific Sanitary Land fill cost*), P_6(*Goal* 7 : *maintenance charges*),
P_7(*Goal*4 : *Liabilities*), P_8(*Goal* 2 : *General expenses*) *as*
16, 14, 12, 10, 8, 6, 4, 2 *respectively*

$MinZ = 56.6, x_1 = 0, x_2 = 1.7333, x_3 = 0, d_1^+ = 15.51, d_2^- = 16.666, d_3^+ = 4.278,$
$d_4^+ = 14.16, d_5^- = 775.3466, d_6^- = 382.666, d_7^- = 1570.13$ *Rest all variables*
$d_1^-, d_2^+, d_3^-, d_4^-, d_5^+, d_6^+, d_7^+, d_8^-, d_8^+$ *are zero*

After the 7 iterations, we get Min Z = 56.6.

Goal 1(personnel cost), Goal 3(assets), Goal 4 (liabilities) is **overachieved,** and Goal 8(Revenue) is **achieved fully** without any deviation or violation. Goal 2(general expenses), Goal 5(infrastructure cost), Goal 6(scientific sanitary landfill cost), Goal 7(maintenance charges) are **underachieved** (Table 2).

Constraints are drawn in the graph with value, RHS, and dual price/Reduced cost. Interpretation of reduced cost of given decision variables as the rate at which the value of objective function (profit) will deteriorate for each unit change in the optimized value of the decision variable with all other data held fixed. The dual price varies according to the priorities and weights of the priorities. Dual price is -4 for fourth constraint and 0.042821333 for eight constraint (Graph 1).

5.2 Case Study 2

Assigning the weights to P_1(*goal* 8 : *revenue*), P_2(*goal*1 : *personnel cost*),
P_3(*goal* 3 : *assets*), P_4(*goal* 5 : *Infrastructure cost*), P_5(*goal* 6 : *Scienctific Sanitary Land fill cost*), P_6(*goal* 7 : *maintenance charges*),
P_7(*goal* 4 : *Liabilities*), P_8(*goal* 2 : *General expenses*) *as* 16, 2, 12, 6, 4, 14, 8, 10

$MinZ = 113.55, x_1 = 0, x_2 = 1.7333, x_3 = 0, d_1^+ = 15.51, d_2^- = 16.666, d_3^+ = 4.278,$
$d_4^+ = 14.16, d_5^- = 775.346, d_6^- = 382.666, d_7^- = 1570.13$
Rest all variables $d_1^-, d_2^+, d_3^-, d_4^-, d_5^+, d_6^+, d_7^+, d_8^-, d_8^+$ *are zero*

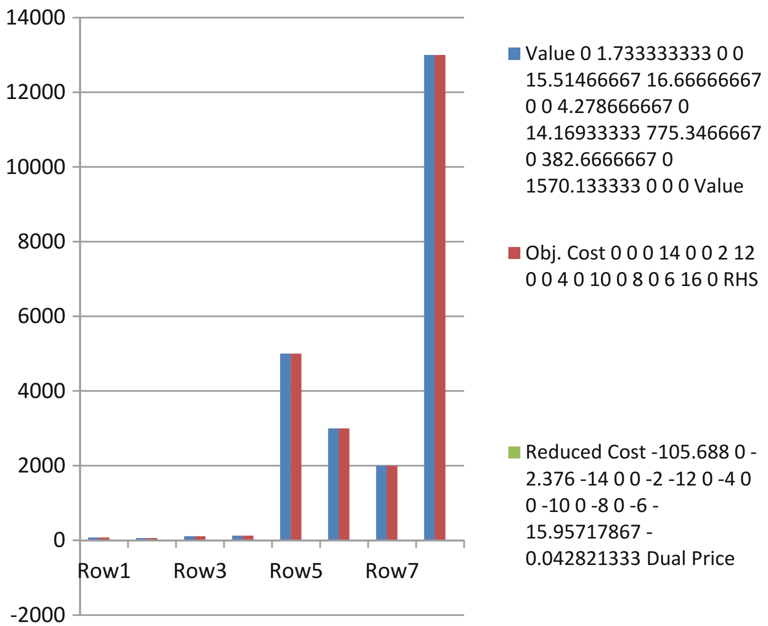
After 7 iterations, we get the solution Zmin = 113.5. Sum of the deviations of the goals is Zmin. Goal 2, Goal 5, Goal 6, and Goal 7 are underachieved, and Goal 1,

Table 2 Results pertaining to goals achievement for case study 1

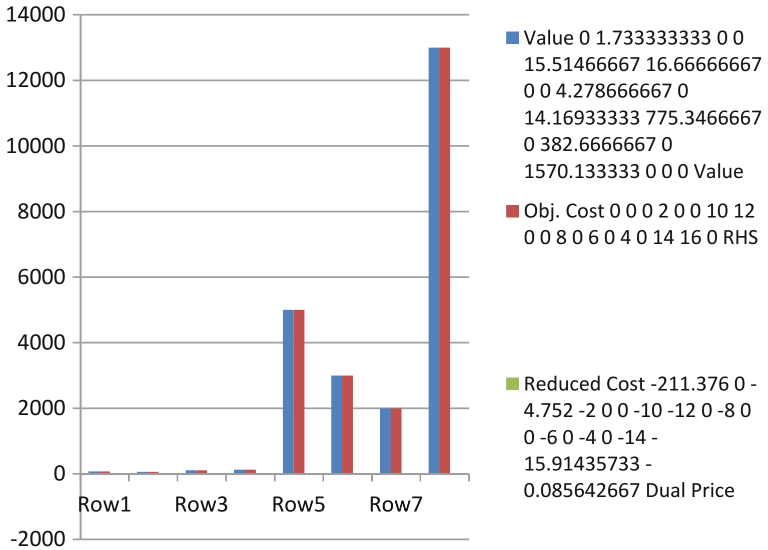
Variable	Value	Obj. Cost	Reduced Cost
X1	0	0	−105.688
X2	1.733333	0	0
X3	0	0	−2.376
X4	0	14	−14
X5	15.51467	0	0
X6	16.66667	0	0
X7	0	2	−2
X8	0	12	−12
X9	4.278667	0	0
X10	0	0	−4
X11	14.16933	4	0
X12	775.3467	0	0
X13	0	10	−10
X14	382.6667	0	0
X15	0	8	−8
X16	1570.133	0	0
X17	0	6	−6
X18	0	16	−15.95717867
X19	0	0	−0.042821333
Constraint	Value	RHS	Dual price
Row1	75	75	0
Row2	60	60	0
Row3	110	110	0
Row4	125	125	−4
Row5	5000	5000	0
Row6	3000	3000	0
Row7	2000	2000	0
Row8	13000	13000	0.042821333

Goal 3, Goal 4 are overachieved, and Goal 8 is achieved fully without any deviation or violation. Manager has to work on underachieved target. Second, fifth, sixth, and seventh goals are analyzed, and the reason for the underachievement can be searched in the unit, and proper allocation can be done according to that (Table 3).

Constraints are drawn in the graph with value, RHS, and dual price/reduced cost. The dual price varies according to the priorities and weights assigned to the priorities. Indication of the reduced cost value shows how much the coefficient of the objective function must be improved before the value of the corresponding variable becomes positive in the optimal solution. In this case study, dual price is -8 for fourth constraint and 0.085642667 for eighth constraint as shown in the graph (Graph 2).



Graph 1 Graphical representation of goals achievement for case study 1



Graph 2 Graphical representation of goals achievement for case study 2

Table 3 Results pertaining to goals achievement for case study 2

Variable	Value	Obj. Cost	Reduced Cost
X1	0	0	−211.376
X2	1.733333	0	0
X3	0	0	−4.752
X4	0	2	−2
X5	15.51467	0	0
X6	16.66667	0	0
X7	0	10	−10
X8	0	12	−12
X9	4.278667	0	0
X10	0	0	−8
X11	14.16933	8	0
X12	775.3467	0	0
X13	0	6	−6
X14	382.6667	0	0
X15	0	4	−4
X16	1570.133	0	0
X17	0	14	−14
X18	0	16	−15.91435733
X19	0	0	−0.085642667
Constraint	Value	RHS	Dual price
Row1	75	75	0
Row2	60	60	0
Row3	110	110	0
Row4	125	125	−8
Row5	5000	5000	0
Row6	3000	3000	0
Row7	2000	2000	0
Row8	13000	13000	0.085642667

5.3 Comparison Study

In the above two case studies, the results obtained in 7 iterations. It need not be 7 iterations in all the cases. When the target value and weights assigned are changed, the number of iterations as well as the results obtained will change. Thus, study can be extended further by changing the weights of the priorities and by giving different targets. From the above study, manager will be able to choose the best solution among different cases and which helps in decision making. Present study helps the manager to identify the deviations from the goals, and manager can find out the reasons for the deviations and proper measures can be taken further. Manager can set the targets according to their requirement of the unit.

	Case study 1			Case study 2		
Variable	Value	Obj. cost	Reduced cost	Value	Obj. cost	Reduced cost
X1	0	0	−105.688	0	0	−211.37
X2	1.7333	0	0	1.7333	0	0
X3	0	0	−2.376	0	0	−4.752
X4	0	14	−14	0	2	−2
X5	15.514	0	0	15.5146	0	0
X6	16.666	0	0	16.6666	0	0
X7	0	2	−2	0	10	−10
X8	0	12	−12	0	12	−12
X9	4.2786	0	0	4.27866	0	0
X10	0	0	−4	0	0	−8
X11	14.169	4	0	14.1693	8	0
X12	775.34	0	0	775.346	0	0
X13	0	10	−10	0	6	−6
X14	382.66	0	0	382.666	0	0
X15	0	8	−8	0	4	−4
X16	1570.13	0	0	1570.13	0	0
X17	0	6	−6	0	14	−14
X18	0	16	−15.957178	0	16	−15.914
X19	0	0	−0.0428213	0	0	−0.0856
Constraint	Value	RHS	Dual price	Value	RHS	Dual price
Row1	75	75	0	75	75	0
Row2	60	60	0	60	60	0
Row3	110	110	0	110	110	0
Row4	125	125	−4	125	125	−8
Row5	5000	5000	0	5000	5000	0
Row6	3000	3000	0	3000	3000	0
Row7	2000	2000	0	2000	2000	0
Row8	13000	13000	0.0428213	13000	13000	0.0856

Reduced cost is different in the above cases, and sum of the deviations (Zmin) is also different in two cases.

6 Conclusion

6.1. First case study is the best among the trails and deviation of the goal is minimum in that case. In result section of case study 1, since Z value is not zero, we can decide at least one of the goals is not satisfied. Goal 1(personnel cost), Goal 3(assets), Goal

4 (liabilities) are **overachieved,** and Goal 8(revenue) is **achieved fully** without any deviation or violation. Goal 2(general expenses), Goal 5(infrastructure cost), Goal 6(scientific sanitary landfill cost), Goal 7(maintenance charges) are **underachieved**. Sum of the deviations of the goals is Min Z = 56.6. Manager has to work on underachieved target. Not achieved goals like second, fifth, sixth, and seventh goals are analyzed and the reason for the underachievement can be searched in the unit and proper allocation can be done according to that. Revenue goal, personnel cost goal, assets goal, infrastructure goals are achieved.

6.2 General conclusions: Proper budget allocation is very important for the improvement of the unit. Improvement of the unit depends on manager's decision. Our study helps the manager to take better decision. On the basis of data, collected solution may be achieved. The use of TORA software a linear program solver helped us to find solution easily. Identifying the deviations while achieving the goals and proper measures can be taken to correct the deviation and it helps the manager to allocate the budget efficiently and effectively. Goal programming technique helps the disposal unit to work with profit. This profit-based business encourages people to take up the business of sales of wastes. This may also help to generate by-products and it helps in the creation of better environment. This also leads to improvement in the economic development of the country, clean, and healthy environment. Thus, our technique indirectly contributes toward the safety of environment and society. Main aim of our study is how goal programming technique is responsible for healthy environment by undertaking the business of wastes. As other business requires planning and proper budgeting, the garbage disposal unit also requires proper budgeting system. Goal programming technique helps in proper allocation of budget in the garbage disposal unit.

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A Validated Model of a Fault-Tolerant System

Veena B. Mendiratta and Kishor S. Trivedi

1 Introduction

In the continuous computing model of fault-tolerant systems, high availability is a key requirement. With appropriate sparing strategies, failures due to redundancy exhaustion are infrequent. In large fault-tolerant systems, however, the recovery and repair durations span several orders of magnitude and can contribute significantly to system unavailability. In this paper, we present a reward based semi-Markov model for predicting the steady-state availability of systems with non-exponential recovery times. The model can also be used for evaluating different recovery options. We also describe how the model was validated against field data. There have been a few availability models of real fault-tolerant systems that have appeared in the literature [4]. Fewer still have been validated [10]. We present here a validated model of a real fault-tolerant system. In addition to taking into account various kinds of recovery modes, we also allow for the possibility of a state being conditionally up or down depending on the length of sojourn in the state.

The remainder of the paper is organized as follows: In Sect. 2, we develop the basic model and its numerical characteristics, in Sect. 3, we consider the extension to software faults, and in Sect. 4, we discuss model validation.

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2 Model

Consider a dual processor fault-tolerant system, in active/active or active/standby mode, where the detection of an error in one of the active units initiates a recovery action. Recovery includes isolation of the faulty unit and reconfiguration of the system with the non-faulty unit. The system is considered operational if at least one of the processors is operational. Successful recovery, therefore, involves a transition from the duplex state to a degraded operational state. After successful recovery, the isolated faulty unit is repaired and brought back into the system. In our model, the duration of the recovery action—isolation and reconfiguration—determines the next state to which the system transitions. Recoveries of short duration cause a transition to the simplex state. Recoveries of longer duration cause a transition to the failure state.

In the semi-Markov model in Fig. 1, state 2 represents the case when both processors are operational. The occurrence of a fault in one of the processors causes a transition to state 1 if the recovery duration is short (fault is covered) and a transition to state 0 if the recovery duration is long. Being in state 1 does not constitute a system outage; however, it may contribute to system unavailability depending on the outage threshold. State 0 is a system failure state. In state 1R, the system is operational but in simplex mode.

The transitions from state 1 to state 1R represent the successful short automatic recoveries for covered faults. In many studies, these are viewed as instantaneous recoveries. The transition rate out of state 1R is the repair rate.

The transitions from state 0 to state 1R fall in two categories. The first category represents the successful long automatic and manual recoveries for covered faults. The second category represents the uncovered faults which, for a large system, are generally characterized by an incomprehensibility of the situation, and hence, a fairly long recovery duration which is unbounded. Clearly, for a highly available

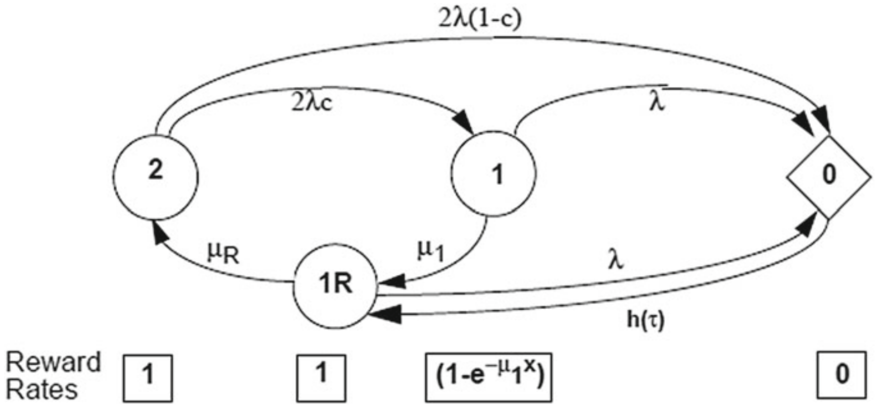

Fig. 1 State-transition diagram for duplex repairable system, hardware faults only

fault-tolerant system the second category of transitions occurs very infrequently. The transition rate from state 0 to state 1R is modeled as a 2-stage hyperexponential hazard function (see Eq. (3.1) in [8]) where the input parameters are the two recovery rates and the relative proportions of the long covered faults and the uncovered faults.

The impact on system unavailability of sojourns in states 1 and 0 is modeled with reward rates. Note that state 1 is considered an upstate if the sojourn in that state is short; otherwise, it is considered a downstate. The threshold for short versus long sojourns is denoted by x (time units). Ordinarily, such a situation will be modeled by splitting the state into two states. This, however, will make the sojourns in these states deterministic and complicate the model. By appropriately assigning reward rates to state 1, we avoid the splitting of states as well as avoid the introduction of deterministic sojourn durations. Since the sojourn time distribution in state 1 is exponential with rate μ_1, we know that an individual sojourn in the state will not exceed the threshold with probability $1 - e^{-\mu_1 x}$ and it will exceed the threshold with probability $e^{-\mu_1 x}$. In other words, state 1 is an upstate $1 - e^{-\mu_1 x}$ fraction of the time while it is a downstate $e^{-\mu_1 x}$ fraction of the time. By attaching a reward rate to the state, we will have modeled this situation. Since the cumulative reward normalized over time represents the steady-state availability, the reward rate in state 1 can be viewed as the probability that the recovery duration is less than the threshold duration (x) that is defined as an outage. The reward rate in state 2 and state 1R is 1. The reward rate in state 0 is 0.

The physical interpretation of the threshold duration x is that it is of the same order of magnitude as the system response time. Having a threshold that is smaller than the system response time is not meaningful for predicting system reliability. Another perspective on this interpretation is that the recovery durations above the threshold represent non-instantaneous recoveries for the system being modeled. The value of the threshold also has performability implications. For example, in a switching system, recovery durations that are smaller than the threshold are considered instantaneous and do not result in a loss of calls that are in transition. On the other hand, recovery durations that are greater than the threshold are considered non-instantaneous and generally lose calls that are in transition but maintain stable calls.

A semi-Markov model such as that in Fig. 1 can be solved by several different techniques. By observing that the sojourn time in only the semi-Markovian state 0 is of phase-type [4], we can expand state 0 into 2 states and obtain a new equivalent Markov model. Such Markovization of semi-Markov models is often used [3, 5, 6, 8–10]. Alternatively, we can directly solve the semi-Markov model. While a transient solution of semi-Markov models is rather involved [1], the steady-state solution is quite straightforward [2]. In fact, we know that the steady-state probabilities of a semi-Markov model are insensitive to the form of the sojourn time distribution and depend only on the mean sojourn times.

Solving for the steady-state probabilities for the SMP in Fig. 1, we obtain

$$\frac{1}{\Pi_2} = 1 + \frac{2\lambda}{\mu_R} + \frac{2\lambda c}{\lambda+\mu_1} + \frac{2\lambda(1-c)}{\mu_2} + \frac{2\lambda^2 c}{(\lambda+\mu_1)\mu_2} + \frac{2\lambda^2}{\mu_R\mu_2}$$

$$\Pi_{1R} = \frac{2\lambda}{\mu_R}\Pi_2 \quad \Pi_1 = \frac{2\lambda c}{\lambda+\mu_1}\Pi_2 \quad \Pi_0 = 1 - \Pi_2 - \Pi_1 - \Pi_{1R}$$

where $\frac{1}{\mu_2} = \frac{p}{\mu_{21}} + \frac{1-p}{\mu_{22}}$, that is $\frac{1}{\mu_2}$ is the mean of the 2-stage hyperexponential distribution that denotes the sojourn time in state 0.

Subsequently, the steady-state availability as a function of the outage threshold x is obtained as

$$A(x) = \Pi_2 + \Pi_1\left(1 - e^{-\mu_1 x}\right) + \Pi_{1R}$$

Table 1 shows the steady-state availability for this system for a range of values of the duration and the proportion of short and long recoveries, where we assume a processor failure rate of $\lambda = 0.0001$, a simplex repair rate of $\mu_R = 0.5$ per hour and vary the value of the outage threshold x between 1 and 10 s. These results are for a system model where the coverage factor is 0.99; the parameters are varied to reflect a change in the proportion of short and long recoveries while the fraction of uncovered faults is fixed at 0.01. Such a model can be used to evaluate the impact of various fault recovery strategies on system availability. Results were obtained both by using the closed-form equations above and by numerical solution via the software package SHARPE [7].

When the proportion of short recoveries is high, then decreasing the recovery rate for the long automatic and manual recoveries does not have a significant impact on system availability. As the proportion of short recoveries is decreased, then the impact of increasing the duration of the long recoveries is more significant. Also, when the proportion of short recoveries is high, then increasing the duration of the short recoveries by an order of magnitude increases the expected system downtime significantly. The impact on availability of increasing the outage threshold x—recovery durations greater than this value contributes toward unavailability—from 1 s to 10 s improves the availability. When the coverage factor is low, a greater proportion of the recovery times exceed 10 s, and therefore, there is a smaller improvement in availability when the outage threshold is increased. Also, the improvement in availability is greater when the short automatic recovery rate is 100 per hour versus 1000 per hour since there are more recoveries that exceed 1 s in the former case.

3 Software Reliability

It is well known that there are significant differences between hardware and software faults. A greater proportion of software faults will impact both units of a duplex system (assuming single-version software), thus leading to more transitions to the system failure state. Based on analysis of field data, we observe that the recovery duration distribution is the same for hardware and software triggered recoveries.

Table 1 Steady-state availability and expected system downtime

Recovery rate (per hour)		Outage threshold $(x) = 1$ s			Outage threshold $(x) = 10$ s		
μ_1	μ_{21}	c = 0.9	c = 0.7	c = 0.5	c = 0.9	c = 0.7	c = 0.5
1000	20	0.9999980 (1.072)	0.9999960 (2.101)	0.9999940 (3.131)	0.9999981 (1.006)	0.9999961 (2.035)	0.9999942 (3.065)
	15	0.9999977 (1.230)	0.9999950 (2.609)	0.9999924 (3.969)	0.9999978 (1.164)	0.9999952 (2.543)	0.9999925 (3.923)
	10	0.9999971 (1.546)	0.9999931 (3.626)	0 9999891 (5.706)	0.9999972 (1.480)	0.9999932 (3.560)	0.9999893 (5.640)
100	20	0.9999963 (1.920)	0.9999947 (2.760)	0.9999931 (3.600)	0.9999967 (1.717)	0.9999951 (2.557)	0.9999935 (3.397)
	15	0.9999960 (2.078)	0.9999938 (3.268)	0.9999915 (4.459)	0.9999964 (1.875)	0.9999942 (3.065)	0.9999919 (4.256)
	10	0.9999954 (2.394)	0.9999918 (4.285)	0.9999882 (6.176)	0.9999958 (2.190)	0.9999922 (4.081)	0.9999886 (5.972)

Notes μ_1 is the recovery rate for the short automatic recoveries which occur with proportion c

μ_{21} is the recovery rate for the long automatic recoveries and manual recoveries for covered faults which occur with proportion $1 - p$; it is one element in calculating the parameter μ_2 for the hyperexponential transition rate from state 0

μ_{22} is the recovery rate for the long uncovered faults which occur with proportion p; it is one element in calculating the parameter μ_2 for the hyperexponential transition rate from state 0

c is the proportion of short recoveries

numbers in () denote expected downtime in minutes per year

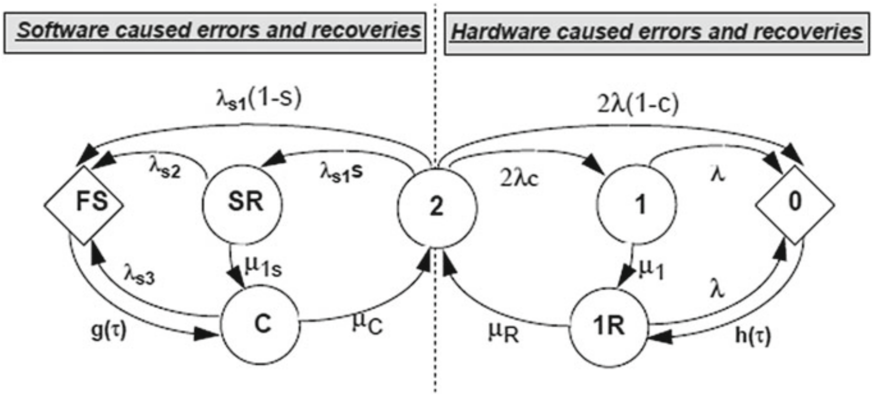

Fig. 2 State-transition diagram for duplex repairable system, hardware and software faults

Another difference is in the factors determining the software failure rate. The software failure rate is a function of the load, the software execution rate, and the software structure. In highly reliable systems, the application software includes checks such as audits and asserts so that most software errors result only in single process purges. In less reliable systems, the application software is not as robust and some of the software errors will escalate to higher recovery levels.

The fault recovery model presented in Fig. 1 captures the hardware faults and the fault recovery software that is invoked by the occurrence of these faults in a fault-tolerant system. We now extend the above model to include software faults as well, and the combined hardware–software model is shown in Fig. 2.

In the semi-Markov model in Fig. 2, the states 0, 1, and 1R are the states associated with hardware caused errors and recovery and are as defined for the model in Fig. 1; the state error free corresponds to state 2, and the states associated with software caused errors and recovery are defined below.

As described for the hardware fault model, the duration of the recovery action determines the next state to which the system transitions. The occurrence of a software fault in one of the processors when the system is in the error-free state (state 2) causes: A transition to state short recovery (SR) if the recovery duration is short (fault is covered) and a transition to state failed due to software errors (FS) if the recovery duration is long. Being in state SR does not constitute a system outage; however, it may contribute to system unavailability depending on the outage threshold. State FS is a system failure state. In state cleanup (C), the system is operational but there may be data cleanup activity occurring in the background.

The transitions from state SR to state C represent the successful short automatic recovery for covered faults. Often these are considered instantaneous recoveries. The transition rate out of state C may be viewed as the software repair rate—data cleanup, etc.

The transitions from state FS to state C fall into two categories: The first category represents the successful long automatic and manual recoveries for covered faults. The second category represents the uncovered faults, which for a large system can have a fairly long and unbounded recovery duration. The transition rate from state FS to state C is modeled, as was done for the hardware faults, as a 2-stage hyperexponential hazard function where the input parameters are the two recovery rates and relative proportions of the long covered faults and the uncovered faults.

Again, the impact on system availability of sojourns in states SR and FS is modeled with reward rates. The reward rate for state SR is modeled analogously to the reward rate for state 1 as described in Sect. 2. The reward rate in state C is 1, and the reward rate in state FS is 0.

The additional model parameters are defined as follows:

s software fault recovery coverage factor, proportion of software faults that occur and are recovered from successfully in near-instantaneous time

λ_{s1} software failure rate when in error-free state

λ_{s2} software failure rate during lowest recovery (when in fast recovery state); this will depend on the proportion of recoveries that have to be escalated when the lowest level recovery does not work or the latent faults that are activated because of the manifestation of the first fault and the resultant recovery action

λ_{s3} software failure rate when in the cleanup state

μ_C recovery rate from the cleanup state

μ_{2s} recovery rate from the long recovery/failed state.

Solving for the steady-state probabilities for the SMP in Fig. 2, we obtain

$$\begin{aligned}\frac{1}{\Pi_2} &= 1 + \frac{1}{\mu_2}\left[2\lambda + 2\lambda c + \frac{\lambda^2 c}{\lambda+\mu_1} + \frac{2\lambda^2 + \lambda\lambda_{s1}}{\mu_R}\right] \\ &\quad + \frac{2\lambda c}{\lambda+\mu_1} + \frac{2\lambda+\lambda_{s1}}{\mu_R} + \frac{\lambda_{s1}s}{\mu_{1s}+\lambda_{s2}} + \frac{1}{\mu_{2s}}[\lambda_{s1}\mu_{1s}(1-s) + \lambda_{s2}\lambda_{s1}] \\ &\quad + \left[1 + \frac{\lambda_{s3}}{\mu_{2s}} - \frac{\mu_C(\lambda+\mu_2)}{\mu_2\mu_R}\right]\cdot\frac{\lambda_{s1}}{(2\lambda_{s3}+\mu_C)}\left[(1-s) + \frac{s(\mu_{1s}-\lambda_{s2})}{(\mu_{1s}+\lambda_{s2})}\right]\end{aligned}$$

$$\Pi_C = \left[\tfrac{\lambda_{s1}(1-s)}{(2\lambda_{s3}+\mu_C)} + \tfrac{\lambda_{s1}s(\mu_{1s}-\lambda_{s2})}{(2\lambda_{s3}+\mu_C)(\mu_{1s}+\lambda_{s2})}\right]\Pi_2$$

$$\Pi_{1R} = \left[\tfrac{2\lambda+\lambda_{s1}}{\mu_R} - \tfrac{\mu_C}{\mu_R}\left(\tfrac{\lambda_{s1}(1-s)}{(2\lambda_{s3}+\mu_C)} + \tfrac{\lambda_{s1}s(\mu_{1s}+\lambda_{s2})}{(2\lambda_{s3}+\mu_C)(\mu_{1s}+\lambda_{s2})}\right)\right]\Pi_2$$

$$\Pi_{SR} = \tfrac{\lambda_{s1}s}{\mu_{1s}+\lambda_{s2}}\Pi_2$$

$$\Pi_1 = \tfrac{2\lambda c}{\lambda+\mu_1}\Pi_2 \quad \Pi_0 + \Pi_{FS} = 1 - \Pi_2 - \Pi_1 - \Pi_{1R} - \Pi_{SR} - \Pi_C$$

where $\frac{1}{\mu_{2s}} = \frac{p}{\mu_{21s}} + \frac{1-p}{\mu_{22s}}$, that is $\frac{1}{\mu_{2s}}$ is the mean of the 2-stage hyperexponential distribution that denotes the sojourn time in state FS.

Subsequently, the steady-state availability as a function of the outage threshold x is obtained as

$$A(x) = \Pi_2 + \Pi_1\left(1 - e^{-\mu_1 x}\right) + \Pi_{SR}\left(1 - e^{-\mu_{1s} x}\right) + \Pi_{1R} + \Pi_C$$

4 Model Validation

The model was validated using the field outage duration and the time of failure data from a large system for over 11,000 incidents. The data was sorted by duration of outage and divided into three data sets such that each data set approximates an exponential distribution. The parameters μ_1, $\mu_{21,}$ and μ_{22} were then estimated from these data. The relative size of each data set allowed us to determine the values of the proportion of short and long recoveries as well as the proportion of recoveries for uncovered faults. Knowing the failure rate for the system and the field repair rate for simplex systems, we ran the model to obtain steady-state availability and expected system downtime. A comparison of the modeled results—expected system downtime—with observed field performance of the system showed a close fit. The variation between the modeled and observed results is about 15%.

Furthermore, the time-to-failure distribution was validated as exponential. This was done by plotting the observed sorted normalized time-to-failure data against quantiles from a standard exponential distribution. The quantile–quantile plot and the fitted line using linear least squares in Fig. 3 show the close fit between the observed data and a standard exponential distribution. Note that the outliers on the right side of the plot represent only about 5% of the data points. In actuality, the outliers represent explainable anomalous behavior in the system under study.

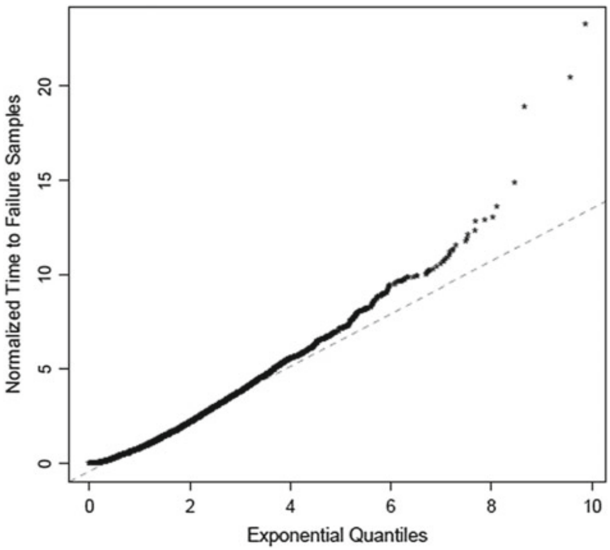

Fig. 3 Exponential probability plot

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Formation and Designing of "Least-Cost Ration Formulation Application of Cattle" Using Excel VBA

Vishal Patil, Radha Gupta, D. Rajendran and Ravinder Singh Kuntal

1 Introduction

Livestock plays an important role in the Indian economy. Livestock sector recorded growth in value of output about 4.8% per annum. The 11th Five-Year Plan expected higher growth in 12th Five-Year Plan. The increased demand for protein foods in the country is the main driver for growth, which also includes small stakeholders and landless farmers; account for major share in ownership of livestock which has recorded 5% growth per annum previously can grow up to 6% per annum [1].

The research on optimizing the ration has been done from many years for cattle and other livestock animals. The optimized ration should meet both nutrient requirements, and the cost of ration should be minimized [2]. Because of availability and cost of feeds, the ration should be formulated using locally available feeds. Manual feeding of ration to cattle may provide all nutrients or may not but when we use optimized ration it will provide digestible feed to a cattle and satisfies the entire nutrient requirement. Nutrient requirement varies for every cattle and depends upon their body weight, milk yield, milk fat, etc. Therefore, finding the optimized ration for least

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cost and maximizing the milk yield is one of the major aspects of dairy farmers and animal feed industry. There are many new methods available for ration formulation like linear programming, goal programming, stochastic programming and nonlinear programming. Selection of method depends upon the objective of optimizing ration. Linear programming is one of the widely used methods to formulate the ration, and it is very effective when the user has only one objective. Many programming software have been developed from many years which uses linear programming for ration (feed) formulation [3, 4] like Feed Formulation (www.kasturi.info/feed.htm). This software is for feed formulation of animals of type layers, broilers, sheep, pigs, fish, etc. It uses linear programming to optimize the ration, and software is developed based on Microsoft.NET Platform. To use this software, Microsoft.NET 2.0 should be present in the system [5].

Win Feed (www.winfeed.com) is software which works in two phases, linear mode: suitable for conventional feed formulation and stochastic mode: specifically for probability-based least-cost feed formulation. It is useful for ruminants and non-ruminants such as poultry, cattle, sheep, horses, dogs, cats, fish, and aquaculture [6].

Feed Assist (An Expert System on Balanced Feeding for Dairy Animals) is an expert system for ration formulation of cattle which uses linear programming. It has huge data Feed Assist is developed using Visual Basic in connection with MS Access [7].

These software can find optimized ration for animal in few seconds* (* after providing all input information properly) according to their standard of feed composition. The above-mentioned software has many limitations like: Win Feed is not freely downloadable; i.e. a user has to purchase it by giving huge amount. Also, even though Feed Formulation software is free but after downloading respective user computer should have minimum specification otherwise it will not work. Feed Assist is a system which is available at NIANP, Bangalore.

When we talk about Excel's worksheet or spreadsheet for optimization of ration compared to any of the software, it is easy to use, user's computer should not have any particular specification, and it is free of cost. Excel has in-built option "Solver Add-in" in which there are three options "Simplex LP, GRG Nonlinear, Evolutionary". For linear problems, Simplex LP can be used; for nonlinear problems, GRG Nonlinear and Evolutionary can be used. In [8], the author formulated least-cost dairy rations using Excel in English for Bangladeshi dairy farmers using "Solver" option.

Considering the above facts, "least-cost ration formulation application for cattle" is developed using Excel VBA. VBA is Visual Basic Application, a computer programming language that allows the user to define the function and automation of specific computer process and calculation. Users do not have to buy any visual basic software. VBA is the standard feature of Microsoft Office. The application which is developed can calculate optimized ration for cattle with feeds and nutrients as constraints and at the same time minimizing the cost as objective function. The solution is given in dry matter basis as well as fresh basis.

2 Materials and Methods

The application calculates optimized ration for different categories of dairy cattle as per the nutrient requirement using locally available feeds. It is developed by using Microsoft Excel VBA. The end-user has to provide the details of animal with respect to parameters like body weight, milk yield, milk fat, status of the animal (pregnant or not pregnant). If the cattle is pregnant, then the user has to select the month of pregnancy. Then select the feeds from the list of ingredients roughages, concentrate, minerals. The application processes the input data and provides an optimized ration at least cost with the help of the locally available feeds. The output solution is provided in tabular form as dry matter basis and as fresh basis in terms of quantity of feed and cost for easy understanding. The developed application has three important steps—collection of data, programming and output solution.

2.1 Collection of Data

The developed application has huge data regarding feeds and nutrient composition of feeds available in different regions, Mandya and Kolar districts of Karnataka, collected from National Institute of Animal Nutritionist and Physiology (NIANP), given in Table 1. Parameters for the composition include dry matter (DM), crude protein (CP), total digestible nutrient (TDN), calcium (Ca), phosphorus (P) and cost of the feeds. Present cost of feeds is considered, and user can change the cost to get realistic costs of the optimized ration.

Nutrient requirements automatically get calculated by the dry matter intake of animal according to NRC 2001 standards. While the range of body weight, milk yield and milk fat has been sourced by NIANP, a database has been saved in Excel file with various tables saved as particular name [9–16].

2.2 Programming

Excel VBA program has been written to calculate optimized ration for dairy cattle based on the nutrient requirement of selected animal. It is a simple worksheet saved as .xlsm form compressed with VBA coding. Schematic diagram is shown in Fig. 1. Based on the input data like body weight, milk yield, milk fat, status of the animal, the application will calculate daily DM intake, CP, TDN, Ca, P which has to be satisfied for the application to get optimized ration. Feed ingredients can be selected from roughages, concentrate and minerals, and facility is given to add new feeds also. Depending upon the availability, minimum and maximum quantity of feed can be changed; otherwise, it will take standard values set by application.

Table 1 Composition of feed ingredient

	Feed name	Cost (Rs)	DM %	CP %	TDN %	Ca %	P %
Roughages	Paddy straw	3	90	3.5	40	0.18	0.08
	CO-4 grass	10	20	8	52	0.38	0.36
	Maize fodder	2	20	8	60	0.53	0.14
	Co Fs 29 sorghum fodder	1	90	7	50	0.3	0.25
	Ragi straw	3.5	90	6	42	0.15	0.09
	Berseem	2	20	15.8	60	1.44	0.14
	Wheat straw	2	90	3.3	42	0.3	0.06
	Maize stover	1.5	90	3	42	0.53	0.14
Concentrate	Maize	15	90	8.1	79.2	0.53	0.41
	Soya DOC	28	90	42	70	0.36	1
	Copra DOC	19.7	90	22	70	0.2	0.9
	Cotton DOC	22	90	32	70	0.31	0.72
	Wheat bran	15	75	12	70	1.067	0.093
	Gram chunies	14	90	17	70	0.28	0.54
	Cotton seed	21	90	16	110	0.3	0.62
	Chickpea husk	10	90	18	45	0.3	0.62
	Concentrate Mix type I	17	90	22	70	0.5	0.45
	Concentrate Mix type II	15	90	20	65	0.5	0.4
Minerals	Calcite	5	97	0	0	36	0
	Grit	4.5	96	0	0	36	0
	MM	60	90	0	0	32	15
	DCP	38	90	0	0	24	16
	Sodabicarb	35	90	0	0	0	0
	Salt	5	90	0	0	0	0
	TM Mix	15	98	0	0	0	0
	Urea	6	95	287.5	0	0	0

The optimization program is developed based on linear programming (LP) for minimizing the cost of ration. The developed linear programming model is given below:

$$Minimize\ Z = \sum_{j=1}^{n} x_j c_j$$

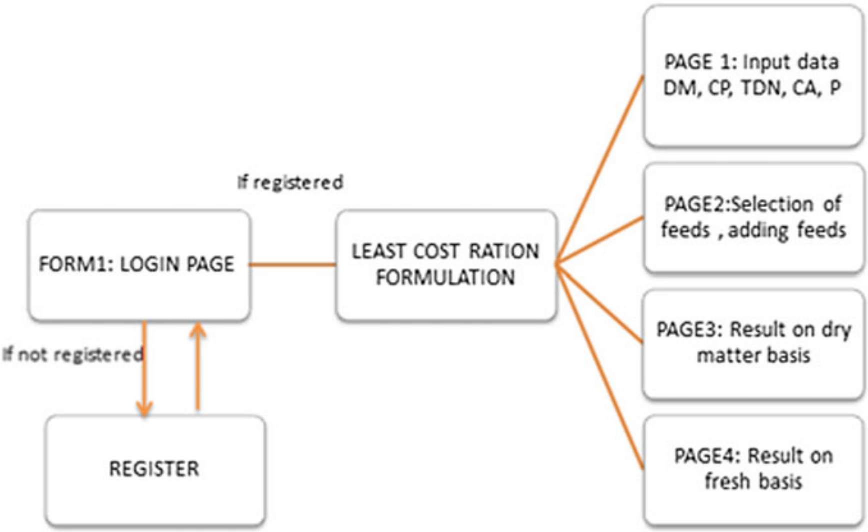

Fig. 1 Scheme of the "least-cost ration formulation application"

Subject to:

$$a_{min} \leq \sum_{j=1}^{n} A_j x_j \leq b_{\max}$$

$$0 \leq x_j \geq c_i \quad \text{where (i, j} = 1, 2, \ldots, \text{n)}$$

where Z is objective function, x_j is the quantity of feed ingredients, c_j is the cost of feed ingredients, $a_{\min}$ is the minimum bound of nutrient, $b_{\max}$ is the maximum bound of nutrient, and A_j are the nutrients.

2.3 Solution and Output

After providing the details of feed ingredients, user should click on "SOLVE" button that gives output solution as dry matter basis; it shows the quantity of each feed to be added in ration, cost of each feed and cost of total dry matter intake. User can take printout of the same which contain detail of animal and the solution of least-cost ration. The output is also given on as fresh basis; if required, this printout also can be taken.

3 Steps for Formulation

a. Once the worksheet is open, sheet 1 will appear which contains user manual and "START" button as shown in Fig. 2. After clicking on "START", login page will appear where user has to register himself by giving minimum information and provision is given to set a username and password.

b. Page 1: Selection of animal details like body weight (or body length and girth), milk yield, milk fat, pregnancy status, and click on "CONSTRAINTS" button. Based on the input data, application will calculate daily DM intake, CP, TDN, Ca, P which has to be satisfied by the application to get optimized ration as shown in Fig. 3.

c. Page 2: Selection of different feed ingredients from roughages, concentrate and minerals or any other feedstuff for which facility is given to add new feed. Depending upon the availability, minimum and maximum quantity of feed can be changed; otherwise, it will take standard values set by application as shown in Fig. 4. Next click on "SOLVE" button.

d. Page 3: It shows the solution of optimized ration on dry matter basis. If the solution is not feasible with the given input, then the message box will appear by saying "PLEASE REFINE THE FEEDS" and then the user has to adjust feed accordingly. In this page, quantity of each feed to be added in ration, cost of each feed and cost of total dry matter intake will be given. Option is given to take printout of the same which contains detail of animal and the solution of least-cost ration.

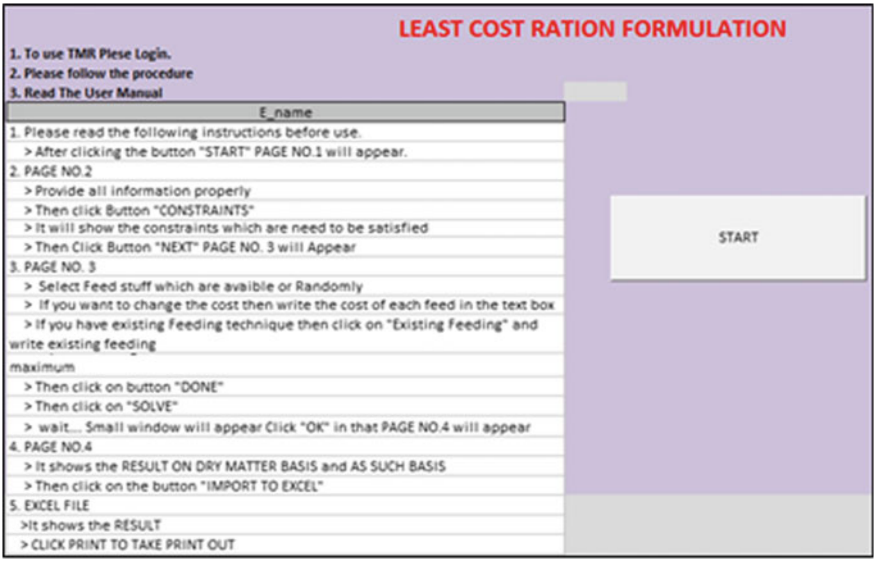

Fig. 2 Screenshot of start-up page

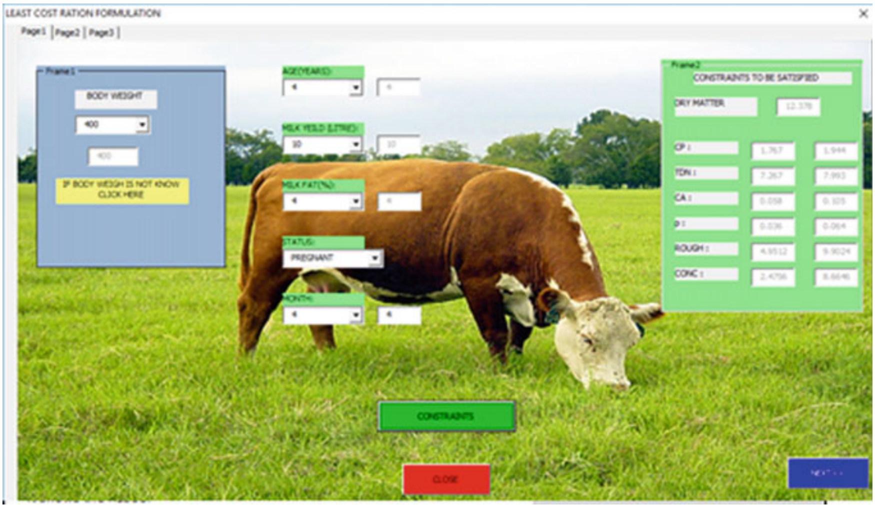

Fig. 3 Screenshot of Page 1

Fig. 4 Screenshot of Page 2

e. Page 4: In this page, solution is given on fresh basis. This page also can be printed. The output solution shows the least-cost ration with available feed, and optimal solution satisfies the entire nutrient requirements like DM, CP, TDN, Ca, P. In the printout, pie diagram is provided to understand the cost breakups of total ration.

4 Results and Discussion

There are many number of software available for optimizing ration for least cost. In this section, we talk about how the developed application is different and more effectively convenient from all other software. In existing software, most of them are not user friendly and farmers have to depend upon expert's help to use it. Majority of the software are developed for industry for commercial purpose where the small dairy farmers have to depend on experts input for getting profit by optimized ration. To overcome this challenge, we have developed "least-cost ration formulation application" which is free of cost and user friendly and the small dairy farmers need not depend on any expert's help.

Data maintenance: If any feed is not listed and it is locally available, then the user can add the feed with nutrient composition while selecting the feedstuff. It helps the user to use available feed in ration and save the cost.

User friendly: It is very easy to use once user is registered. User can get optimized ration in only two steps—selecting the animal details and selecting the feeds—and then click on "SOLVE" button to get the solution.

Display and printing: Once the user gets the solution, the provision is given to take print of the solution on dry matter basis as well as fresh basis with feed quantity, price and total DM intake per kg.

System requirements: Any system with Microsoft Office can be used. No particular requirement for hardware or RAM. In Microsoft Excel, macros should be enabled.

The developed application is tested in NIANP under the guidance of the expert nutritionist, and the results of some specific categories of animal are given in Table 2.

5 Conclusion

The development of Excel VBA application for minimizing the cost of ration for cattle livestock based on linear programming model can be used effectively by farmers as this application does not require any advanced software and expertise. By providing basic inputs like weight, age, milk yield, milk fat, feeds available, pregnancy status, it will calculate the requirement of dry matter and energy, protein, calcium and also the least-cost animal ration in which it is easy for farmers to decide the quantity of each feed to be mixed in ration. In addition, this application also shows whether there is a profit or loss in the ration. The results obtained by this application are validated from Sr. Scientist of National Institute of Animal Nutrition and Physiology (NIANP), Bangalore. Hence, we conclude that this application can be used by the dairy farmers very effectively.

Table 2 Formulated optimized ration for different categories of animal by developed application

Animal details	Nutrient requirements (kg)	Feeds selected	Suggested feed quantity with price and total cost			Nutrients from feeds
		Name	**Quantity (kg)**	**Price (Rs)**	**Cost (Rs)**	
Body weight—400 Milk yield—0 Milk fat—0 Pregnant month—4	DM—8.658 TDN—0.869 CP—4.047 Ca—0.026 P—0.016	Paddy straw	1.73	**3**	5.19	DM—8.658 TDN—0.869 CP—4.451 Ca—0.039 P—0.029
		CO-4 grass	1.46	**10**	14.65	
		Maize fodder	1.57	**2**	3.13	
		Co Fs 29 sorghum fodder	1.30	**1**	1.30	
		Ragi straw	0.87	**3.5**	3.03	
		Maize	0.26	**15**	3.90	
		Soya DOC	0.61	**28**	17.05	
		Chickpea husk	0.43	**10**	4.33	
		Concentrate mix type I	0.31	**17**	5.28	
		MM	0.03	**60**	1.95	
		Salt	0.09	**5**	0.43	
Body weight—350 Milk yield—10 Milk fat—4 Pregnant month—0	DM—11.553 TDN—1.223 CP—6.327 Ca—0.049 P—0.031	Paddy straw	1.747	**3**	5.24	DM—11.553 TDN—1.223 CP—6.327 Ca—0.049 P—0.031
		CO-4 grass	0.347	**10**	3.47	
		Maize fodder	3.466	**2**	6.93	
		Co Fs 29 sorghum fodder	1.733	**1**	1.73	
		Ragi straw	1.155	**3.5**	4.04	
		Maize	0.347	**15**	5.20	
		Soya DOC	0.341	**28**	9.56	
		Chickpea husk	0.578	**10**	5.78	
		Concentrate mix type I	1.776	**17**	30.20	
		MM	0.017	**60**	1.04	
		Salt	0.046	**5**	0.23	
Body weight—400 Milk yield—10 Milk fat—4 Pregnant month—4	DM—12.378 TDN—1.767 CP—7.267 Ca—0.058 P—0.036	Paddy straw	0.271	**3**	0.81	DM—12.378 TDN—1.767 CP—7.267 Ca—0.058 P—0.045
		CO-4 grass	0.371	**10**	3.71	
		Maize fodder	3.713	**2**	7.43	
		Co Fs 29 sorghum fodder	1.857	**1**	1.86	
		Ragi straw	1.238	**3.5**	4.33	
		Maize	0.498	**15**	7.47	
		Soya DOC	1.262	**28**	35.34	
		Chickpea husk	0.619	**10**	6.19	
		Concentrate mix type I	2.476	**17**	42.09	
		MM	0.024	**60**	1.42	
		Salt	0.050	**5**	0.25	

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New Stable Numerical Inversion of Generalized Abel Integral Equation

Shweta Pandey, Sandeep Dixit and S. R. Verma

1 Introduction

Zeilon [1] studied the particular vital condition of the generalized Abel integral equation first time for finite segment. Various reversal techniques [2–5] have been produced with particular degree of convergence for approximating the generalized Abel integral equation numerically.

In this paper, Bernstein polynomials multiwavelets approach utilized for tackling following generalized Abel integral equation with singular kernel $1/\left(u^{\beta} - t^{\beta}\right)^{\alpha}$

$$p(u)\int_{a}^{s}\frac{t^{\beta-1}\varepsilon(t)dt}{(u^{\beta}-t^{\beta})^{\alpha}} + q(u)\int_{s}^{b}\frac{t^{\beta-1}\varepsilon(t)dt}{(t^{\beta}-u^{\beta})^{\alpha}} = I(u), \quad (0 < \alpha < 1),\ (a \le s \le b), \tag{1}$$

where $(0 < \alpha < 1)$ and $\beta \ge 1$ and the coefficients $p(u)$ and $q(u)$ do not vanish together, and Eq. (1) was recently studied by Chakrabarti [6].

Gakhov [7] considered the generalized Abel integral equation previously, with unique assumptions and the drawback of Gakhov's technique was that singular integrals having strong singularities of the form $(t-u)^{-1}$ must be allowed [6, 7] while solving an integral equation with weak singularity of the form $(t-u)^{-\alpha}$ $(0 < \alpha < 1)$.

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Chakrabarti [6] acquired an answer with just weak singular integrals and escaped the strong singular integrals of the form $(t - u)^{-1}$. For physical models, numerical arrangements are yet required since experimental data (intensity $I(u)$) might be disturbed by noise.

Two special types of Abel integral equations obtained from (1), by taking

(i) $\alpha = 1/2, p(u) = 1, q(u) = 0, a = 0, \beta = 1$ and
(ii) $\alpha = 1/2, p(u) = 0, q(u) = 2, b = 1, \beta = 2$ are as per the following:

$$\int_0^u \frac{\varepsilon(t)dt}{(u-t)^{1/2}} = I(u) \tag{2}$$

and

$$2\int_u^1 \frac{t\,\varepsilon(t)\,dt}{(t^2-u^2)^{1/2}} = I(u). \tag{3}$$

Equation (3) is inverted analytically by [8]

$$\varepsilon(t) = \frac{-1}{\pi}\int_t^1 \frac{1}{(u^2-t^2)^{1/2}}\frac{dI(u)}{du}du \quad 0 \le t \le 1 \tag{4}$$

similarly Eq. (2) is inverted.

On assessing the emissivity $\varepsilon(t)$ (solution function) if the intensity $I(u)$ (data function) is given with discrete points approximately, not well posed due to frequency errors (very small or high) in the calculated intensity $I(u)$ can arise from experimental errors could set out into massive errors within the reconstructed emissivity $\varepsilon(t)$, since differentiation of the measured data is required for these formulae to evade this problem, a derivative-free third analytical formula was given by Deutsch and Beniaminy [9] after the two analytic explicit inversion formulae given by Abel [10] whose direct implementation enlarges the experimental noise.

Bernstein polynomials operational matrix of integration was used for finding the solution of generalized Abel integral Eq. (3) in [11, 12]. To invert Eq. (3) numerically, a similar approach was executed to develop another almost operational matrix of integration which inspires us for the application of another approach by combining the two integrals (2) and (3) to solve the Eq. (1) numerically. We proposed a new stable technique to solve generalized Abel integral Eq. (1) numerically, with the help of newly constructed Bernstein polynomials multiwavelets. In our strategy, first we reduced Eq. (1) to algebraic equations using operational matrix and expand emissivity ($\varepsilon(t)$) with unknown coefficient.

The steadiness of the technique was examined by incorporating a fixed perturbation (η) in information data function ($I(u)$). The dependability of our algorithm was showed by finding the pointwise error, and suitable numerical examples are given with figures to illustrate the accuracy and convergence of the recommended approach even with suitable perturbations (η).

2 Wavelets and Bernstein Polynomials Multiwavelets

Wavelet is a new tool which originated from mathematics and was quickly embraced by an immense number of scientific fields. By utilizing the properties of dilation and translation of a single archetype function known as mother wavelet $\psi(t)$, a given signal can be broken down into too many functions, which is a fundamental principle of wavelet transform. The continuous variation of the dilation and translation parameters c and d comes into the following continuous wavelets form [13].

$$\psi_{c,d}(t) = |c|^{-1/2}\psi\left(\frac{t-d}{c}\right), \quad c, d \in R, \quad c \neq 0,$$

and when the parameters c and d are regulated to different values as $c = 2^{-k}$, $d = n^{2^{-k}}$, then new discrete wavelets family is acquired as $\psi_{k,n}(z) = 2^{k/2}\psi(2^k z - n), \quad k, n \in Z$, in above equation $\int_R \psi(z)dz = 0$. The situation when $\psi_{k,n}(z)$ constitutes an orthonormal basis of $L^2(R)$ is of our interest.

Bernstein polynomials, named after their founder S. N. Bernstein, are given as:

$$B_{i,n}(y) = \binom{n}{i} y^i (1-y)^{n-i}, \quad \forall\ i = 0, 1, 2, \ldots, n$$

characterized over the interval [0, 1].

A few significant characteristics of Bernstein polynomials are:

- Recurrence formula to discover Bernstein polynomial of lower degree (degree < m) from Bernstein polynomials of degree m

$$B_{i,m-1}(x) = \left(\frac{m-i}{m}\right) B_{i,m}(x) + \left(\frac{i+1}{m}\right) B_{i+1,m}(x)$$

- $B_{i,n} \geq 0$ for $x \in [0, 1]$ and $B_{n-i,n}(x) = B_{i,n}(1-x)$.
- Bernstein polynomials always form a unit partition

$$\sum_{i=0}^{n} B_{i,n}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} = (1-t+t)^n = 1$$

$P(x) = \sum_{i=0}^{m} \alpha_i B_{i,m}(x)$ is representation of any polynomial $P(x)$ in $\Re[x]$ whose degree is m, α_i is called Bezier or Bernstein coefficients, and $P(x)$ at that point is known as Bernstein polynomial of degree m.

Bernstein polynomials multiwavelets $\psi_{m,n}(z) = \psi(k, m, n, z)$ have four parameters—translation parameter $m = 0, 1, 2, \ldots, 2^k - 1$, dilation parameter '$k$' that can take any positive integer value, Bernstein polynomial order 'n', and normalized time 'z'. They are characterized on the interval [0, 1) as [14]

$$\psi_{m,n}(z) = \begin{cases} 2^{k/2} b_n(2^k z - m) & \frac{m}{2^k} \leq z < \frac{m+1}{2^k} \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

where $n = 0, 1, \ldots, N$, $m = 0, 1, 2, \ldots, 2^{k-1}$, the coefficient $2^{k/2}$ is for orthonormality, the estimation of dilation parameter is $c = 2^{-k}$, and the estimation of translation parameter is $d = m2^{-k}$. Here, $b_n(z)$ represents an order 'n' orthonormal Bernstein polynomial.

For $N = 5$, six orthonormal Bernstein polynomials of order five are ascertained utilizing Gram–Schmidt process given as follows:

$$\begin{gathered} b_0(z) = \sqrt{11}(1 - z)^5, \\ b_1(z) = 3(z - 1)^4(11z - 1), \\ b_2(z) = -\sqrt{7}(z - 1)^3(1 - 20z + 55\,\mathrm{z}^2), \\ b_3(z) = \sqrt{5}(z - 1)^2(-1 + 27z - 135z^2 + 165z^3), \\ b_4(z) = \sqrt{3}(1 - 33z + 248z^2 - 696z^3 + 810z^4 - 330z^5), \\ b_5(z) = -1 + 35z - 280z^2 + 840z^3 - 1050z^4 + 462z^5. \end{gathered}$$

With help of these orthonormal Bernstein polynomials for $N = 5$, and taking dilation parameter $k = 0$, six Bernstein polynomials multiwavelets can be constructed as,

$$\psi_{0,0}(z) = \begin{cases} b_0(z) & 0 \leq z < 1 \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi_{0,1}(z) = \begin{cases} b_1(z) & 0 \leq z < 1 \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi_{0,2}(z) = \begin{cases} b_2(z) & 0 \leq z < 1 \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi_{0,3}(z) = \begin{cases} b_3(z) & 0 \leq z < 1 \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi_{0,4}(z) = \begin{cases} b_4(z) & 0 \leq z < 1 \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi_{0,5}(z) = \begin{cases} b_5(z) & 0 \leq z < 1 \\ 0 & \text{otherwise}, \end{cases}$$

Similarly for $N = 5$, $k = 1$, twelve Bernstein polynomials multiwavelets can be obtained as,

$$\psi_{0,0}(z) = \begin{cases} \sqrt{2}\, b_0(2z) & 0 \leq z < \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{0,1}(z) = \begin{cases} \sqrt{2}\, b_1(2z) & 0 \leq z < \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{0,2}(z) = \begin{cases} \sqrt{2}\, b_2(2z) & 0 \leq z < \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{0,3}(z) = \begin{cases} \sqrt{2}\, b_3(2z) & 0 \leq z < \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{0,4}(z) = \begin{cases} \sqrt{2}\, b_4(2z) & 0 \leq z < \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{0,5}(z) = \begin{cases} \sqrt{2}\, b_5(2z) & 0 \leq z < \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{1,0}(z) = \begin{cases} \sqrt{2}\, b_0(2z-1) & \frac{1}{2} \leq z < 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{1,1}(z) = \begin{cases} \sqrt{2}\, b_1(2z-1) & \frac{1}{2} \leq z < 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{1,2}(z) = \begin{cases} \sqrt{2}\, b_2(2z-1) & \frac{1}{2} \leq z < 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{1,3}(z) = \begin{cases} \sqrt{2}\, b_3(2z-1) & \frac{1}{2} \leq z < 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{1,4}(z) = \begin{cases} \sqrt{2}\, b_4(2z-1) & \frac{1}{2} \leq z < 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{1,5}(z) = \begin{cases} \sqrt{2}\, b_5(2z-1) & \frac{1}{2} \leq z < 1 \\ 0 & \text{otherwise,} \end{cases}$$

3 Function Approximation

As $f(r) \in L^2[0,\ 1]$, where $L^2[0,\ 1]$ is Hilbert space and $\{\psi_{mn}\}$ are an orthonormal basis, we may expand $f(r)$ as follows

$$f(r) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn}\, \psi_{mn}(r), \tag{6}$$

where $c_{mn} = \langle f(r),\ \psi_{mn}(r) \rangle$ and $\langle , \rangle$ symbolize the inner product on the space $L^2(\Re)$.

By terminating the infinite series (6) at points $m = 2^k - 1$ and $n = N$, we have

$$f(r) \approx \sum_{m=0}^{2^k-1} \sum_{n=0}^{N} c_{mn}\, \psi_{mn}(r) = C^T\, \Psi(r) \tag{7}$$

which is an approximate representation of $f(r)$, where C and Ψ matrices have order $2^k(N+1) \times 1$.

$$C = \left[c_{00}, c_{01}, \ldots, c_{0N}, c_{10}, \ldots, c_{1N}, \ldots, c_{(2^k-1)0}, \ldots, c_{(2^k-1)N}\right]^T \tag{8}$$

$$\Psi(r) = [\psi_{00}(r), \psi_{01}(r), \ldots, \psi_{0N}(r), \psi_{10}(r), \ldots, \psi_{1N}(r), \ldots, \psi_{(2^k-1)0}(r), \ldots, \psi_{(2^k-1)N}(r)]^T \tag{9}$$

The integration of $\Psi(r)$ is approximated by Bernstein wavelet series by Bernstein polynomials multiwavelets coefficient matrix W

$$\int_0^r \Psi(r)dr = W_{2^k(N+1)\times 2^k(N+1)}\Psi(r),$$

where W is $2^k(N+1)$-order square matrix called Bernstein multiwavelets-based operational matrix of integration.

4 Solution of Generalized Abel Integral Equation

To get the solution of generalized Abel integral Eq. (1) by utilizing Bernstein polynomials multiwavelets, taking $a = 0,\ b = 1,\ \alpha = 1/2$ and by changing the variables, generalized Abel integral Eq. (1) reduces to

$$I(\sqrt{u}) = p(u) \int_0^u \frac{\varepsilon(\sqrt{t})}{\sqrt{u-t}}dt + q(u) \int_u^1 \frac{\varepsilon(\sqrt{t})}{\sqrt{t-u}}dt, 0 \le u \le 1 \tag{10}$$

then from Eq. (10) we have

$$I_1(u) = p(u)\int_0^u \frac{\xi(t)}{\sqrt{u-t}}dt + q(u)\int_u^1 \frac{\xi(t)}{\sqrt{t-u}}dt \tag{11}$$

where $I_1(u) = I(\sqrt{u})$ and $\xi(u) = \varepsilon(\sqrt{u})$.

Using Eq. (7), the anticipated intensity $I_1(u)$ and emissivity $\xi(t)$ are approximated as

$$\xi(t) = C^T\psi(t) \tag{12}$$

and

$$I_1(u) = F^T\psi(u) \tag{13}$$

where F is known matrix.

Putting values of $\xi(t)$ and $I_1(u)$ from Eqs. (12) and (13) into (11), then we have

$$F^T\psi(u) = p(u)\int_0^u \frac{C^T\psi(t)}{\sqrt{u-t}}dt + q(u)\int_u^1 \frac{C^T\psi(t)}{\sqrt{t-u}}dt \tag{14}$$

Calculation of integrals in Eq. (14) involves integrals of the form

$$\int_0^u \frac{t^n}{\sqrt{u-t}}dt, \quad \text{and} \quad \phi_n = \int_u^1 \frac{t^n}{\sqrt{t-u}}dt \tag{15}$$

Above integrals in Eq. (15) are calculated using the following recursive formulae

$$\int_0^u \frac{t^n}{\sqrt{u-t}}dt = \frac{\sqrt{\pi}\,t^{(n+\frac{1}{2})}\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \tag{16}$$

and

$$\phi_n = \frac{1}{2n+1}(\phi_0 + 2nu\phi_{n-1}), \quad \phi_0 = 2\sqrt{1-u} \tag{17}$$

from Eqs. (9), (15), (16), and (17), we get

$$\int_0^u \frac{\psi(t)}{\sqrt{u-t}}dt = W_1\psi(t) \tag{18}$$

and

$$\int_{u}^{1} \frac{\psi(t)}{\sqrt{t-u}} dt = W_2 \psi(t) \tag{19}$$

where W_1 and W_2 are $2^k(N+1) \times 2^k(N+1)$-order square matrix, called Bernstein multiwavelets-based operational matrix of integration for generalized Abel integral equation. On putting the values of (18) and (19) into (14), we get

$$C^T = F^T (p(u)W_1 + q(u)W_2)^{-1}. \tag{20}$$

Consequently, the approximate solutions $\xi(t)(=\varepsilon(\sqrt{t}))$ of the Eq. (1) are acquired by utilizing the estimations of C^T from above Eq. (20) to (12).

5 Numerical Results

To examine the convergence of our method, we have tried our technique on a few well-known test profiles that usually occur in test functions and utilized as a part of various research problems. Absolute errors $\Delta\xi(t_i)$, are defined by $\Delta\xi(t_j) = \left|\xi(t_j) - \xi 1(t_j)\right|$, and average deviation σ denoted by σ_X is:

$$\sigma_X = \left\{ \frac{1}{X} \sum_{j=1}^{X} [\xi(t_j) - \xi 1(t_j)]^2 \right\}^{1/2} = \left\{ \frac{1}{X} \sum_{j=1}^{X} \Delta\xi^2(t_j) \right\}^{1/2} = \|\Delta\xi\|_2,$$

which is calculated to demonstrate the accuracy of the proposed method, and σ_X are calculated by assigning $X = 1000, 500$, where $\xi 1(t_j)$ and $\xi(t_j)$ are, respectively, the calculated approximate and exact analytical solution at point t_j and $I_1^{\eta}(u)$ is obtained by adding a random perturbations η to $I_1(u)$ such that $I_1^{\eta}(u_j) = I_1(u_j) + \eta\omega_j$, where $I_1(u)$ denotes exact intensity and $I_1^{\eta}(u)$ is disturbed intensity (intensity with noise η), and ω_j is the uniform random variable which belongs to $[-1, 1]$ where $u_j = jh, j = 1, \ldots, X, Xh = 1$ and $\underset{1 \le j \le X}{Max} \left| I_j^{\eta}(u) - I_j(u) \right| \le \eta$.

By choosing different noise levels η_j as $\eta_0 = 0$, $\eta_1 = \sigma_X$ (for $X = 1000$) and $\eta_2 = \sigma_X$ (for $X = 500$). Correlation between exact emissivity $\xi(t)$ and calculated emissivities $\xi 1(t)$ (for $N = 5$ and $k = 0$), $\xi 2(t)$ (for $N = 5$ and $k = 1$) is drawn through figures, where k is the dilation argument for the taken Bernstein polynomials multiwavelets. Next, we calculate the corresponding error $E1(t)$ without noise and $E2(t)$ and $E3(t)$ with noise terms η_1 and η_2, which are presented in data function $I_1(u)$ for $X = 1000, 500$, respectively, and the impact of X is also drawn through figures. In both the examples, the series (7) is truncated at level $n = 5$, and $m = 0, 1$ (On taking dilation argument $k = 0$ and $k = 1$ in $m = 2^k - 1$).

5.1 Example 1

In first example, we consider Eq. (11) with $p(u) = 1,\ q(u) = 1$ for the pair

$$I_1(u) = \frac{16}{15}u^{5/2} + \frac{4}{3}u^{3/2} + \frac{2}{3}(1-u)^{3/2}(1+2u) + 2u(1-u)^{1/2}(1+u) + \frac{2}{5}(1-u)^{3/2} \quad \text{for } 0 \le u \le 1$$
$$\xi(t) = t + t^2 \quad \text{for } 0 \le t \le 1. \tag{21}$$

Equations (12) and (20) provide the desired calculated approximate solution; for $k = 0$, it is given by $\xi 1(t) = C^T \psi(t)$, and errors associated with the above example are $E1(t) = |\xi(t) - \xi 1(t)|$, $E2(t) = \left|\xi(t) - \xi 1'(t)\right|$, $E3(t) = \left|\xi(t) - \xi 1''(t)\right|$, where $\xi 1'(t), \xi 1''(t)$ are the values of approximate solution with perturbation η_1, η_2 for $X = 1000, 500$, respectively; similarly for $k = 1$, the desired calculated approximate solution is $\xi 2(t) = C^T \psi(t)$, and errors associated with the above example are $E1(t) = |\xi(t) - \xi 2(t)|$, $E2(t) = \left|\xi(t) - \xi 2'(t)\right|$, $E3(t) = \left|\xi(t) - \xi 2''(t)\right|$, where $\xi 2'(t), \xi 2''(t)$ are the values of approximate solution with perturbation η_1, η_2 for $X = 1000, 500$.

The corresponding values for C^T without noise in $I_1(u)$ for $k = 0, 1$ are given as:

$$C^T = [0.98708, 0.303577, 0.475601, 0.561682, 0.527861, 0.333339],$$
$$C^T = [0.3142815, 0.91566, 0.135183, 0.153866, 0.141301, 0.89904, 0.35216, 0.432607, 0.471492, 0.460757, 0.390692, 0.235741]$$

respectively, whereas the values for C^T for two different noise terms $\eta_1 = 0.001$ and $\eta_2 = 0.001$ in $I_1(u)$ for $k = 0, 1$ are given as:

$$C^T = [0.98729, 0.303448, 0.475631, 0.561742, 0.527859, 0.333339],$$
$$C^T = [0.314, 0.9162, 0.135115, 0.153943, 0.141283, 0.89912, 0.352147, 0.432693, 0.471496, 0.460708, 0.390727, 0.235732]$$

and

$$C^T = [0.98752, 0.303556, 0.475701, 0.561626, 0.527816, 0.33329],$$
$$C^T = [0.3138, 0.91671, 0.135111, 0.153991, 0.141156, 0.9015646, 0.35199, 0.432735, 0.471362, 0.461036, 0.390664, 0.2357]$$

respectively.

Figure 1 compares $\xi(t)$ (exact emissivity) with $\xi 1(t)$ and $\xi 2(t)$ (approximate emissivities without noise) for dilation parameters $k = 0$ and $k = 1$, respectively, for the generalized Abel integral equation, (where k is the dilation argument for the given Bernstein polynomials multiwavelets and order of Bernstein polynomial is $N = 5$),

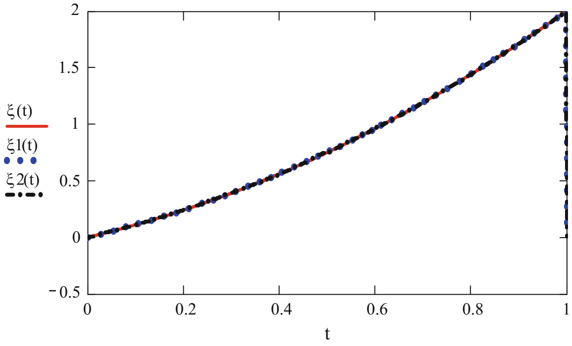

Fig. 1 Comparison of emissivities; exact emissivity $\xi(t)$; reconstructed emissivity $\xi 1(t)$ (with dilation parameter $k = 0$); and reconstructed emissivity $\xi 2(t)$ (with dilation parameter $k = 1$) for $N = 5$

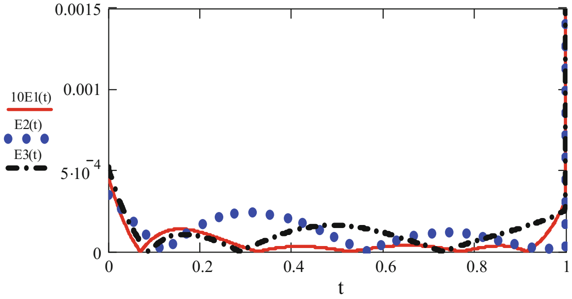

Fig. 2 Comparison of absolute errors, $E1(t)$ (with $\eta_0 = 0$), $E2(t)$ (with $\eta_1 = 0.001$), and $E3(t)$ (with $\eta_2 = 0.002$) for $N = 5$ and dilation parameter $k = 0$

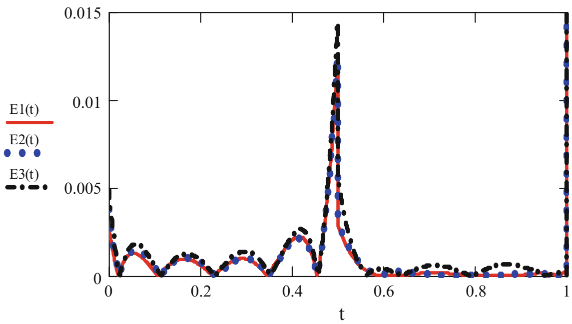

Fig. 3 Comparison of absolute errors, $E1(t)$ (with $\eta_0 = 0$), $E2(t)$ (with $\eta_1 = 0.001$), and $E3(t)$ (with $\eta_2 = 0.002$) for $N = 5$ and dilation parameter $k = 1$

whereas the correlation between absolute errors $E1(t)$, $E2(t)$, and $E3(t)$ is drawn by choosing different noise levels, and $\eta_0 = 0$, $\eta_1 = 0.001$, and $\eta_2 = 0.002$ showing the effect of X are drawn in Fig. 2 and Fig. 3 for $k = 0$ and $k = 1$, respectively.

We used the Bernstein polynomial multiwavelets technique and solved the above problem (21) with $N = 5$ and $k = 0, 1$ to get the required solution, $\sigma_{1000} = 0.0632456$; $\sigma_{500} = 0.0894427$ and $\sigma_{1000} = 0.0632645$; $\sigma_{500} = 0.0894556$ are values of averages deviations for dilation parameters $k = 0$ and $k = 1$, respectively.

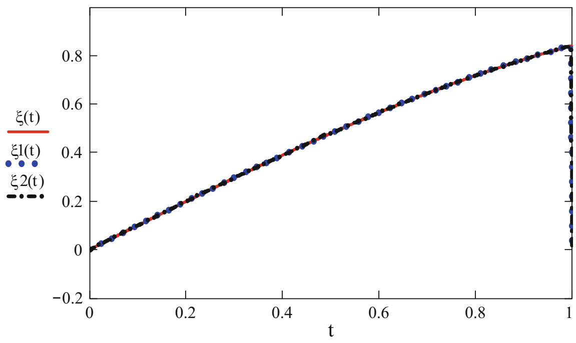

Fig. 4 Comparison of emissivities; exact emissivity $\xi(t)$; reconstructed emissivity $\xi 1(t)$ (with dilation parameter $k = 0$); and reconstructed emissivity $\xi 2(t)$ (with dilation parameter $k = 1$) for $N = 5$

5.2 *Example 2*

In this example, we consider Eq. (11) with $p(u) = \frac{3}{4}e^u, \quad q(u) = e^{2u} + 1/\sqrt{2\pi}$, for the pair $\xi(t) = \sin(t)$, for $0 \le t \le 1$,

$$I_1(u) = \left[{}_1F_2\left(1; \frac{5}{2}; \frac{7}{4}; -\frac{u^2}{4}\right) e^u u^{3/2} + (e^{2u} + 1) C\left(\frac{\sqrt{2-2u}}{\sqrt{\pi}}\right) \sin u + S\left(\frac{\sqrt{2-2u}}{\sqrt{\pi}}\right) \cos u \right], \tag{22}$$

where $C(u)$ and $S(u)$ are known as Fresnel integrals, defined by

$$C(u) = \int_0^u \cos\left(\frac{\pi t^2}{2}\right) dt, \ \mathrm{S}(u) = \int_0^u \sin\left(\frac{\pi t^2}{2}\right) dt$$

respectively.

Equations (12) and (20) provide the desired calculated approximate solution; for $k = 0$, it is given by $\xi 1(t) = C^T \psi(t)$, and errors associated with the above example are $E1(t) = |\xi(t) - \xi 1(t)|$, $E2(t) = \left|\xi(t) - \xi 1'(t)\right|$, $E3(t) = \left|\xi(t) - \xi 1''(t)\right|$, where $\xi 1'(t), \xi 1''(t)$ are the values of approximate solution with perturbation η_1, η_2 for $X = 1000, 500$, respectively; similarly for $k = 1$, the desired calculated approximate solution is $\xi 2(t) = C^T \psi(t)$, and errors associated with the above example are $E1(t) = |\xi(t) - \xi 2(t)|$, $E2(t) = \left|\xi(t) - \xi 2'(t)\right|$, $E3(t) = \left|\xi(t) - \xi 2''(t)\right|$, where $\xi 2'(t), \xi 2''(t)$ are the values of approximate solution with perturbation η_1, η_2 for $X = 1000, 500$.

Figure 4 compares $\xi(t)$ (exact emissivity) with $\xi 1(t)$ and $\xi 2(t)$ (approximate emissivities without noise) for dilation parameters $k = 0$ and $k = 1$, respectively, for the generalized Abel integral equation, (where k is the dilation argument for the given Bernstein polynomials multiwavelets and order of Bernstein polynomial is $N = 5$), whereas the correlation between absolute errors $E1(t)$, $E2(t)$, and $E3(t)$ is drawn by choosing different noise levels, and $\eta_0 = 0$, $\eta_1 = 0.001$, and $\eta_2 = 0.002$ showing the effect of X are drawn in Fig. 5 and Fig. 6 for $k = 0$ and $k = 1$, respectively.

We used the Bernstein polynomial multiwavelets technique and solved the above problem (22) with $N = 5$ and $k = 0, 1$ to get the required solution, $\sigma_{1000} =$

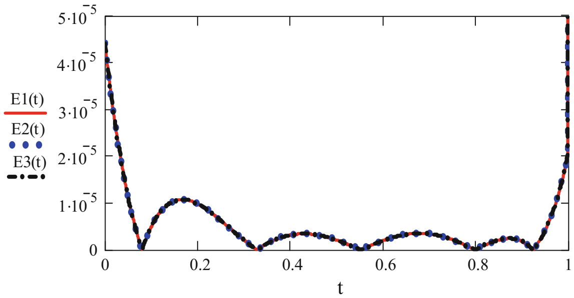

Fig. 5 Comparison of absolute errors, $E1(t)$ (with $\eta_0 = 0$), $E2(t)$ (with $\eta_1 = 0.001$), and $E3(t)$ (with $\eta_2 = 0.002$) for $N = 5$ and dilation parameter $k = 0$

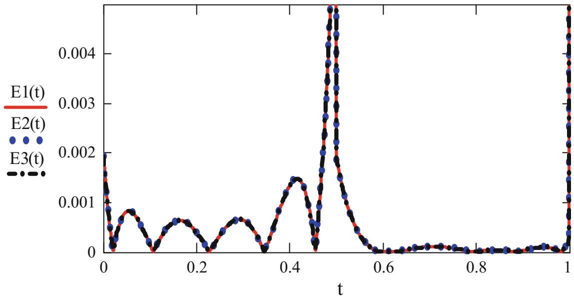

Fig. 6 Comparison of absolute errors, $E1(t)$ (with $\eta_0 = 0$), $E2(t)$ (with $\eta_1 = 0.001$), and $E3(t)$ (with $\eta_2 = 0.002$) for $N = 5$ and dilation parameter $k = 1$

0.02661; $\sigma_{500} = 0.037632$ and $\sigma_{1000} = 0.026629$; $\sigma_{500} = 0.037645$ which are the values of averages deviations for deviations with dilation parameters $k = 0$ and $k = 1$, respectively.

6 Conclusions

For two unique estimations of k (dilation parameter) that is $k = 0$ and $k = 1$, we have constructed operational matrix of integration for Bernstein polynomials multi-wavelets and utilized them to suggest a new steady approach for finding the solution of generalized Abel integral equation numerically, and furthermore, our technique demonstrates the comparison between the solution for two different dilation parameters $k = 0$ and $k = 1$. The stability with respect to the data is restored and good result is obtained, even for tiny sample intervals and high perturbation in the data, the selection of only six orthonormal polynomials of degree 5 makes the method easy and straightforward to use.

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Empirical Analysis of Probabilistic Bounds

R. Swarnalatha and V. Kumaran

1 Introduction

Let $A_1, A_2, \ldots, A_n$ be arbitrary events in an arbitrary probability space. Let γ be the random variable denoting the number of events that occur out of these n events. The complete knowledge of the probability distribution of γ is not assumed. Known only the first few binomial moments of the random variable γ, sharp bounds to the probability of union of events in terms of the given binomial moments can be obtained by formulating the problem into a linear programming problem known as discrete moment problem [15]. In [15], closed form sharp bounds are given when first 2, 3, and 4 binomial moments are known.

The bounds to the probability of union of events in terms of the binomial moments given by Boole in [1] were proved to be sharp by Frechet in [8]. Using operations research techniques and other techniques, bounds are presented to the probability of union of events when the first few binomial moments are known in [2, 5–7, 9–11, 13–15, 19, 23]. Bounds to the same are also found from the disaggregated version of the problem in [4, 16, 24]. Also, several applications of the bounds obtained from the discrete moment problem and the disaggregated version of it are presented in [3, 22]. In [17], new methods are proposed to solve the univariate continuous and discrete moment problems. Without assuming the complete knowledge of the probability distribution but just assuming its shape, bounds are presented in [12, 18, 20, 21].

In [12, 21], the complete knowledge of the probability distribution of γ is not assumed. Instead, the shape of the distribution of the random variable γ is assumed and the closed form bounds to the probability of union of events are found, when

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any number of binomial moments are known by identifying the inverse of the matrix corresponding to the dual feasible basis.

In this paper, the tightness of the probabilistic bounds obtained in [12] with respect to the monotonicity of the distribution is empirically analyzed. Also, the probability distribution corresponding to the optimal basis is studied for different monotonic functions with different nature of monotonicity.

The outline of the paper is as follows. In Sect. 2, the mathematical formulation of the problem is given. In Sect. 3, the structures of the dual feasible bases are presented. In Sects. 4 and 5, the primal feasibility conditions and the bounds are presented when the probability distribution of the random variable γ is assumed to be monotonically increasing and decreasing, respectively. And in Sect. 6, the empirical analysis is carried out and discussed.

2 Problem Formulation

Let the kth binomial moment of the events $A_1, A_2, \ldots, A_n$ be denoted by S_k and is given by the following equation.

$$S_k = \sum_{1\le i_1<\cdots<i_k\le n} P\left(A_{i_1}\ldots A_{i_k}\right),\ k = 1, \ldots, n. \tag{1}$$

Let $S_0 = 1$, then (see [15])

$$S_k = E\left[\binom{\gamma}{k}\right],\ k = 0, \ldots, n. \tag{2}$$

Let P($\gamma = k$) = p_k, $k = 0, \ldots, n$, then (2) can be written as

$$S_k = \sum_{i=0}^{n} \binom{i}{k} p_i,\ k = 0, \ldots, n. \tag{3}$$

When the distribution is supposed to be monotonically increasing, the discrete moment problem takes the following form: (see [12])

$$min(max)\ nv_0 + \sum_{i=1}^{n} (n-i+1)\, v_i$$

subject to

$$\sum_{i=0}^{n} (n-i+1)\, v_i = 1 \tag{4}$$

$$\sum_{i=0}^{n}\left[\binom{i}{j}+\cdots+\binom{n}{j}\right]v_i = S_j,\ \ j=1,\ldots,m$$

$$v_i \geq 0,\ \ i=0,\ldots,n$$

When the distribution is supposed to be monotonically decreasing, the discrete moment problem takes the following form: (see [12])

$$min(max)\ \sum_{i=1}^{n} i\,v_i$$

subject to

$$\sum_{i=0}^{n}(i+1)\,v_i = 1 \tag{5}$$

$$\sum_{i=0}^{n}\left[\binom{0}{j}+\cdots+\binom{i}{j}\right]v_i = S_j,\ \ j=1,\ldots,m$$

$$v_i \geq 0,\ \ i=0,\ldots,n$$

In this paper, we empirically analyze the bounds and the probability distribution generated from the optimal basis with respect to the monotonicity of the distribution taken.

3 Structures of Dual Feasible Bases of Relaxed Problems

Let $a_0, a_1, \ldots, a_n$ be the columns of the coefficient matrix A, corresponding to the variables $v_0, v_1, \ldots, v_n$, respectively. Let B be a basis and I_B be the subscript set of those columns of A that form B. Let C be the $n+1$-component vector consisting of coefficients of the variables in the objective function and C_B be the $m+1$-component vector consisting of basic components of C.

Any basis B consists of $m+1$ variables. And these $m+1$ variables are chosen among the $n+1$ variables $\{v_0, v_1, \ldots, v_n\}$. In other words, $n-m$ variables among these $n+1$ variables are discarded. Let $t_1, t_2, \ldots, t_{n-m}$ be subscripts of those $n-m$ discarded variables and $t_1 < t_2 < \cdots < t_{n-m}$.

From the structures of dual feasible bases given in [18], it could be easily seen that $I_B = \{0, 1, \ldots, n\} - \{t_1, t_2, \ldots, t_{n-m}\}$ is dual feasible if $t_1, t_2, \ldots, t_{n-m}$ satisfy the following conditions (see [12]).

Note that these conditions over t_i are true for $M = n$, $M = 0$, and $1 \leq M \leq n-1$.

	$m+1$ *even*	$m+1$ *odd*
min. problem	$1 \le t_1 < t_2 < \cdots < t_{n-m} < n$ *also* t_i *is odd for odd* i *and* t_i *is even for even* i	$1 \le t_1 < t_2 < \cdots < t_{n-m} \le n$ *also* t_i *is odd for odd* i *and* t_i *is even for even* i

max. problem	*If* $t_1 \neq 0$, *then*
	$1 < t_1 < t_2 < \cdots < t_{n-m} \le n$
	also t_i *is even for odd* i
	and t_i *is odd for even* i
	If $t_1 = 0$, *then*
	$1 \le t_2 < t_3 < \cdots < t_{n-m} \le n$
		If $t_1 \neq 0$, *then*
		$1 < t_1 < t_2 < \cdots < t_{n-m} < n$
		also t_i *is even for odd* i
		and t_i *is odd for even* i
		If $t_1 = 0$, *then*
		$1 \le t_2 < t_3 < \cdots < t_{n-m} \le n$

4 The Case of Increasing Distribution

Let $t_0 = -1$ and $t_{n-m+1} = n+1$. Then $B_\uparrow = \left(b_{\uparrow ij}\right)$, the matrix corresponding to the basis structure $\{0, 1, \ldots, n\} - \{t_1, t_2, \ldots, t_{n-m}\}$, is

$$\binom{j+l-1}{i-1} + \cdots + \binom{n}{i-1},\ 1 \le i \le m+1,\ t_l - (l-2) \le j \le t_{l+1} - l,\ where\ l\ :\ 0\ to\ n-m.$$

But $\binom{j+l-1}{i-1} + \cdots + \binom{n}{i-1} = \binom{n+1}{i} - \binom{j+l-1}{i}$. Hence,

$$\left(b_{\uparrow ij}\right) =$$
$$\binom{n+1}{i} - \binom{j+l-1}{i},\ 1 \le i \le m+1,\ t_l - (l-2) \le j \le t_{l+1} - l,\ where\ l\ :\ 0\ to\ n-m.$$

The matrix $D_\uparrow = \left(d_{\uparrow ij}\right)$ given below is the inverse of the basis matrix $B_\uparrow$ (see [12]).

$$\left(d_{\uparrow ij}\right) = (-1)^{i+j+l}\, \eta_{t_1,\ldots,t_{n-m}}(i+l-1),\ t_l - (l-2) \le i \le t_{l+1} - l,$$
$$1 \le j \le m+1, \qquad (l\ :\ 0\ to\ n-m),$$

where $\eta_{t_1,\ldots,t_{n-m}}(i+l-1) = \binom{j}{i+l-1} + \sum_{p=1}^{n-m} (-1)^p \prod_{k=1,k\neq p}^{n-m} \frac{[t_k-(i+l-1)]}{|t_k-t_p|} \frac{\binom{j}{t_p}\binom{n+1}{i+l-1}}{\binom{n+1}{t_p}}$.

Throughout this paper let us denote $\eta_{t_1,\ldots,t_{n-m}}(i)$ by $\eta_{n-m}(i)$ for the sake of brevity.

4.1 Bounds for the Case of Increasing Distribution

A dual feasible basis is optimal only when it is primal feasible also. Considering the minimization problem (4), the primal feasibility conditions for the dual feasible basis $\{0, 1, \ldots, n\} - \{t_1, t_2, \ldots, t_{n-m}\}$ where t_i is odd(even) for odd(even) i are given below:

$$\sum_{j=1}^{m+1} (-1)^{i+j+l} \left[\eta_{n-m}(i+l-1)\right] S_{j-1} \ge 0,$$
$$t_l - (l-2) \le i \le t_{l+1} - l,\ l\ :\ 0\ to\ n-m, \qquad (6)$$

where $\eta_{n-m}(i+l-1) = \binom{j}{i+l-1} + \sum_{p=1}^{n-m} (-1)^p \prod_{k=1,k\neq p}^{n-m} \frac{[t_k-(i+l-1)]}{|t_k-t_p|} \frac{\binom{j}{t_p}\binom{n+1}{i+l-1}}{\binom{n+1}{t_p}}$.

These conditions hold good for both odd and even $m+1$.

The corresponding closed form lower bound for the probability of union of n events is given below:

$$P(\gamma \geq 1) \geq 1 + \sum_{j=1}^{m+1} (-1)^j \left[1 + \sum_{p=1}^{n-m} (-1)^p \prod_{k=1,k\neq p}^{n-m} \frac{t_k}{|t_k - t_p|} \frac{\binom{j}{t_p}}{\binom{n+1}{t_p}} \right] S_{j-1}. \quad (7)$$

The lower bound is optimal when the chosen basis is both dual and primal feasible.

Consider the dual feasible bases of maximization problem with $t_1 \neq 0$. To get the upper bound from such dual feasible bases, we proceed in the same way as above. The primal feasibility conditions and the bound expressions are the same as (6) and (7), respectively. But here t_i is odd(even) for even(odd) i.

Now consider any basis B, for which $t_1 = 0$ $(I_B \subset \{1, \ldots, n\})$ of maximization problem. The primal feasibility conditions are (for both odd and even $m+1$).

$$\sum_{j=1}^{m+1} (-1)^{i+j+l} \left[\eta_{n-m}(i+l-1)\right] S_{j-1} \geq 0,$$
$$t_l - (l-2) \leq i \leq t_{l+1} - l,\ l\ :\ 1\ to\ n-m. \quad (8)$$

The corresponding closed form upper bound is given by $P(\gamma \geq 1) \leq 1$.

5 The Case of Decreasing Distribution

Let $t_0 = -1$ and $t_{n-m+1} = n+1$. Then $B_{\downarrow} = (b_{\downarrow ij})$, the matrix corresponding to the basis structure $\{0, 1, \ldots, n\} - \{t_1, t_2, \ldots, t_{n-m}\}$, is

$$\binom{0}{i-1} + \cdots + \binom{j+l-1}{i-1},\ 1 \leq i \leq m+1,\ t_l - (l-2) \leq j \leq t_{l+1} - l,\ where\ l\ :\ 0\ to\ n-m.$$

But $\binom{0}{i-1} + \cdots + \binom{j+l-1}{i-1} = \binom{j+l}{i}$. Hence,

$$(b_{\downarrow ij}) = \binom{j+l}{i},\ 1 \leq i \leq m+1,\ t_l - (l-2) \leq j \leq t_{l+1} - l,\ where\ l\ :\ 0\ to\ n-m.$$

The matrix $D_{\downarrow} = (d_{\downarrow ij})$ given below is the inverse of the basis matrix $B_{\downarrow}$. (see [12])

$$(d_{\downarrow ij}) = (-1)^{i+j+l} \theta_{t_1,\ldots,t_{n-m}}(i+l),\ t_l - (l-2) \leq i \leq t_{l+1} - l,$$
$$1 \leq j \leq m+1, \qquad (l\ :\ 0\ to\ n-m),$$

where $\theta_{t_1,\ldots,t_{n-m}}(i+l) = \binom{j}{i+l} + \sum_{p=1}^{n-m}(-1)^p \prod_{k=1,k\neq p}^{n-m} \frac{[t_k+1-(i+l)]}{|t_k-t_p|} \frac{\binom{j}{t_p+1}\binom{n+1}{i+l}}{\binom{n+1}{t_p+1}}$.

Throughout this paper let us denote $\theta_{t_1,\ldots,t_{n-m}}(i)$ by $\theta_{n-m}(i)$ for brevity.

5.1 Bounds for the Case of Decreasing Distribution

Consider the minimization problem (5). The primal feasibility conditions for the dual feasible basis $\{0, 1, \ldots, n\} - \{t_1, t_2, \ldots, t_{n-m}\}$ where t_i is odd(even) for odd(even) i are given below:

$$\sum_{j=1}^{m+1}(-1)^{i+j+l}\left[\theta_{n-m}(i+l)\right] S_{j-1} \geq 0,$$
$$t_l - (l-2) \leq i \leq t_{l+1} - l,\ l\ :\ 0\ to\ n-m, \tag{9}$$

where $\theta_{n-m}(i+l) = \binom{j}{i+l} + \sum_{p=1}^{n-m}(-1)^p \prod_{k=1,k\neq p}^{n-m} \frac{[t_k+1-(i+l)]}{|t_k-t_p|} \frac{\binom{j}{t_p+1}\binom{n+1}{i+l}}{\binom{n+1}{t_p+1}}$.

These conditions hold good for both odd and even $m+1$.

The corresponding closed form lower bound for the probability of union of n events is given below:

$$P(\gamma \geq 1) \geq 1 + \sum_{j=1}^{m+1}(-1)^j \left[1 + \sum_{p=1}^{n-m}(-1)^p \prod_{k=1,k\neq p}^{n-m} \frac{t_k+1}{|t_k-t_p|} \frac{\binom{j}{t_p+1}}{\binom{n+1}{t_p+1}}\right] S_{j-1}. \tag{10}$$

Consider the dual feasible bases of maximization problem with $t_1 \neq 0$. To get the upper bound from the basis when $t_1 \neq 1$, we proceed in the same way. The primal feasibility conditions and the bound expressions are the same as (9) and (10), respectively. But when $t_1 \neq 1$, t_i is odd(even) for even(odd) i.

For the dual feasible bases of maximization problem with $t_1 = 0$, the bound expression is same as (10) for the appropriate selection of t_i (see [12]).

6 Numerical Examples

In this section, we present the bounds and the measure of deviation of the optimal probability distribution from the actual probability distribution. Let p_i represent the probabilities of the actual probability distribution taken in question and z_i be the probability distribution generated from the optimal basis. The measure $\sum_i (p_i - z_i)^2$ gives the measure of deviation of the optimal probability distribution from the actual probability distribution. The bounds and the measure $\sum_i (p_i - z_i)^2$ are given for some particular monotonic distributions of size $n = 11$ and 51 in Table 1.

Table 1 Comparing the measure of deviation obtained for different monotonic distributions with different monotonicity for $n = 11$ and 51

	$n = 11$											
	When the distribution is monotonically increasing											
	Distribution 1: $\sum_{i=1}^{11} p_i = 0.955822$				Distribution 2: $\sum_{i=1}^{11} p_i = 0.99827$				Distribution 3: $\sum_{i=1}^{11} p_i = 0.988177$			
	Max. Problem		Min. Problem		Max. Problem		Min. Problem		Max. Problem		Min. Problem	
m	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$
2	1.0	0.00536626	0.944481	0.00180082	1.0	0.00204394	0.996055	0.00624326	1.0	0.00256169	0.970694	0.00471493
3	0.970035	0.00121579	0.948344	0.00143519	1.0	0.0000213066	0.997448	0.0000253759	1.0	0.00193531	0.975879	0.00361011
4	0.962953	0.00113013	0.951707	0.000710983	0.999362	0.0000205231	0.997766	0.0000581681	0.999787	0.00192733	0.983163	0.0016232
5	0.958681	0.0005448	0.953511	0.0003813	0.998801	0.0000206952	0.998036	9.17557e−6	0.990761	0.000836486	0.984785	0.000758723
6	0.957149	0.000262675	0.95435	0.0003706	0.998475	7.10675e−6	0.998133	6.80404e−6	0.98984	0.000813629	0.987442	0.000329313
7	0.956427	0.000226323	0.955337	0.000253755	0.998361	5.22544e−6	0.998197	3.76191e−6	0.9883	0.000124429	0.987733	0.000307709
8	0.956123	0.000300265	0.955757	0.0000248043	0.998307	4.50188e−6	0.998252	1.13476e−6	0.988255	0.000145849	0.988177	2.18051e−25
9	0.955869	0.0000814816	0.955794	0.00006542	0.998277	1.6052e−6	0.998263	1.92252e−6	0.988177	5.91599e−25	0.988177	5.23406e−25
10	0.955828	0.0000267754	0.955817	0.0000158272	0.998271	7.09694e−7	0.998269	5.78337e−7	0.988177	5.74343e−25	0.988177	1.69304e−25
	When the distribution is monotonically decreasing											
	Distribution 4: $\sum_{i=1}^{11} p_i = 0.873372$				Distribution 5: $\sum_{i=1}^{11} p_i = 0.0723761$				Distribution 6: $\sum_{i=1}^{11} p_i = 0.863105$			
	Max. Problem		Min. Problem		Max. Problem		Min. Problem		Max. Problem		Min. Problem	
m	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$
2	0.885001	0.00562601	0.813393	0.00951446	0.117491	0.00703301	0.0665415	0.0000901559	0.872661	0.011241	0.781304	0.0174136
3	0.87949	0.0058113	0.854146	0.00228475	0.082853	0.000581311	0.0703168	0.0000299615	0.864999	0.00410992	0.835542	0.00682339
4	0.876459	0.000667192	0.865274	0.00159912	0.0751779	0.0000825661	0.0715124	0.0000216051	0.863369	0.000867526	0.854913	0.0030734
5	0.874658	0.000352583	0.870641	0.000524384	0.0731758	0.000020924	0.0720054	0.0000112129	0.863147	0.000480482	0.861333	0.00144647
6	0.874348	0.00021109	0.87209	0.000566558	0.0725804	8.99761e−6	0.0721904	7.86251e−6	0.86311	0.00012472	0.862847	0.000438057
7	0.874074	0.000380992	0.872923	0.000174301	0.0724351	5.15101e−6	0.0723015	3.20519e−6	0.863105	0.000024828	0.863074	0.000127506
8	0.873533	0.00010497	0.873142	0.000214245	0.0723833	6.25488e−7	0.0723326	7.69565e−6	0.863105	1.2826e−23	0.863102	0.000029102
9	0.873404	0.000036275	0.873278	0.00028771	0.0723784	7.19208e−7	0.0723707	1.06361e−6	0.863105	1.01155e−23	0.863105	1.46191e−24
10	0.873367	0.000048826	0.87338	0.000015918	0.0723769	4.33539e−7	0.072751	7.83415e−7	0.863105	3.22531e−24	0.863105	1.82469e−24

(continued)

Table 1 (continued)

	$n = 51$											
	When the distribution is monotonically increasing											
	Distribution 1: $\sum_{i=1}^{51} p_i = 0.998286$				Distribution 2: $\sum_{i=1}^{51} p_i = 0.999453$				Distribution 3: $\sum_{i=1}^{51} p_i = 0.994831$			
	Max. Problem		Min. Problem		Max. Problem		Min. Problem		Max. Problem		Min. Problem	
m	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$
2	1.0	0.00163687	0.99281	0.00185339	1.0	0.166518	0.994084	0.16704	1.0	0.00150616	0.989811	0.00189996
3	1.0	0.00329566	0.99437	0.00363176	1.0	0.00129957	0.996794	0.00143079	1.0	0.00469849	0.990937	0.00571449
4	1.0	0.00073995	0.99584	0.000759338	1.0	0.161332	0.99736	0.161564	1.0	0.000604958	0.992823	0.000613185
5	1.0	0.000774721	0.996472	0.000797206	1.0	0.000314065	0.998066	0.000320865	1.0	0.000439911	0.993323	0.000377986
6	1.0	0.000392419	0.997037	0.000478752	1.0	0.154833	0.998323	0.155011	1.0	0.000452437	0.993592	0.00050981
7	1.0	0.000330149	0.997353	0.000372654	1.0	0.000129367	0.998621	0.000149692	1.0	0.000417289	0.993834	0.000448712
	When the distribution is monotonically decreasing											
	Distribution 4: $\sum_{i=1}^{51} p_i = 0.964233$				Distribution 5: $\sum_{i=1}^{51} p_i = 0.585411$				Distribution 6: $\sum_{i=1}^{51} p_i = 0.967045$			
	Max. Problem		Min. Problem		Max. Problem		Min. Problem		Max. Problem		Min. Problem	
m	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$	UB	$\sum_i (p_i - z_i)^2$	LB	$\sum_i (p_i - z_i)^2$
2	0.969401	0.00166833	0.86445	0.0126567	0.919054	0.142325	0.523528	0.00488383	0.972001	0.00157776	0.899609	0.00642495
3	0.967684	0.0011051	0.920828	0.00324461	0.848832	0.114811	0.556502	0.00143277	0.970601	0.00152075	0.935351	0.00157787
4	0.966569	0.00066962	0.941537	0.0014887	0.766091	0.078587	0.571012	0.000571719	0.969149	0.00057567	0.946291	0.00150387
5	0.965942	0.000533313	0.95231	0.00734192	0.666152	0.0201467	0.577302	0.000320629	0.968567	0.000273234	0.956043	0.000843289
6	0.965472	0.000412453	0.957043	0.0005569	0.622356	0.00496023	0.580683	0.00021542	0.968312	0.000387546	0.962092	0.000374236
7	0.965182	0.000305659	0.960076	0.000336858	0.603249	0.00145469	0.582512	0.000159596	0.968072	0.000394421	0.964131	0.000106265

The complete details of the probability distribution used in Table 1, for $n = 11$, are given below. The probability distribution taken for $n = 51$ is given as graphs in Fig. 1.

Monotonically increasing distributions: (for $n = 11$)

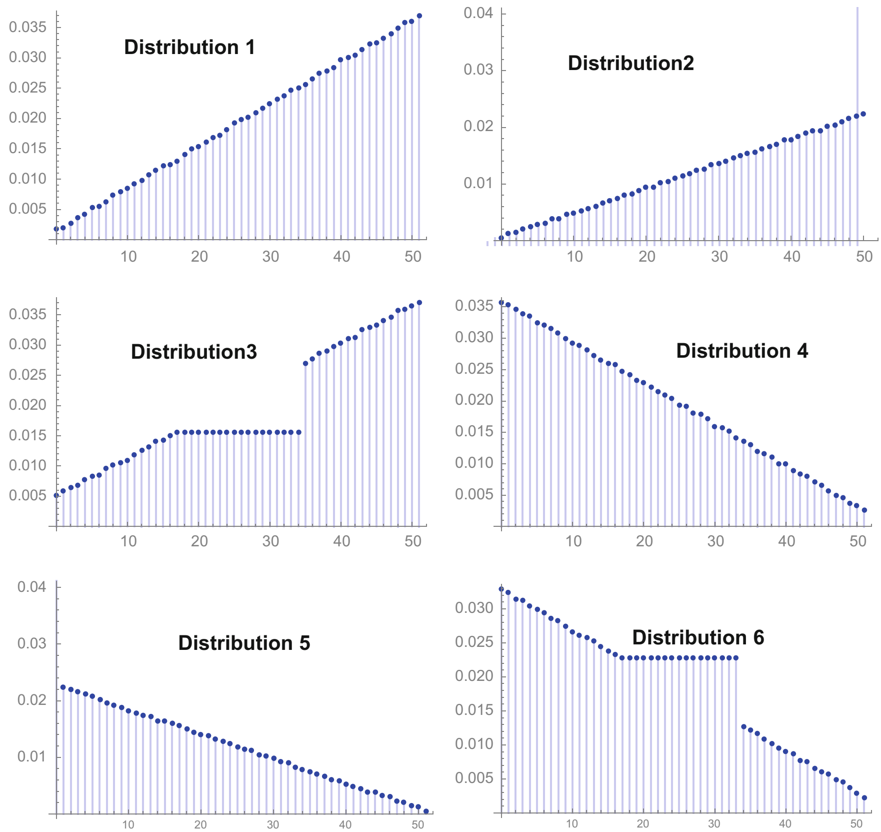


Fig. 1 Monotonically increasing and decreasing distributions used in Table 1 for $n = 51$

Distribution 1:
$p_0 = 0.0441779$; $p_1 = 0.0518822$; $p_2 = 0.0624065$; $p_3 = 0.0647823$; $p_4 = 0.0759011$; $p_5 = 0.0796526$; $p_6 = 0.0853452$; $p_7 = 0.0944972$; $p_8 = 0.103421$; $p_9 = 0.106504$; $p_{10} = 0.114037$; $p_{11} = 0.117393$.
Distribution 2:
$p_0 = 0.0017297$; $p_1 = 0.00232233$; $p_2 = 0.00380578$; $p_3 = 0.00469149$; $p_4 = 0.00593368$; $p_5 = 0.00696946$; $p_6 = 0.00789625$; $p_7 = 0.00861336$; $p_8 = 0.0098376$; $p_9 = 0.0105402$; $p_{10} = 0.0114955$; $p_{11} = 0.926165$.
Distribution 3:
$p_0 = 0.0118234$; $p_1 = 0.0245958$; $p_2 = 0.0408115$; $p_3 = 0.0558881$; $p_4 = 0.0688818$; $p_5 = 0.0820979$; $p_6 = 0.100621$; $p_7 = 0.108476$; $p_8 = 0.126701$; $p_9 = 0.126701$; $p_{10} = 0.126701$; $p_{11} = 0.126701$.
Monotonically decreasing distributions: (for $n = 11$)
Distribution 4:
$p_0 = 0.126628$; $p_1 = 0.121531$; $p_2 = 0.115248$; $p_3 = 0.107762$; $p_4 = 0.0918901$; $p_5 = 0.085301$; $p_6 = 0.0809117$; $p_7 = 0.0675205$; $p_8 = 0.0645845$; $p_9 = 0.0547059$; $p_{10} = 0.0461053$; $p_{11} = 0.0378116$.
Distribution 5:
$p_0 = 0.927624$; $p_1 = 0.0116666$; $p_2 = 0.0104373$; $p_3 = 0.0102649$; $p_4 = 0.00915516$; $p_5 = 0.007645$; $p_6 = 0.00667127$; $p_7 = 0.00514484$; $p_8 = 0.00426389$; $p_9 = 0.00389097$; $p_{10} = 0.0020566$; $p_{11} = 0.00117962$.
Distribution 6:
$p_0 = 0.136895$; $p_1 = 0.136895$; $p_2 = 0.136895$; $p_3 = 0.136895$; $p_4 = 0.0902154$; $p_5 = 0.0834857$; $p_6 = 0.0738748$; $p_7 = 0.0595357$; $p_8 = 0.0537315$; $p_9 = 0.0420723$; $p_{10} = 0.031877$; $p_{11} = 0.0176263$.

In Table 3, we considered different monotonic distributions of different monotonicity, especially with different levels of steepness and studied the effect of the steepness of the distributions considered over the bounds, the percentage relative errors $100 \times \frac{|optimal\ bound - actual\ value|}{actual\ value}$ and the measure $\sum_i (p_i - z_i)^2$. The monotonic distributions used in Table 3 are given in Fig. 2.

Discussions and Conclusions
With the increase in the value of m, the range of the bounds decreases. In other words, the bounds become more tighter for larger m. Also for large n, the method presented in this paper is more elegant for small and large m rather than the values of m around $\frac{n}{2}$. From the Tables 1, 2 and 3, it could be easily seen that for the increasing distributions, the range of the optimal bounds increases with the decrease in the steepness of the distribution. This in fact is easily seen from the increasing distributions taken in Table 3. More the steepness of the distribution, less the relative error. Also, we could observe a different effect of the steepness of the distribution for the decreasing ones. For the decreasing distributions, the effect of the steepness of the distribution over the bounds and the relative errors are exactly opposite to that of the increasing distributions. That is, for decreasing distributions, more the steepness more the relative error. Also for decreasing distributions, the rate of decrease of the relative error after certain stage of m increases with the increase in the steepness of the

Table 2 Percentage relative error of the bounds presented in Table 1 for $n = 51$

	When the distribution is monotonically increasing					
	Distribution 1: $\sum_{i=1}^{51} p_i = 0.998286$		Distribution 2: $\sum_{i=1}^{51} p_i = 0.999453$		Distribution 3: $\sum_{i=1}^{51} p_i = 0.994831$	
	Max. Problem	Min. Problem	Max. Problem	Min. Problem	Max. Problem	Min. Problem
m	% RE	% RE	% RE	% RE	% RE	% RE
2	0.171695	0.548527	0.054707	0.537195	0.519567	0.50459
3	0.171695	0.392248	0.054707	0.266092	0.519567	0.391467
4	0.171695	0.245047	0.054707	0.209416	0.519567	0.201896
5	0.171695	0.181669	0.054707	0.138763	0.519567	0.151563
6	0.171695	0.125116	0.054707	0.113063	0.519567	0.124601
7	0.171695	0.093465	0.054707	0.083253	0.519567	0.100197
	When the distribution is monotonically decreasing					
	Distribution 4: $\sum_{i=1}^{51} p_i = 0.964233$		Distribution 5: $\sum_{i=1}^{51} p_i = 0.585411$		Distribution 6: $\sum_{i=1}^{51} p_i = 0.967045$	
	Max. Problem	Min. Problem	Max. Problem	Min. Problem	Max. Problem	Min. Problem
m	% RE	% RE	% RE	% RE	% RE	% RE
2	0.535944	10.348	56.9931	10.5709	0.512437	6.97348
3	0.357866	4.50152	44.9977	4.93822	0.367656	3.27744
4	0.242215	2.3538	30.8637	2.45968	0.217543	2.14612
5	0.177173	1.23659	13.7923	1.38516	0.157379	1.13771
6	0.128463	0.745724	6.311	0.807552	0.131005	0.512235
7	0.098353	0.431165	3.04704	0.495257	0.106132	0.301421

Table 3 Table showing the effect of steepness of the distribution over the bounds, the relative errors and the measure of deviation

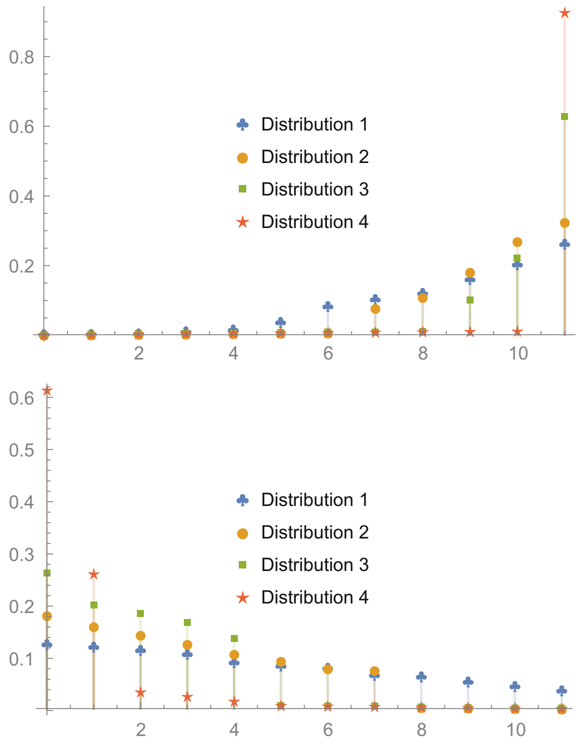
	When the distribution is monotonically increasing								
		Distribution 1: $\sum_{i=1}^{11} p_i = 0.99827$		Distribution 2: $\sum_{i=1}^{11} p_i = 0.99827$		Distribution 3: $\sum_{i=1}^{11} p_i = 0.99827$		Distribution 4: $\sum_{i=1}^{11} p_i = 0.99827$	
m		Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.
2	Bound	0.990455	1.0	0.994872	1.0	0.995646	1.0	0.996055	1.0
	% RE	0.782858	0.173270	0.340442	0.173270	0.262872	0.173270	0.221892	0.173270
	$\sum_i (p_i - z_i)^2$	0.019039	0.019372	0.017123	0.016965	0.069381	0.052337	0.006243	0.002044
3	Bound	0.995303	1.0	0.996769	1.0	0.996822	1.0	0.997448	1.0
	% RE	0.297225	0.173270	0.150369	0.173270	0.145072	0.173270	0.082356	0.173270
	$\sum_i (p_i - z_i)^2$	0.009972	0.008162	0.015196	0.016394	0.012227	0.007161	0.000025	0.000021
4	Bound	0.996989	1.0	0.997400	1.0	0.997390	1.0	0.997766	0.999362
	% RE	0.128372	0.173270	0.087226	0.173270	0.088175	0.173270	0.050523	0.109343
	$\sum_i (p_i - z_i)^2$	0.003963	0.004444	0.003651	0.002814	0.014494	0.000670	0.000058	0.000020
5	Bound	0.997583	1.0	0.997696	0.999328	0.997930	0.999352	0.998036	0.998801
	% RE	0.068823	0.173270	0.575170	0.106000	0.034063	0.108364	0.023432	0.053113
	$\sum_i (p_i - z_i)^2$	0.005352	0.001355	0.007984	0.004642	0.000217	0.006506	9.17e−6	0.000021
6	Bound	0.997930	0.999335	0.998030	0.998809	0.998081	0.998542	0.998133	0.998475
	% RE	0.034122	0.106646	0.024022	0.053976	0.018936	0.027183	0.013746	0.020467
	$\sum_i (p_i - z_i)^2$	0.002883	0.001999	0.003115	0.003705	0.000459	0.000036	6.80e−6	7.10e−6
7	Bound	0.998085	0.998696	0.998149	0.998520	0.998175	0.998390	0.998197	0.998361
	% RE	0.018522	0.042605	0.012114	0.025033	0.009516	0.012017	0.007310	0.009109
	$\sum_i (p_i - z_i)^2$	0.001328	0.002157	0.0007041	0.003743	9.12e−6	0.000047	3.76e−6	5.22e−6
8	Bound	0.998180	0.998389	0.998185	0.998334	0.998245	0.998309	0.998252	0.998307
	% RE	0.009086	0.011849	0.008589	0.006404	0.002523	0.003874	0.001831	0.003636
	$\sum_i (p_i - z_i)^2$	0.000551	0.000489	0.000929	0.000070	4.53e−6	5.2e−6	1.13e−6	4.50e−6
9	Bound	0.998209	0.998312	0.998261	0.998288	0.998263	0.998277	0.998263	0.998277
	% RE	0.006163	0.004165	0.000907	0.001781	0.000762	0.000707	0.000762	0.000707
	$\sum_i (p_i - z_i)^2$	0.000269	0.000285	4.11e−6	0.000036	1.92e−6	1.60e−6	1.92e−6	1.60e−6
10	Bound	0.998245	0.998281	0.998269	0.998271	0.998269	0.998271	0.998269	0.998271
	% RE	0.002532	0.001065	0.000131	0.000100	0.000090	0.000100	0.000090	0.000100
	$\sum_i (p_i - z_i)^2$	0.000450	0.0000797	1.21e−6	7.09e−7	5.78e−7	7.09e−7	5.78e−7	7.09e−7

(continued)

Table 3 (continued)

	When the distribution is monotonically decreasing								
		Distribution 1: $\sum_{i=1}^{11} p_i = 0.0.873372$		Distribution 2: $\sum_{i=1}^{11} p_i = 0.817375$		Distribution 3: $\sum_{i=1}^{11} p_i = 0.737375$		Distribution 4: $\sum_{i=1}^{11} p_i = 0.386375$	
m		Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.
2	Bound	0.813393	0.883612	0.757870	0.844255	0.644911	0.777310	0.231684	0.472346
	% RE	6.867540	1.172440	7.280070	3.288550	12.53960	5.415870	40.0364	22.2506
	$\sum_i (p_i - z_i)^2$	0.009514	0.003329	0.008889	0.014262	0.0332668	0.012199	0.076237	0.030639
3	Bound	0.854146	0.879127	0.787623	0.832417	0.723102	0.770009	0.331970	0.423404
	% RE	2.20132	0.658938	3.64	1.84022	1.93569	4.42577	14.0808	9.58375
	$\sum_i (p_i - z_i)^2$	0.00228475	0.001759	0.010620	0.005026	0.0011826	0.019386	0.030662	0.008576
4	Bound	0.865274	0.876459	0.802426	0.824852	0.730903	0.753086	0.360458	0.396491
	% RE	0.92717	0.35349	1.829	0.91475	0.877706	2.13074	6.70764	2.61834
	$\sum_i (p_i - z_i)^2$	0.001599	0.000667	0.007379	0.001727	0.000687	0.003958	0.017491	0.001427
5	Bound	0.870641	0.875453	0.813609	0.823062	0.733696	0.742170	0.379789	0.389386
	% RE	0.312713	0.2383	0.46085	0.695766	0.498973	0.650325	1.70453	0.779236
	$\sum_i (p_i - z_i)^2$	0.000524	0.000698	0.001107	0.002880	0.001343	0.001053	0.002741	0.000318
6	Bound	0.872090	0.874563	0.815710	0.819457	0.735742	0.738383	0.384484	0.387068
	% RE	0.14677	0.13642	0.20378	0.254707	0.221445	0.136738	0.489453	0.179430
	$\sum_i (p_i - z_i)^2$	0.000566	0.000605	0.000615	0.000915	0.001209	0.000189	0.000621	0.000063
7	Bound	0.872923	0.874074	0.816542	0.817953	0.736716	0.737826	0.385979	0.386621
	% RE	0.051361	0.080419	0.101989	0.070682	0.089404	0.0611199	0.102554	0.0637042
	$\sum_i (p_i - z_i)^2$	0.000174	0.000380	0.000522	0.000274	0.000544	0.000195	0.000107	0.000044
8	Bound	0.873142	0.873533	0.817187	0.817659	0.737292	0.737595	0.386292	0.386497
	% RE	0.026293	0.018462	0.023053	0.034738	0.011247	0.029877	0.021465	0.031607
	$\sum_i (p_i - z_i)^2$	0.000214	0.000104	0.000233	0.000401	0.000036	0.000367	0.000036	0.000068
9	Bound	0.873278	0.873404	0.817320	0.817420	0.737362	0.737396	0.386362	0.386396
	% RE	0.010722	0.003672	0.006814	0.005466	0.001751	0.002818	0.003341	0.005379
	$\sum_i (p_i - z_i)^2$	0.000287	0.000036	0.000225	0.000111	5.72e−6	0.000019	5.72e−6	0.000019
10	Bound	0.873367	0.873380	0.817362	0.817380	0.737374	0.737376	0.386374	0.386376
	% RE	0.000544	0.000952	0.001594	0.000523	0.000177	0.000097	0.000338	0.000186
	$\sum_i (p_i - z_i)^2$	0.000015	0.000048	0.00011	0.000012	1.20e−6	3.66e−7	1.20e−6	3.66e−7

Fig. 2 Monotonically increasing and decreasing distributions used in Table 3 for $n = 11$



distribution. The data from all the tables conclude that the steepness of the monotonic function has more effect on the bounds rather than the strictness of the monotonic distribution. Also, it can be easily seen that the measure $\sum_i (p_i - z_i)^2$ decreases with the increase in m. For small m, $\sum_i (p_i - z_i)^2$ is affected by the steepness of the monotonicity rather than the strictness of the monotonic distribution. But for larger m such as $n-1, n-2, \ldots$, the strictness of the monotonic distribution also has considerable effect on the measure.

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A Graph-Theoretical Approach for Comparison Between the Pair of Allied *Ragas Bhupali* and *Deshkar* of North Indian Classical Music

Nandini Sarma and Pranita Sarmah

1 Introduction

In music theory, graphs have been used for discovering patterns in music as they provide a visual way to analyse a melodic sequence [1]. As such, graph theory has been implicitly incorporated in many formal aspects of modern music like artificial intelligence and musicology, cognition, modelling and algorithmic composition. The musical event considered for analysis is generally found to be constituted of a finite number of states. The transitions between the states usually occur at discrete points of time according to specified probabilities called transitional probabilities. The transition probabilities are then presented in terms of a finite matrix. However, the graph-theoretical approach can also be considered as an alternative for describing the Markov chain that can be observed from the transition of the finite states. A number of important properties of the Markov chain can then be deduced from the pictorial representation. A *raga* is a highly patterned fundamental feature of Indian classical music. It involves repetition either in the form of individual sequences of notes, phrases or ornaments. Hence, graph theory can be employed for analysing and comparing such patterns inherent in *raga* compositions. In North Indian classical music, there exist *ragas* that closely resemble one another in some way or the other. Such *ragas* are called allied *ragas*. Although most of these *ragas* use the same set of notes, they differ in other subtler aspects. These differences may occur due to several reasons. A raga [2] dictates how each of its notes is to be used. Some pitches will get more emphasis than others, some will be used one way in an ascending melody and another way in a descending melody, and some will be used in certain

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types of ornaments. And these rules differ from *raga* to *raga*. The result is that each *raga* is a collection of scalar patterns, phrases, motifs and ornaments, which may be used together to construct music in that *raga*. Under such circumstances, analysis of a *raga* would seem to be inadequate without considering its resemblances or subtle differences with other allied *ragas*.

In this paper, a comparison of a pair of allied *ragas* of North Indian classical music has been presented to understand the unique features of a *raga* by analysing its similarities as well as dissimilarities with its allied *raga*. As a musical example, the pair of allied *ragas*—*Bhupali* and *Deshkar*—have been selected for comparison. Then graph theory has been applied to understand and analyse the difference between the musical structures of the pair with respect to the characteristics.

(a) *Arohana–Avarohana*
(b) Catch Phrase
(c) *Alap*

Definitions of musical graph, musical walk, multi-musical graph, musical cycle and connectivity of musical graph are then used for explaining the various digraphs of music theory. The transition of musical notes in the *alap* of each of the *ragas* is modelled as a Markov chain. The weight matrices corresponding to *alap* of *Bhupali* and *Deshkar* are then obtained, and the estimated mean absolute difference of weights is also derived. For analysis based on the *alap*s of the pair of ragas, the sample musical data corresponding to *raga Bhupali* and *Deshkar* have been obtained. The analysis is then carried out by considering the sample *alap* of each of the raga compositions as random sequences of musical notes. The movement of notes in the sample *alap* of both the *ragas* is then modelled as a Markov chain.

This paper consists of the following sections. A brief review of the literature is presented in Sect. 2. The source of musical data and the method of encoding the melodic data into numerical form is presented Sects. 3 and 4. The materials and methods used in the paper are discussed in Sect. 5. Section 6 consists of a description of the pair of *allied ragas* selected for comparison. Section 7 compares *raga Bhupali* and *raga Deshkar* by applying the concepts of graph theory. Finally, a discussion of the results of the comparisons is presented at the end of the paper.

2 Review of Literature

Peusner [1] introduced a graphical method for mapping the sequential aspects of melody into a static, global presentation of music. In the process, the mapping made explicit some of the aesthetic information buried in the score of the melody, although the visual patterns loosely correlated to the normal way in which a listener segmented a melody. Although the visual patterns loosely correlated to the normal way in which a listener segmented a melody. The graph introduced new grouping elements that were not part of the listener's perception. Szeto and Wong [3, 4] proposed a graph-theoretical approach to facilitate pattern matching in post-tonal music analysis with pitch-class set theory in which music perception is taken into account by incorporating stream segregation. A piece of music is modelled as a graph, with each musical note presented as a vertex and the relationship between a pair of musical notes as an edge. The relationship is determined by stream segregation. According to the proposed matching conditions, searching for a musical pattern is equivalent to searching for a special sub-graph called maximal matched CRP sub-graph. The comparisons were made between the patterns identified by the graph-theoretical approach and those by the musicologists [5, 6]. Santhi identified an application of graph theory in music information technology to generate all the 72 *Melakarthas* automatically. An undirected graph was constructed by using the *Melakartha* algorithm with seven *swaras* and its variations as its vertices. By applying the Breadth Search algorithm to the constructed graph, the authors then generated all possible paths representing the 72 *Melakartha ragas.* Haus [7, 8] proposed a new and useful model that allowed retrieval of music contents. The authors adopted an approach which looked into a thematic fragment globally. Given a thematic fragment (TF) M on "*n*" distinct notes and of length "*m*", the authors described the construction of a musical graph representing M, a weighted Eulerian-oriented multi-graph G(M) = (VG, AG, …) with *n* vertices and *m* arrows where Vg has a metric space structure (Vg, d). A similarity function between graphs was also defined. Then some necessary conditions for the inclusions of TFs by graph invariants and metrics on VG were given.

3 Source of Data

The melodic data of the present work are based upon the musical notations of North Indian classical ragas provided by the eminent musical scholar Pt Vishnu Narayan Bhatkhande in his two major works [9], *Kramik Pustak Mallika* and *Hindustani Sangeet Paddhati*.

4 Data Encoding

Indian classical music is based on seven basic or pure (*Shuddha*) *notes* arranged in increasing order of pitch. These are *Sa*, *Re*, *Ga*, *ma*, *Pa*, *Dha*, *Ni* or *Shuddha*—*Sa*, *Re*, *Ga*, *ma*, *Pa*, *Dha*, *Ni*—and are denoted as S, R, G, m, P, D and N. Now out of these seven notes, the notes "S" and "P" are immovable. These two notes form the tonal foundation for all the Indian classical music. The remaining five notes, viz. R, G, M, P, D and N on the other hand can be modified in pitch. They have alternate forms. Out of the five alternate forms, four are *komal* or flat, viz. (r, g, d and n) or (*Komal Re*, *Komal Ga*, *Komal Dha* and *Komal Ni*) and one sharp or *teevra,* viz. (M) or (*Teevra ma*). Thus, the full twelve tone scale is represented as S, r, R, g, G, m, M, P, d, D, n, N, S. The S at the end of this scale denoted by "S" would be double in frequency compared to the first S at the beginning of the scale. The interval between these

NOTES	LOWER OCTAVE	MIDDLE OCTAVE	HIGHER OCTAVE
S	-1	1	13
r	-2	2	14
R	-3	3	15
g	-4	4	16
G	-5	5	17
m	-6	6	18
M	-7	7	19
P	-8	8	20
d	-9	9	21
D	-10	10	22
n	-11	11	23
N	-12	12	24

Fig. 1 Numerical representation of *notes*

two notes is an octave. Similarly, one can define an octave in terms of each of the remaining eleven notes mentioned above. Music is generally developed within the three octaves, viz. lower octave, middle octave and higher octave in increasing order of pitch. The middle octave starts after the end of the lower octave, and similarly, the higher octave begins after the end of the middle octave. As such a musical note in middle octave would be double in frequency in comparison to its frequency in the lower octave. Similarly, the same note in higher octave will be double in frequency as compared to its frequency in the middle octave. Each of these octaves will consist of all the twelve notes defined above as such we get a total of 36 notes, 12 in each octave. In the present work, the 36 notes have been redefined in numerical form in such a way that numbers have been used as symbols for representing the notes arranged according to increasing order of pitch. The encoding of musical notes into numbers has been performed for the purpose of computation. The distribution of these 36 notes in each of the three octaves along with their corresponding encoded numerical symbols is illustrated in the figure below (Fig. 1).

5 Materials and Methods

Following are some musical definitions corresponding to North Indian classical music.

(a) ***Arohana***: The *Arohana* describes the pattern in which the notes of a *raga* ascend *in pitch* or scale. Any ascending sequence of notes in improvised portions of the *raga* is found to strictly [10] follow this pattern defined in *Arohana*.

(b) ***Avarohana***: The *Avarohana* is a description of the way the notes of a *raga* descends the scale. Every descending sequence of notes in a particular portion of a *raga* strictly follows this pattern of descending in scale defined in *Avarohana*.
(c) ***Pakad (Catch Phrase)***: In Hindustani music, a *pakad* or catch phrase is a generally accepted musical phrase (or set of phrases) thought to encapsulate the essence of a particular *raga* [11, 12]. The *pakad* contains the melodic theme of the *raga*. The *pakad* also contains information about *gayaki* or *chalan* (the way the notes are to be ordered and played or sung). Usually, the *pakad* is formed from short convolutions of the *Arohana* and *Avarohana*, while in some cases it is quite different from them. The *pakad* for a particular raga need not be unique; its sole purpose is to clarify what raga it is.
(d) ***Alap***: [13, 14] The *alap* is basically the opening section in the rendition of a typical North Indian classical *raga* where the various possibilities of exposing a *raga* are explored. It gives valuable [15] hints on aspects such as appropriate starting note, the typical phrases and the notes around which the raga may be elaborated and some other principle characteristics like *Arohana*, *Avarohana* and catch phrase. As such it is the appropriate choice for summarizing the grammatical structure of a *raga*.
(e) ***Note***: A musical note is a type [16] of notation which is used to represent the pitch and duration corresponding to a musical sound. Musical notes are combined together in order of pitch to construct musical scales. In fact, music in its simplest form is monotonic, i.e. a sequence of single frequencies or notes played one at a time [17].
(f) ***Octave***: There are three voice registers [17] recognized by our musicologists, namely the "lower" and "middle" which produces normal voice and the "higher" which produces top or head *notes*. These are actually the "*Octaves*" each consisting of seven *notes*. In the North Indian classical music, these seven basic *notes,* viz. S, R, G, M, P, D and N are categorized in increasing order of *pitch*. The eighth *note* is S' which repeats itself at the end of scale and whose frequency is exactly double to that of the first S at the beginning of the scale. The interval between these two *notes* is called an *octave*. The division of *octave* is based on normal and natural range of human voice. Music is developed mainly within these three *octaves,* viz. lower (l), middle (m) and higher (h).
(g) ***Vadi note (Sonant)***: The *vadi* note is generally defined as "the note which, compared with the other notes used in the *raga*, is sounded most often with clarity", [18]. It literally means "the note which speaks" [19, 20]. This is the most important or dominating note in a *raga*, which is a sort of key to the unfolding of its characteristics. It is usually the note which is repeated the greatest number of times, and often it is the note on which the singer can pause for a significant time.
(h) ***Samvadi note (Consonant)***: [20] The *Samvadi* or the Consonant is second important note in the *raga*, after the *vadi* note. It is located in that half of the octave, which does not include the *vadi*. Lastly, the Sonant–Consonant pairs bear a fourth or a fifth relationship between them. Thus, the Consonant (*Samvadi*) is the fourth or fifth *note* from the Sonant (*vadi*).

Following definitions are used for the purpose of explaining digraphs of music theory.

(i) ***Musical Graph***: A musical graph is a diconnected digraph with only one dicomonent that describes the pattern of movements of notes (nodes) in a musical piece (complete or part). Symbolically [21], a musical graph M is an ordered quadruplet ($V(M)$, $E(M)$, $\psi(M)$, $C(M)$) consisting a non-empty set $V(M) = \{i\}$ of vertices, a set $E(M) = \{e_{ij}\}$ of arcs which is disjoint from $V(M)$, an incidence function $\psi(M)=\{i, j\}$ that associates with each arc of M, an ordered pair of (not necessarily distinct) vertices (i, j) of M. And $C(M)$ represents a complete musical graph corresponding to M. Symbolically, $C(M) = \{i(e_{i1}, e_{i2}, \ldots, e_{ij}, \ldots, e_{ik})\ 1, 2, \ldots, i, j, \ldots, k\}$ which represents the connections of the node i with nodes (1, 2, …, $i, j, \ldots, k$) by the arcs $e_{i1}, e_{i2}, \ldots, e_{ij}, \ldots, e_{ik}$, respectively, for all i, j and k.

(j) ***Musical Walk***: A musical walk from vertex i to vertex j is [21] an alternating sequence of vertices and arcs, beginning with vertex i and ending with vertex j such that each arc is oriented from the vertex preceding it to the vertex following it. It contains no self-loops, and any tail node in a musical walk can have not more than two heads and vice versa. Symbolically, a musical walk M is an ordered quadruplet, viz. ($V(M)$, $E(M)$, $\psi(M)$, $C(M)$), where $V(M) = \{i\}$ is a non-empty set of vertices. $E(M) = \{e_{ij}\}$ is a non-empty set of arcs which is disjoint from $V(M)\cdot$ $\psi(M) = \{i, j\}$ is an incidence function that associates with each arc of M, an ordered pair of vertices (i, j) in two consecutive positions. And $C(M)$ represents a complete musical graph corresponding to M.

Symbolically, $C(M) = \{i\ (e_{ii+1}, e_{ii\text{-}1})\ i+1, i-1\}$ which represents the connections of the node i with nodes $(i+1, i-1)$ by the arcs e_{ii+1} and $e_{ii\text{-}1,}$ respectively, for all i.

Note: A musical walk is a musical graph, but the converse is not necessarily true.

(k) ***Multi-Musical Graph***: A multi-musical graph is a musical graph with $V(M) = \{i\}$, $\psi(M) = \{(i, j)\}$, $E(M) = \{e_{ij}\}$ and $C(M) = \{i(e_{ij}, e_{ji}, e_{ij}, e_{ji}, e_{ij}, e_{ji}, \ldots), i, j, i, j, i, j, \ldots\}$ for some i, j in the graph.

(l) ***Musical Cycle***: A musical cycle of a particular length corresponding to a note (node) in a musical graph is defined to be a path through which it returns to itself for the first time in a musical event with minimum number of arcs. The number of arcs in a musical cycle is the length of the *cycle*.

(m) ***Strength of a Node***: Strength $S(a)$ of a node a is defined as the sum of number of arcs converging to it (I_a) and the number of arcs diverging out of it (O_a). Mathematically

$$S(a) = (I_a + O_a) \text{ for } S(a) \in I, \tag{1}$$

where I is a set of integers.

(n) ***Rank of the Strength***: Let $a1, a2, \ldots, ak$ are k notes (nodes) in a musical graph with strengths $S(a1), S(a2), \ldots, S(ak)$, respectively. If the strengths are arranged in order of magnitude such that $S_{(a1)} < S_{(a2)} < \ldots < S_{(ai)} < \ldots < S_{(ak)}$, then $S_{(ai)}$ is said to be strength of rank i.

(o) ***Connectivity of a Musical Graph***: The connectivity of a musical graph M may be defined as the total number of arcs in the graph divided by total number of nodes.

6 Description of the Pair of Allied *Ragas*

6.1 Raga Bhupali

Raga Bhupali is one of the sweetest evening melodies, derived from the *Carnaticraga Mohanam*. This *raga* belongs to *Kalyan thaat* [22] and forbids the use of *Madhyam* (*Ma*) and *Nishad* (*Ni*). The set of notes used in this *raga* is given by $A = \{(S_l, R_l, G_l, P_l, D_l); (S_m, R_m, G_m, P_m, D_m); (S_h, R_h, G_h, P_h, D_h)\}$. (**Note**: Here the symbols "*l*", "*m*" and "*h*" indicate lower, middle and higher octaves, respectively. Thus, S_m denotes the note S sung or played in the middle octave. The rest of the notes can also be similarly defined.) Equivalently A may be rewritten as

$$A = \{ (-1, -3, -5, -8, -10); (1, 3, 5, 8, 10); (13, 15, 17, 20, 22)\}$$

The most important features for identification of any Indian classical *raga* are

1. *Arohana* (*Ar*)
2. *Avarohana* (*Av*) and
3. *Pakad* (*catch phrase*).

In case of *raga Bhupali*:

Arohana: $S_m, R_m, G_m, P_m, D_m, S_h$
Avarohana: $S_h, D_m\ P_m, G_m, R_m, S_m$
Pakad: $G_m, R_m, S_m\ D_l, S_mR_mG_m, P_mG_m, D_mP_mG_m, R_m, S_m$
Thaat: *Kalyan*
Vadi: *Ga*
Samvadi: *Dha*
Time of Singing: Evening.

6.2 Raga Deshkar

Vir [23] *raga Deshkar* is a pretty melody and is the counterpart of *Bhupali* which also forbids the use of the notes *Madhyam* (*Ma*) and *Nishad* (*Ni*). The distinguishing features of both the *ragas* lie in giving prominence to the *vadi* (Sonant) note. *Deshkar* takes *Dha* as *vadi* and is a morning tune. *Bhupali* takes *Ga* as *vadi* and is an evening tune. The *Dha* and *Pa* combination is a pleasing characteristic. This *raga* is derived from the *Bilawal thaat*. The set of notes used in Deshkar is same as that of Bhupali and is given by $B = \{-8, -10, 1, 3, 5, 8, 10, 13, 15, 17, 20\}$. The main characteristics of *Deshkar* are listed below.

Arohana: $S_mR_mG_m$, P_m, D_mS_h
Avarohana: S_hD_m, P_m, $G_mP_mD_mP_m$, $G_mR_mS_m$
Pakad: D_m, P_m, G_mP_m, G_mR_m S_m
Thaat: *Bilawal*
Vadi: *Dha*
Samvadi: *Ga*
Time of Singing: Day.

7 Comparison of *Raga Bhupali* and *Raga* Deshkar

7.1 Comparison of Arohana *and* Avarohana

It is clear from the above discussion that [24] *Bhupali* and *Deshkar* use the same set of notes although they belong to different *thaats*. The graphical presentation of *Arohana* and *Avarohana* of *Bhupali* and *Deshkar* are given as follows in Figs. 2 and 3 respectively.

The above digraph M_{11} is a *musical graph* and may be represented by the quadruplet $(V(M_{11}), E(M_{11}), \psi(M_{11}), C(M_{11}))$ where

$V(M_{11}) = \{S_m, R_m, G_m, P_m, D_m, S_h\}$. Without loss of generality, we may re-label the states and as such

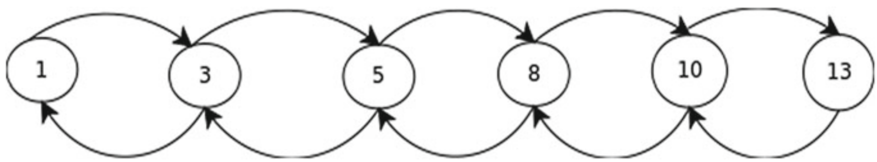

Fig. 2 Digraph M_{11} corresponding to *Arohana* and *Avarohana* of *raga Bhupali*

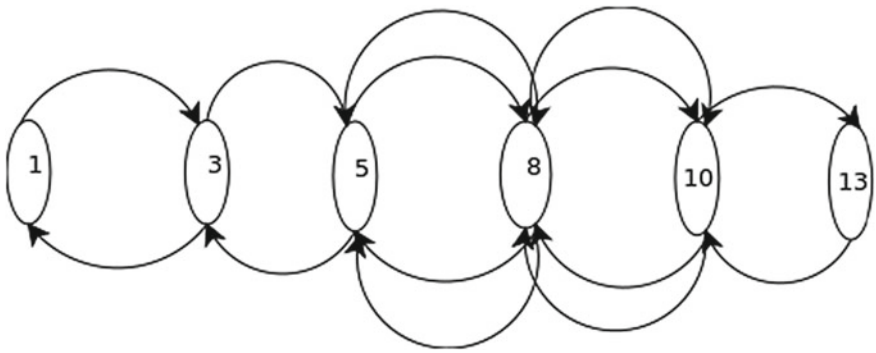

Fig. 3 Digraph M_{21} corresponding to *Arohana* and *Avarohana* of *raga Deshkar*

$V(M_{11}) = \{1, 3, 5, 8, 10, 13\}$.

And hence $E(M_{11}) = \{e_{13}, e_{3\,5}, e_{5\,8}, e_{8\,10}, e_{10\,13}, e_{13\,10}, e_{10\,8}, e_{8\,5}, e_{5\,3}, e_{3\,1}\}$.

Where arc e_{ij} represents the connection from ith node to jth node, where i, j = 1, 3, 5, 8, 10, 13. The incidence function corresponding to M_{11} is given by

$\psi\ (M_{11}) = \{(1, 3), (3, 5), (5, 8), (8, 5), (5, 3), (3, 1)\}$
$C(M_{11})$ represents the *complete musical graph* and is defined by

$C(M_{11}) = \{1(e_{1\,3})3\}\{3(e_{3\,5})5\}\{5(e_{5\,8})8\}\{8(e_{8\,10})10\}\{10(e_{10\,13})13\}\{13(e_{13\,10})10\}$ $\{10(e_{10\,8})8\}\{8\ (e_{8\,5})5\}\{5(e_{5\,3})3\}\{3(e_{3\,1})1\}$.

It is seen that $C(M_{11})$ represents a *musical walk*. Since in this graph the arcs intersect at their ends, they are planar and can be presented in a plane. The adjacency matrices corresponding to digraph M_{11} and M_{12} is given as follows in Tables 1 and 2 respectively.

It is observed that the digraph M_{21} is a *multi-musical graph*. M_{21} may be described by the quadruplet ($V(M_{21})$, $E(M_{21})$, $\psi(M_{21})$, $C(M_{21})$) as shown below
Observations:
From the pairs (M_{11}, A_{11}) and (M_{21}, A_{21}), we observe that

Table 1 Adjacency matrix A_{11} corresponding to digraph M_{11}

Notes	1	3	5	8	10	13	Outdegree (O)
1	0	1	0	0	0	0	**1**
3	1	0	1	0	0	0	**2**
5	0	1	0	1	0	0	**2**
8	0	0	1	0	1	0	**2**
10	0	0	0	1	0	1	**2**
13	0	0	0	0	1	0	**1**
Indegree(*I*)	**1**	**2**	**2**	**2**	**2**	**1**	
Polarity = *I* − *O*	**0**	**0**	**0**	**0**	**0**	**0**	
Strength = *I* +*O*	**2**	**4**	**4**	**4**	**4**	**2**	**Total strength = 20**

Table 2 Adjacency matrix $\boldsymbol{A_{21}}$ corresponding to digraph M_{21} is given as follows

	1	3	5	8	10	13	Outdegree (O)
1	0	1	0	0	0	0	**1**
3	1	0	1	0	0	0	**2**
5	0	1	0	2	0	0	**3**
8	0	0	2	0	2	0	**4**
10	0	0	0	2	0	1	**3**
13	0	0	0	0	1	0	**1**
Indegree(*I*)	**1**	**2**	**3**	**4**	**3**	**1**	
Polarity = *I* − *O*	**0**	**0**	**0**	**0**	**0**	**0**	
Strength = *I* +*O*	**2**	**4**	**6**	**8**	**6**	**2**	**Total strength = 28**

$V(M_{21})$	{1, 3, 5, 8, 10, 13}
$E(M_{21})$	$\{e_{1\,3}, e_{3\,5}, e_{5\,8}, e_{8\,10}, e_{10\,13}, e_{13\,10}, e_{10\,8}, e_{8\,5}, e_{5\,3}, e_{3\,1}\}$
$\psi(M_{21})$	{(1, 3), (3, 5), (5, 8).(5, 8), (8, 10).(8, 10), (10, 13), (13, 10), (10, 8),(10, 8), (8, 5).(8, 5), (5, 3), (3, 1)}
$C(M_{21})$	[{1 $(e_{1\,3})$3}{3 $(e_{3\,5})$5}{5 $(e_{5\,8})(e_{5\,8})$8}{8 $(e_{8\,10})(e_{8\,10})$10}{10 $(e_{10\,13})$13}{13$(e_{13\,10})$ 10}{10 $(e_{10\,8})(e_{10\,8})$ 8}{8 $(e_{8\,5})(e_{8\,5})$5}{5 $(e_{5\,3})$3}{3 $(e_{3\,1})$1}]

(1) Graph M_{11} is a *musical graph,* whereas M_{21} is a *multi-musical* graph.
(2) M_{11} is a *musical walk,* whereas M_{21} is not a *musical walk.*
(3) Again from the adjacency matrices A_{11} and $A_{21,}$ it is found that outdegree corresponding to *vadi note* (*Ga*) of *Bhupali* is 2, whereas that of *Deshkar* (*Dha*) is 3. However, the polarities of both the *vadi notes* remain the same and are equal to zero.
(4) Strength of *vadi note* of *Deshkar* (*Dha*) is greater than that of *vadi* note (*Ga*) of *Bhupali.*
(5) Total strength of *Deshkar* is greater than that of *Bhupali*.

7.2 Comparison of Catch Phrase (Pakad)

The *catch phrases* of *Bhupali* and *Deshkar* are given by $\{G_m, R_m, S_mD_l, S_mR_mG_m, P_mG_m, D_mP_mG_m, R_m, S_m\}$ and $\{D_m, P_m, G_mP_m, G_mR_mS_m\}$. The size of a *catch phrase* (i.e. total number of independent phrases in the catch phrase) of *Bhupali* is found to be 8, whereas that of *Deshkar* is only 4. If S_i denote the number of independent phrases with length i in the *catch phrase* of a *raga*, then for *raga Bhupali* $S_1 = 4$, $S_2 = 2$ and $S_3 = 2$. Again for *raga Deshkar,* we observe that $S_1 = 2$, $S_2 = 1$ and $S_3 = 1$. The *catch phrase* which is an essential feature of *raga* identification may also be presented by a digraph using the rules of *Arohana* and *Avarohana*. This may be called *root to catch the catch phrase*.

7.3 Comparison of Root to Catch the Catch Phrase

The *roots to catch the catch phrase* of *Bhupali* and *Deshkar* along with *Arohana* and *Avarohana* may be represented by the following digraphs, i.e. M_{12} and M_{22}, respectively (Figs. 4 and 5).

Both the digraphs M_{12} and M_{22} are *musical graphs*. M_{22} is a *musical walk,* whereas M_{12} is not. It can be observed that:

$V(M_{12}) = \{-10, 1, 3, 5, 8, 10\}$

$C(M_{12})$ = [{−10 ($e_{-10\ 1}$)1}{1 ($e_{1\ 3}$)3}{3 ($e_{3\ 5}$)5}{5 ($e_{5\ 8}$, $e_{5\ 10}$)8, 10}{10 ($e_{10\ 8}$)8}{8 ($e_{8\ 5}$)5}{5 ($e_{5\ 3}$) 3}{3 (e_{31})1}{1 ($e_{1\text{-}10}$) −10}]

$V(M_{22})$ = {1, 3, 5, 8, 10}

$C(M_{22})$ = [{1 ($e_{1\ 3}$)3}{3 ($e_{3\ 5}$)5}{5 ($e_{5\ 8}$)8}{8 ($e_{8\ 10}$)10}{10 ($e_{10\ 8}$)8}{8 ($e_{8\ 5}$)5}{5 ($e_{5\ 3}$)3}{3 ($e_{3\ 1}$)1}]

Thus $V(M_{12})$ = {−10, $V(M_{22})$} and

$C(M_{12})$ ∩ $C(M_{22})$ = [{1 ($e_{1\ 3}$)3}{3 ($e_{3\ 5}$)5}{5($e_{5\ 8}$)8}{10 ($e_{10\ 8}$)8}{8 ($e_{8\ 5}$)5}{5 ($e_{5\ 3}$)3}{3($e_{3\ 1}$)1}]

The adjacency matrices A_{12} and A_{22} corresponding to digraphs M_{12} and M_{22} are given as follows (Tables 3 and 4):

It may be observed that C(M_{12}) ∩ C(M_{22}) = [{1($e_{1\ 3}$)3}{3($e_{3\ 5}$)5}{5($e_{5\ 8}$)8}{8 ($e_{8\ 5}$)5} {5 ($e_{5\ 3}$)3}{3 ($e_{3\ 1}$) 1}] is a *musical graph* and is a sub-graph of M_{12}.

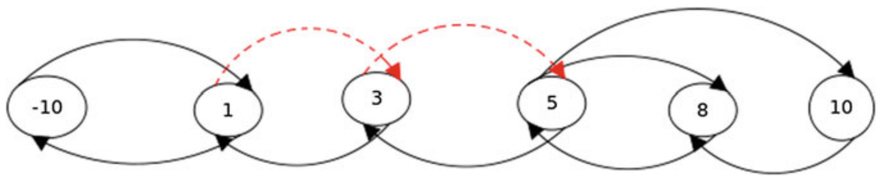

Fig. 4 Digraph M_{12} corresponding to root to catch the *catch phrase* of *raga Bhupali*

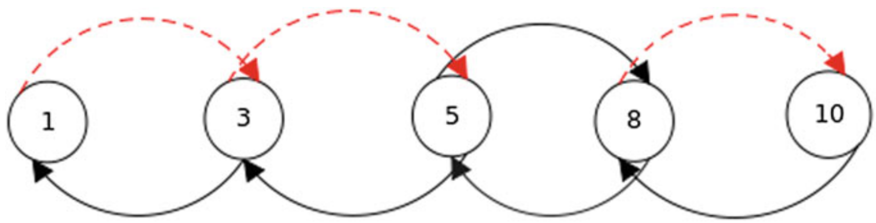

Fig. 5 Digraph M_{22} corresponding to root to catch the *catch phrase* of *raga Deshkar*

Table 3 Adjacency matrix A_{12} corresponding to digraph M_{12}

	−10	1	3	5	8	10	Outdegree (O)
−10	0	1	0	0	0	0	**1**
1	1	0	1	0	0	0	**2**
3	0	1	0	1	0	0	**2**
5	0	0	1	0	1	1	**3**
8	0	0	0	1	0	0	**1**
10	0	0	0	0	1	0	**1**
Indegree(I)	**1**	**2**	**2**	**2**	**2**	**1**	
Polarity = $I - O$	**0**	**0**	**0**	**−1**	**1**	**0**	
Strength = $I + O$	**2**	**4**	**4**	**5**	**3**	**2**	**Total strength = 20**

Table 4 Adjacency matrix A_{22} corresponding to digraph M_{22}

	1	3	5	8	10	Outdegree (O)
1	0	1	0	0	0	**1**
3	1	0	1	0	0	**2**
5	0	1	0	1	0	**2**
8	0	0	1	0	1	**2**
10	0	0	0	1	0	**1**
Indegree(I)	**1**	**2**	**2**	**2**	**1**	
Polarity = $I - O$	**0**	**0**	**0**	**0**	**0**	**Total strength = 16**
Strength = $I + O$	**2**	**4**	**4**	**4**	**2**	

Observations:

(1) From the adjacency matrices A_{12} and A_{22}, it is observed that the *vadi note* of *raga Bhupali*, i.e. the node 5 of M_{12}, has highest outdegree and lowest polarity. Again for *Deshkar*, the outdegree of the nodes 3, 5 and 8 is same having the same polarity.

(2) The strength of the *vadi note* of *raga Bhupali*, i.e. the node 5 of M_{12}, is greater than that of *vadi note* of *raga Deshkar*.

(3) Total strength of *raga Bhupali* is greater than that of *raga Deshkar* with respect to roots to catch the *catch phrase.*

(4) The connectivity of *raga Bhupali* is $10/6 = 1.6$, whereas connectivity of *raga Deshkar* is also $8/5 = 1.6$.

7.3.1 Musical Cycles

Both the *ragas* have some common nodes, viz. 1, 3, 5, 8, 10. The *musical cycles* corresponding to each common node for both the *ragas* are given below. Let $M_b(M_{12})_i$

denote the *musical cycle* starting with node i for $i=1, 3, 5, 8, 10$ for *raga Bhupali* corresponding to digraph M_{12}.

7.3.2 Musical Cycles $M_b(M12)i$ Corresponding to Digraph M12

Raga Bhupali: $\boldsymbol{M_b(M_{12})_i}$

(1) $\boldsymbol{i = 1}$

1→3→1

1→3→5→3→1

1→3→5→8→5→3→1

1→3→5→10→8→5→3→1

(2) $\boldsymbol{i = 3}$

3→1→3

3→5→3

3→5→8→5→3

3→5→10→8→5→3

(3) $\boldsymbol{i = 5}$

5→8→5

5→3→5

5→10→8→5

(4) $\boldsymbol{i = 8}$

8→5→8

8→5→10→8

8→5→3→5→8

8→5→3→1→3→5→8

(5) $\boldsymbol{i = 10}$

10→8→5→10

10→8→5→3→5→10

10→8→5→3→1→3→5→10

Since a portion of M_{22} is a sub-graph of M_{12}, all the *musical cycles* of both the *ragas* are either equal or sub-graph or super-graph of each other. On the other hand, M_{12} being a super-graph of M_{22} provides more number of *musical cycles* and thus produces more variability in *raga Bhupali*.

7.4 *Comparison Corresponding to Alap*

In this section, Markov chain has been employed to study the occurrence pattern of movement of *notes* in *alap* of *raga Bhupali* and *Deshkar*. Let $X_n = i$ be a random variable where i denotes a *note* occurring in *alap* of *raga* (*Bhupali* or *Deshkar*) as an effect of nth transition for i belonging to set (A_b or A_d) and $n=1, 2, 3, \ldots,$. Here $A_b = \{-10, 1, 3, 5, 8, 10, 13, 15, 17, 20\}$ and $A_d = \{-8, -10, 1, 3, 5, 8, 10, 13, 15, 17, 20\}$ stand for sets of *notes* used in *alap* of *Bhupali* and *Deshkar,* respectively. Further, it is assumed that

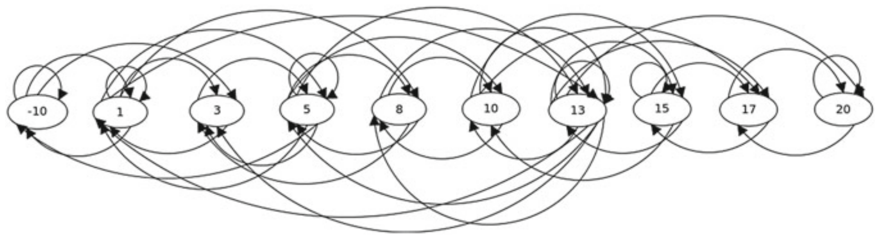

Fig. 6 Digraph M_{13} corresponding to *alap* of *raga Bhupali*

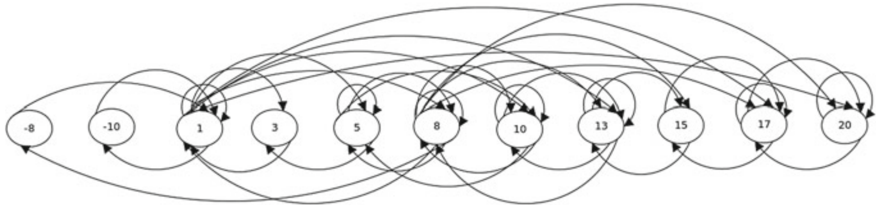

Fig. 7 Digraph M_{23} corresponding to *alap* of *raga Deshkar*

$P\{X_{n+1} = |X_1 = k\ X_2 = l, \ldots, X_n = i\} = P\{X_{n+1} = j|X_n = i\} = p_{ij}$ such that $\sum_i \sum_j p_{ij} = 1$. Thus, $\{X_n = i, n = 1, 2, 3, \ldots\}$ may be assumed to be a Markov chain with state space $S_b = \{A_b\}$ or $S_d = \{A_d\}$. The parameters p_{ij} are estimated by the method of maximum likelihood. Thus, $\hat{p}_{ij} = \frac{n_{ij}}{n_i}$ with n_{ij} being number of times the system visited the state j provided it entered state i initially and n_i is the total number of times the system is in state i. The transition probabilities p_{ij} are treated as weights w_{ij} in digraph theory. An estimated weight has one-to-one correspondence with arcs e_{ij} of a digraph. To obtain n_{ij} and n_i, data have been collected from the source mentioned in Sect. 3 of the paper.

The *alap* of *raga Bhupali* and *Deshkar* may be represented by the following digraphs (i.e. M_{13} and M_{23}, respectively) (Figs. 6 and 7).

The digraph M_{13} corresponding to *alap* of *raga Bhupali* may be described by the quadruplet

$\{V(M_{13}), E(M_{13}), \psi(M_{13}), C(M_{13})\}$ as shown below

Similarly, the graph M_{23} corresponding to *alap* of *raga Deshkar* may be described by the quadruplet $\{V(M_{23}), E(M_{23}), \psi(M_{23}), C(M_{23})\}$as shown below

(Tables 5 and 6).

Observations on M_{13} and M_{23}:

(1) Both M_{13} and M_{23} are not *simple digraphs* and have only one dicomponent each. Both are connected *musical graph.*

(2) Total number of *nodes* in M_{13} is 10 with total strength equal to 86, whereas total number of nodes in M_{23} is 11 with total strength equal to 80. Hence, *raga Bhupali* has 86/2 = 43 connections (arcs), whereas *Deshkar* has 80/2 = 40 arcs. Therefore, the *connectivity* of M_{13} and M_{23} is 43/10 = 4.3 and 40/11 =

V(M13)	{−10, 1, 3, 5, 8, 10, 13, 15, 17, 20},
E(M13)	$\{e_{-10\text{-}10}, e_{-10\,1}, e_{-10\,3}, e_{1\,1}, e_{1\,3}, e_{1\,5}, e_{1\,8}, e_{1\,13}, e_{1\text{-}10}, e_{35}, e_{31}, e_{5\,5}, e_{5\,8},$
	$e_{5\,10}, e_{5\,13}, e_{5\,3}, e_{5\,1}, e_{5\,-10}, e_{8\,10}, e_{8\,13}, e_{8\,5}, e_{8\,3}, e_{10\,13}, e_{10\,15}, e_{10\,17}, e_{10\,8},$
	$e_{13\,13}, e_{13\,15}, e_{13\,17}, e_{13\,20}, e_{13\,10}, e_{13\,8}, e_{13\,5}, e_{13\,3}, e_{13\,1}, e_{15\,15}, e_{15\,17}, e_{15\,13},$
	$e_{15\,10}, e_{17\,20}, e_{17\,15}, e_{20\,20}, e_{20\,17}\},$
ψ(M13)	{(−10, −10), (−10, 1), (−10, 3), (1, 1), (1, 3), (1, 5), (1, 8), (1, 13), (1, −10),
	(3, 5), (3, 1), (5, 5), (5, 8), (5, 10), (5, 13), (5, 3), (5, 1), (5, 10), (8, 10), (8, 13),
	(8, 5), (8, 3), (10, 13), (10, 15), (10, 17), (10, 8), (13, 13), (13, 15), (13, 17),
	(13, 20), (13, 10), (13, 8), (13, 5), (13, 3), (13, 1), (15, 15), (15, 17), (15, 13),
	(15, 10), (17, 20), (17, 15), (20, 20), (20, 17)}
C(M13)	[{−10 $(e_{-10\text{-}10}, e_{-10\,1}, e_{-10\,3})$ −10, 1, 3}{1$(e_{1\,1}, e_{1\,3}, e_{1\,5}, e_{1\,8}, e_{1\,13}, e_{1\text{-}10})$
	1, 3, 5, 8, 13, −10}{3(e_{35}, e_{31})5, 1}{5 $(e_{5\,5}, e_{5\,8}, e_{5\,10}, e_{5\,13}, e_{5\,3}, e_{5\,1}, e_{5\,-10})$
	5, 8, 10, 13, 3, 1, −10}{8 $(e_{8\,10}, e_{8\,13}, e_{8\,5}, e_{8\,3})$10, 13, 5, 3}{10$(e_{10\,13}, e_{10\,15},$
	$e_{10\,17}, e_{10\,8})$ 13, 15, 17, 8}{13$(e_{13\,13}, e_{13\,15}, e_{13\,17}, e_{13\,20}, e_{13\,10}, e_{13\,8}, e_{13\,5},$
	$e_{13\,3}, e_{131})$ 13, 15, 17, 20, 10, 8, 5, 3, 1)}{15$(e_{15\,15}, e_{15\,17}, e_{15\,13}, e_{15\,10})$15,
	17, 13, 10}{17$(e_{17\,20}, e_{17\,15})$20, 15}{20$(e_{20\,20}, e_{20\,17})$20, 17}]

V(M23)	{−8, −10, 1, 3, 5, 8, 10, 13, 15, 17, 20}
E(M23)	$\{e_{-8\,1}, e_{-10\,1}, e_{1\,1}, e_{1\,3}, e_{1\,5}, e_{1\,8}, e_{1\,10}, e_{1\,13}, e_{1\,17}, e_{1\,20}, e_{1\,-10}, e_{3\,1}, e_{5\,5}, e_{5\,8}, e_{5\,10}, e_{5\,3}, e_{8\,8}, e_{8\,10}, e_{8\,13}, e_{8\,15}, e_{8\,17}, e_{8\,20}, e_{8\,5}, e_{8\,1}, e_{8\,-8}, e_{10\,10}, e_{10\,13}, e_{10\,8}, e_{10\,5}, e_{13\,13}, e_{13\,15}, e_{13\,10}, e_{13\,8}, e_{15\,17}, e_{15\,13}, e_{17\,17}, e_{17\,20}, e_{17\,15}, e_{20\,20}, e_{20\,17}\}$
ψ(M23)	{(−8, 1), (−10, 1), (1, 1), (1, 3), (1, 5), (1, 8), (1, 10), (1, 13), (1, 17), (1, 20), (1, −0), (3, 1), (5, 5), (5, 8), (5, 10), (5, 3), (8, 8), (8, 10), (8, 13), (8, 15), (8, 17), (8, 20), (8, 5), (8, 1), (8, −8), (10, 10), (10, 13), (10, 8), (10, 5), (13, 13), (13, 15), (13, 10), (13, 8), (15, 17), (15, 13), (17, 17), (17, 20), (17, 15), (20, 20), (20, 17)}
C(M23)	[{−8(e-$_{8\,1}$) 1}{−10 (e_{-101})1}{1$(e_{1\,1}, e_{1\,3}, e_{1\,5}, e_{18}, e_{1\,10}, e_{1\,13}, e_{1\,17}, e_{1\,20}, e_{1\,-10})$1, 3, 5, 8, 10, 13, 17, 20, −10}{3$(e_{3\,1},)$1}{5$(e_{5\,5}, e_{5\,8}, e_{5\,10}, e_{5\,3})$5, 8, 10, 3} {8$(e_{8\,8}, e_{8\,10}, e_{8\,13}, e_{8\,15}, e_{8\,17}, e_{8\,20}, e_{8\,5}, e_{8\,1}, e_{8\,-8})$8, 10, 13, 15, 17, 20, 5, 1, −8} {10$(e_{10\,10}, e_{10\,13}, e_{10\,8}, e_{10\,5},)$ 10, 13, 8, 5}{13$(e_{13\,13}, e_{13\,15}, e_{13\,10}, e_{13\,8},)$13, 15, 10, 8} {15$(e_{15\,17}, e_{15\,13},)$17, 13}{17$(e_{17\,17}, e_{17\,20}, e_{17\,15},)$17, 20, 15}{20$(e_{20\,20}, e_{20\,17},)$20, 17}]

3.6, respectively, which implies that *raga Bhupali* has greater connectivity than *raga Deshkar*.

(3) *Note* 13 (S_h) of *Bhupali*, *notes* 1 (S_m) and 8 (P_m) of *Deshkar* demonstrate highest strength, whereas *vadi note* 5 (G_m) of *Bhupali* and *vadi note* 10 (D_m) of *Deshkar* produce strength of rank2 in M_{13} and $M_{23,}$ respectively.

(4) The graphs M_{13} and M_{23} have a set of common nodes, viz. {−10, 1, 3, 5, 8, 10, 13, 15, 17, 20} with respective weight matrices W_1 and W_2 as shown in tables below (Tables 7, 8 and 9).

The estimated mean absolute difference of weights of *raga Bhupali* and *Deshkar* is given by $w = \sum_i \sum_j |Iw_{1ij} - w_{2ij}I|/n$, where n = total number of elements in

Table 5 Adjacency matrices A_{13} corresponding to digraphs M_{13}

	−10	1	3	5	8	10	13	15	17	20	Outdegree (O)
−10	1	1	1	0	0	0	0	0	0	0	**3**
1	1	1	1	1	1	0	1	0	0	0	**6**
3	0	1	0	1	0	0	0	0	0	0	**2**
5	1	1	1	1	1	1	1	0	0	0	**7**
8	0	0	1	1	0	1	1	0	0	0	**4**
10	0	0	0	0	1	0	1	1	1	0	**4**
13	0	1	1	1	1	1	1	1	1	1	**9**
15	0	0	0	0	0	1	1	1	1	0	**4**
17	0	0	0	0	0	0	0	1	0	1	**2**
20	0	0	0	0	0	0	0	0	1	1	**2**
Indegree(I)	**3**	**5**	**5**	**5**	**4**	**4**	**6**	**4**	**4**	**3**	**Total strength = 86**
Polarity = $I - O$	**0**	**−1**	**3**	**−2**	**0**	**0**	**−3**	**0**	**2**	**1**	
Strength = $I + O$	**6**	**11**	**7**	**12**	**8**	**8**	**15**	**8**	**6**	**5**	

Table 6 Adjacency matrices A_{23} corresponding to digraphs M_{23}

	−8	−10	1	3	5	8	10	13	15	17	20	Outdegree (O)
−8	0	0	1	0	0	0	0	0	0	0	0	**1**
−10	0	0	1	0	0	0	0	0	0	0	0	**1**
1	0	1	1	1	1	1	1	1	0	1	1	**9**
3	0	0	1	0	0	0	0	0	0	0	0	**1**
5	0	0	0	1	1	1	1	0	0	0	0	**4**
8	1	0	1	0	1	1	1	1	1	1	1	**9**
10	0	0	0	0	1	1	1	1	0	0	0	**4**
13	0	0	0	0	0	1	1	1	1	0	0	**4**
15	0	0	0	0	0	0	0	1	0	1	0	**2**
17	0	0	0	0	0	0	0	0	1	1	1	**3**
20	0	0	0	0	0	0	0	0	0	1	1	**2**
Indegree(I)	**1**	**1**	**5**	**2**	**4**	**5**	**5**	**5**	**3**	**5**	**4**	**Total strength = 80**
Polarity = $I - O$	**0**	**0**	**−4**	**1**	**0**	**−4**	**1**	**1**	**1**	**2**	**2**	
Strength = $I + O$	**2**	**2**	**14**	**3**	**8**	**14**	**9**	**9**	**5**	**8**	**6**	

Table 7 W_1

	−10	1	3	5	8	10	13	15	17	20	Outdegree (O)
−10	0.11	0.33	0.56	0	0	0	0	0	0	0	3
1	0.15	0.32	0.40	0.04	0.04	0	0.04	0	0	0	6
3	0	0.52	0	0.48	0	0	0	0	0	0	2
5	0.01	0.03	0.31	0.14	0.36	0.11	0.03	0	0	0	7
8	0	0	0.05	0.56	0	0.36	0.02	0	0	0	4
10	0	0	0	0	0.5	0	0.46	0.02	0.02	0	4
13	0	0.04	0.02	0.04	0.05	0.38	0.13	0.27	0.05	0.02	9
15	0	0	0	0	0	0.03	0.67	0.03	0.27	0	4
17	0	0	0	0	0	0	0	0.76	0	0.24	2
20	0	0	0	0	0	0	0	0	0.63	0.38	2
Indegree(I)	**3**	**5**	**5**	**5**	**4**	**4**	**6**	**4**	**4**	**3**	
Polarity = $I - O$	**0**	**−1**	**3**	**−2**	**0**	**0**	**−3**	**0**	**2**	**1**	
Strength = $I + O$	**6**	**11**	**7**	**12**	**8**	**8**	**15**	**8**	**6**	**5**	**Total strength = 76**

Table 8 W_2

	−8	−10	1	3	5	8	10	13	15	17	20	Outdegree(O)
−8	0	0	1	0	0	0	0	0	0	0	0	1
−10	0	0	1	0	0	0	0	0	0	0	0	1
1	0	0.04	0.18	0.04	0.18	0.29	0.11	0.04	0	0.11	0.04	9
3	0	0	1	0	0	0	0	0	0	0	0	1
5	0	0	0	0.42	0.07	0.49	0.02	0	0	0	0	4
8	0.01	0	0.01	0	0.39	0.23	0.29	0.02	0.01	0.02	0.01	9
10	0	0	0	0	0.02	0.75	0.4	0.19	0	0	0	4
13	0	0	0	0	0	0.14	0.55	0.24	0.07	0	0	4
15	0	0	0	0	0	0	0	0.91	0	0.09	0	2
17	0	0	0	0	0	0	0	0	0.62	0.31	0.08	3
20	0	0	0	0	0	0	0	0	0	0.5	0.5	2
Indegree(I)	**1**	**1**	**5**	**2**	**4**	**5**	**5**	**5**	**3**	**5**	**4**	
Polarity = $I - O$	**0**	**0**	**−4**	**1**	**0**	**−4**	**1**	**1**	**1**	**2**	**2**	
Strength = $I + O$	**2**	**2**	**14**	**3**	**8**	**14**	**9**	**9**	**5**	**8**	**6**	**Total strength = 80**

Table 9 Absolute difference matrix $|W_1 - W_2|$

	−10	1	3	5	8	10	13	15	17	20
−10	0.11	0.66	0.56	0	0	0	0	0	0	0
1	0.11	0.14	0.36	0.14	0.25	0.11	0	0	0.11	0.04
3	0	0.48	0	0.48	0	0	0	0	0	0
5	0.01	0.03	0.11	0.07	0.13	0.09	0.03	0	0	0
8	0	0	0.04	0.17	0.23	0.07	0	0.01	0.02	0.01
10	0	0	0	0.02	0.25	0.04	0.27	0.02	0.02	0
13	0	0.04	0.02	0.04	0.09	0.17	0.11	0.20	0.05	0.02
15	0	0	0	0	0	0.03	0.24	0.03	0.18	0
17	0	0	0	0	0	0	0	0.14	0.31	0.16
20	0	0	0	0	0	0	0	0	0.13	0.12

$I\mathrm{W}_1 - \mathrm{W}_2 I$. The variance corresponding to absolute difference of weights is given by $0.01593771 = 0.02$. The estimated mean absolute difference of weights of the *alap* of *raga Bhupali* and *Deshkar* is given by $= 7.27/100 = 0.073$. Hence, $\mathrm{w} = 0.0073$.

8 Conclusion

Bhupali and *Deshkar* differ significantly corresponding to *Arohana* and *Avarohana*. The graphical structure of both the *ragas* corresponding to *Arohana* and *Avarohana* reveals that *Deshkar* has greater strength than that of *Bhupali.* On the other hand, analysis of *root to catch the catch phrase* of both the graphs shows that graph M_{22} corresponding to *Deshkar* is a sub-graph of M_{12} corresponding to *Bhupali*. Again the connectivity of both the graphs (M_{22} and M_{12}) is same. M_{22} is a sub-graph of M_{12}, i.e. all the *musical cycles* of both the *ragas* are either equal or sub-graph or super-graph of each other. On the other hand, M_{12} being a super-graph of M_{22} provides more number of *musical cycles* and thus produces more variability in *raga Bhupali*. Again in the case of graphs (M_{13} and M_{23}) corresponding to *alap* of *Bhupali* and *Deshkar*, it has been observed that the total strength of *Bhupali* is more than that of total strength of *Deshkar*. Again the connectivity of *Bhupali* is also slightly higher than that of *Deshkar*.

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A General Class of Tests for Testing Homogeneity of Location Parameters Against Ordered Alternatives

Manish Goyal and Narinder Kumar

1 Introduction

Let there be $k(k \geq 2)$ independent populations and $X_{l1}, \ldots, X_{ln_l}$ be a random sample of size n_l from lth population with an absolute continuous cumulative distribution function (cdf) of lth population as $F_l(x) = F(x - \theta_l)$, $l = 1, \ldots, k$. Now, the hypothesis of the proposed test is to check if all the populations are alike or there is an ordering in their location parameters $\theta_l' s$, i.e. to test

$$H_0 : \theta_1 = \cdots = \theta_k$$

against the ordered alternative

$$H_1 : \theta_1 \leq \cdots \leq \theta_k.$$

This type of multi-sample location problem with ordered alternative is encountered in many situations, as in economics, psychology, zoology, botany, dose-level testing.

The first test for such problems was considered by Jonckheere [1] and Terpstra [2]. This test is further generalized by Amita and Kochar [3], which is based on linear combination of two-sample test statistics proposed by Deshpande and Kochar [4]. Kumar et al. [5] proposed a test based on linear combination of two-sample test statistics proposed by Shetty and Govindarajulu [6], and this test is further generalized by Kumar et al. [7]. Kumar et al. [8] proposed a new class of test statistics which

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is generalization of Amita and Kochar [3] test. Gaur [9] proposed a test based on linear combination of two-sample test statistics proposed by Öztürk [10]. Gaur [11] proposed a new class of test based on linear combination of two-sample U-statistics based on sub-sample extremes. For other additional references, one may refer to monographs Barlow et al. [12], Randles and Wolfe [13], Hettmansperger [14], Sprent and Smeeton [15] and Gibbons and Chakraborti [16].

Kumar [17] proposed a class of tests for two-sample location problem which is a generalization of tests proposed by Deshpande and Kochar [4], Kumar et al. [8], Wilcoxon [18], Mann–Whitney [19] and Kumar [20]. In this paper, we proposed a general class of distribution-free tests for multi-sample location problem with ordered alternative based on linear combination of two-sample test statistics proposed by Kumar [17].

In Sect. 2, we introduce the test statistics and its asymptotic distribution is developed in Sect. 3. To get maximum efficiency of the test, optimal choice of weights is discussed in Sect. 4. Comparisons of the proposed test with respect to some other competing tests are studied in Sect. 5. Implementation of the test is discussed using real-life data set example in Sect. 6. Power of the test is assessed by simulation study in Sect. 7, and conclusions are given in Sect. 8.

2 Proposed Test Statistics

Suppose c and d are fixed sub-sample size such that $1 \leq c, d \leq \min(n_1, \ldots, n_k)$. Let i and j be fixed positive integer such that $1 \leq i \leq c$ and $1 \leq j \leq d$. Now, define a kernel for lth and mth population such that $l < m$ for $l, m = 1, \ldots, k$, as:

$$h_{i,j}(X_{l1}, \ldots, X_{lc}; X_{m1}, \ldots, X_{md}) = \begin{cases} 2 \text{ if } X^{(l)}_{i:c} < X^{(m)}_{j:d} \; and \; X^{(l)}_{c-i+1:c} < X^{(m)}_{d-j+1:d} \\ 1 \text{ if } X^{(l)}_{i:c} < X^{(m)}_{j:d} \; or \; X^{(l)}_{c-i+1:c} < X^{(m)}_{d-j+1:d} \\ 0 \text{ otherwise} \end{cases}$$

where $X^{(l)}_{i:c}\left(X^{(l)}_{c-i+1:c}\right)$ is the ith($(c-i+1)$th) order statistics for a sub-sample of size c for the observations from the lth population and likewise $X^{(m)}_{j:d}\left(X^{(m)}_{d-j+1:d}\right)$ is the jth($(d-j+1)$th) order statistics for a sub-sample of size d for the observations from the mth population.

The U-statistics associated with kernel $h_{i,j}$ is:

$$U^{(l,m)}_{c,d;i,j} = \left[\binom{n_l}{c}\binom{n_m}{d}\right]^{-1} \sum h_{i,j}(X_{l\alpha_1}, \ldots, X_{l\alpha_c}; X_{m\beta_1}, \ldots, X_{m\beta_d}),$$

where Σ donates the sum over all possible combinations $(\alpha_1, \ldots, \alpha_c)$ of c integers chosen from $(1, \ldots, n_l)$ and all possible combinations of d integers $(\beta_1, \ldots, \beta_d)$

chosen from $(1, \ldots, n_m)$. Clearly, $U^{(l,m)}_{c,d;i,j}$ is the test statistics proposed by Kumar [17] for lth and mth populations.

The proposed class of test statistics is:

$$W_{c,d;i,j} = \sum_{l=1}^{k-1} a_l U^{(l,l+1)}_{c,d;i,j},$$

where a_l, $l = 1, \ldots, k-1$ are suitably chosen real constants. Test is to reject H_0 for large values of $W_{c,d;i,j}$.

In particular,

1. For $c = d = 1;\ i = j = 1$, the test statistics corresponds to Jonckheere [1] test.
2. For $i = 1,\ j = 1$ or $i = c,\ j = d$, the test statistics corresponds to Amita and Kochar [3] test.
3. For $c = d = 3;\ i = j = 2$, the test statistics corresponds to Kumar et al. [5] test.
4. For $c = d =$ any odd positive integer, $i = j = (c+1)/2$, the test statistics corresponds to Kumar et al. [7] test.
5. For $c = d;\ i = j$, the test statistics corresponds to Kumar et al. [8] test.

Thus, this paper is the extended version of the previous work.

3 Distribution of the Proposed Test Statistics

The expectation of two-sample test statistics $U^{(l,l+1)}_{c,d;i,j}$ is:

$$\begin{aligned}
E\left[U^{(l,l+1)}_{c,d;i,j}\right] &= \left[\binom{n_l}{c}\binom{n_{l+1}}{d}\right]^{-1} \sum E\left[h_{i,j}\left(X_{l\alpha_1}, \ldots, X_{l\alpha_c}; X_{(l+1)\beta_1}, \ldots, X_{(l+1)\beta_d}\right)\right] \\
&= E\left[h_{i,j}\left(X_{l\alpha_1}, \ldots, X_{l\alpha_c}; X_{(l+1)\beta_1}, \ldots, X_{(l+1)\beta_d}\right)\right] \\
&= P\left(X^{(l)}_{i:c} < X^{(l+1)}_{j:d}\right) + P\left(X^{(l)}_{c-i+1:c} < X^{(l+1)}_{d-j+1:d}\right) \\
&= \int_{-\infty}^{\infty} F^{(l)}_{i:c}(y)\, d\left[F^{(l+1)}_{j:d}(y)\right] + \int_{-\infty}^{\infty} F^{(l)}_{c-i+1:c}(y)\, d\left[F^{(l+1)}_{d-j+1:d}(y)\right]
\end{aligned}$$

where

$$F^{(l)}_{i:c}(y) = \sum_{t=i}^{c} \binom{c}{t} (F_l(y))^t (1 - F_l(y))^{c-t},$$

$$F^{(l)}_{c-i+1:c}(y) = \sum_{t=c-i+1}^{c} \binom{c}{t} (F_l(y))^t (1 - F_l(y))^{c-t}$$

and $d[]$ stands for the derivative of the expression inside the brackets. Here,

$$d\Big[F_{j:d}^{(l+1)}(y)\Big] = \binom{d}{j} j(F_{l+1}(y))^{j-1}(1-F_{l+1}(y))^{d-j}d[F_{l+1}(y)],$$

$$d\Big[F_{d-j+1:d}^{(l+1)}(y)\Big] = \binom{d}{j} j(F_{l+1}(y))^{d-j}(1-F_{l+1}(y))^{j-1}d[F_{l+1}(y)].$$

Under H_0,

$$E\Big[U_{c,d;i,j}^{(l,l+1)}\Big] = 1, \text{ for all } c, d; i, j.$$

The expectation of test statistics $W_{c,d;i,j}$ is:

$$E\big[W_{c,d;i,j}\big] = \sum_{l=1}^{k-1} a_l\, E\Big[U_{c,d;i,j}^{(l,l+1)}\Big] = \sum_{l=1}^{k-1} a_l.$$

Consider the random vector $\underline{U'} = \Big(U_{c,d;i,j}^{(1,2)}, U_{c,d;i,j}^{(2,3)}, \ldots, U_{c,d;i,j}^{(k-1,k)}\Big)$. As $U_{c,d;i,j}^{(l,m)}$ is two-sample U-statistics, so using the results given by Lehmann [21], the following theorem on limiting normality of $\Big\{U_{c,d;i,j}^{(l,m)}\Big\}$ can be easily proved (also see Randles and Wolfe [13] p. 104).

Theorem 1 *Let $N = \sum_{l=1}^{k} n_l$. The asymptotic normality of $N^{1/2}\big[\underline{U} - E\big[\underline{U}\big]\big]$ as $N \to \infty$ in such a way that $n_l/N \to p_l, 0 < p_l < 1$, for $l = 1, \ldots, k$ is multivariate normal with mean vector zero and dispersion matrix $\sum = \big(\big(\sigma_{l,m}\big)\big)$, given as:*

$$\big(\big(\sigma_{l,m}\big)\big) = \begin{cases} \frac{c^2}{p_l}\xi_{l,l+1;l,l+1}^{(l)} + \frac{d^2}{p_{l+1}}\xi_{l,l+1;l,l+1}^{(l+1)} & \textit{for } l = m = 1, \ldots, k-1 \\ \frac{cd}{p_{l+1}}\xi_{l,l+1;l+1,l+2}^{(l+1)} & \textit{for } m = l+1; l = 1, \ldots, k-2 \\ \frac{cd}{p_l}\xi_{l-1,l;l,l+1}^{(l)} & \textit{for } m = l-1; l = 2, \ldots, k-1 \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

where

$$\xi_{l,l+1;l,l+1}^{(l)} = E\Big[\Big\{\psi_{l,l+1}^{(l)}(x)\Big\}^2\Big] - E^2\Big[U_{c,d;i,j}^{(l,l+1)}\Big]$$

$$\xi_{l,l+1;l,l+1}^{(l+1)} = E\Big[\Big\{\psi_{l,l+1}^{(l+1)}(x)\Big\}^2\Big] - E^2\Big[U_{c,d;i,j}^{(l,l+1)}\Big]$$

$$\xi_{l,l+1;l+1,l+2}^{(l+1)} = E\Big[\Big\{\psi_{l,l+1}^{(l+1)}(x)\Big\}\Big\{\psi_{l+1,l+2}^{(l+1)}(x)\Big\}\Big] - E\Big[U_{c,d;i,j}^{(l,l+1)}\Big]E\Big[U_{c,d;i,j}^{(l+1,l+2)}\Big]$$

and

$$\psi^{(l)}_{l,m}(x) = E\big[h_{i,j}(x, X_{l2}, \ldots, X_{lc}; X_{m1}, X_{m2}, \ldots, X_{md})\big]$$
$$\psi^{(m)}_{l,m}(x) = E\big[h_{i,j}(X_{l1}, X_{l2}, \ldots, X_{lc}; x, X_{m2}, \ldots, X_{md})\big].$$

After some involved calculations, it can be seen that under H_0, $\sum_0 = \big((\sigma^0_{l,m})\big)$, *given as:*

$$\big((\sigma^0_{l,m})\big) = \begin{cases} c^2\rho_{c,d;i,j}\left(\frac{1}{p_l} + \frac{1}{p_{l+1}}\right) \textit{for}\, l = m = 1, \ldots, k-1 \\ -c^2\rho_{c,d;i,j}\left(\frac{1}{p_{l+1}}\right) \quad \textit{for}\, m = l+1; l = 1, \ldots, k-2 \\ -c^2\rho_{c,d;i,j}\left(\frac{1}{p_l}\right) \quad \textit{for}\, m = l-1; l = 2, \ldots, k-1 \\ 0 \quad \textit{otherwise} \end{cases} \tag{2}$$

where $\rho_{c,d;i,j}$ *is given by Kumar [17] as:*

$$\begin{aligned}\rho_{c,d;i,j} + 1 = \binom{d}{j}^2 j^2 a^2_{c,d;i,j}\Bigg[& -\frac{2A_{c,d;i,j}}{a_{c,d;i,j}}\sum_{r=i+j-1}^{c+d-1}\binom{c+d-1}{r}\beta(r+1, c+d-r) \\ & + \sum_{r=i+j-1}^{c+d-1}\sum_{s=i+j-1}^{c+d-1}\binom{c+d-1}{r}\binom{c+d-1}{s}\beta(r+s+1, 2c+2d-r-s-1) \\ & + 2\sum_{r=i+j-1}^{c+d-1}\sum_{s=c+d-i-j+1}^{c+d-1}\binom{c+d-1}{r}\binom{c+d-1}{s}\beta(r+s+1, 2c+2d-r-s-1) \\ & + \sum_{r=c+d-i-j+1}^{c+d-1}\sum_{s=c+d-i-j+1}^{c+d-1}\binom{c+d-1}{r}\binom{c+d-1}{s} \\ & \times\ \beta(r+s+1, 2c+2d-r-s-1) \\ & + \frac{A^2_{c,d;i,j}}{a^2_{c,d;i,j}} - \frac{2A_{c,d;i,j}}{a_{c,d;i,j}}\sum_{r=c+d-i-j+1}^{c+d-1}\binom{c+d-1}{r}\beta(r+1, c+d-r)\Bigg]\end{aligned}$$

with

$$\begin{aligned}A_{c,d;i,j} = & \sum_{r=c-i+1}^{c-1}\binom{c-1}{r}\beta(r+d-j+1,\ c-r+j-1) \\ & + \sum_{r=i}^{c-1}\binom{c-1}{r}\beta(r+j, c+d-r-j) + 2\binom{c-1}{i-1}\beta(i+j-1, c+d-i-j+1)\end{aligned}$$

and

$$a_{c,d;i,j} = \binom{c-1}{i-1}\binom{c+d-1}{i+j-1}^{-1}(i+j-1)^{-1}.$$

In case, all sample sizes are equal, i.e. $p_l = 1/k$ *for all* $l = 1, \ldots, k$*; then,* $\big((\sigma^0_{l,m})\big)$ *becomes*

$$\big((\sigma^*_{l,m})\big) = \begin{cases} 2kc^2\rho_{c,d;i,j} & for\, i = j = 1, \ldots, k-1 \\ -kc^2\rho_{c,d;i,j} & for\ j = i+1; i = 1, \ldots, k-2 \\ -kc^2\rho_{c,d;i,j} & for\ j = i-1; i = 2, \ldots, k-1 \\ 0 & otherwise. \end{cases} \tag{3}$$

As $W_{c,d;i,j}$ *is the linear combination of the components of* $\underline{U}$*, so using the transformation theorem given by Serfling ([22], p. 122), the following theorem on asymptotic distribution of* $W_{c,d;i,j}$ *can be easily proved.*

Theorem 2 *The asymptotic distribution of* $N^{1/2}\big[W_{c,d;i,j} - E\big[W_{c,d;i,j}\big]\big]$ *as* $N \to \infty$ *is normal with mean zero and variance* $\underline{a}' \sum \underline{a}$*, where* $\underline{a} = (a_1, \ldots, a_{k-1})'$.

Under H_0*,*

$$E\big[W_{c,d;i,j}\big] = \sum_{l=1}^{k-1} a_l \ and$$

$$\underline{a}' \sum \underline{a} = 2kc^2\rho_{c,d;i,j}\left\{\sum_{l=1}^{k-1} a_l^2 - \sum_{l=1}^{k-2} a_l a_{l+1}\right\},$$

when all the sample sizes are equal, i.e. $p_l = 1/k$ *for all* $l = 1, \ldots, k$.

4 Optimal Choice of Weights

Since the test statistics $W_{c,d;i,j}$ is the weighted linear combination of two-sample U-statistics, so in this section, we consider the problem of obtaining the optimal weights, a_l's, which results in test statistics, $W_{c,d;i,j}$, to have maximum efficiency for the sequence of Pitman-type alternatives defined as:

$$H_N : F_l(x) = F\big(x - N^{-1/2}\theta_l\big), l = 1, \ldots, k.$$

For efficiency comparisons, we shall consider the case of equal sample sizes and equal spaced alternatives, i.e. $p_l = 1/k$ and $\theta_l = l\theta, \theta > 0$ for $l = 1, \ldots, k$. Then, the alternative H_N becomes

$$H_N' : F_l(x) = F\Big(x - N^{-\frac{1}{2}} l\theta\Big), l = 1, \ldots, k. \tag{4}$$

The following theorem gives the asymptotic distribution of $\underline{U}$, under the sequence of alternatives H_N'.

Theorem 3 *Let X_{lm} be independent random variables with cdf $F_l(x)$, $l = 1, \ldots, k$, where $F_l(x)$ is given in (4); then, the limiting distribution of $N^{1/2}\left[\underline{U} - I_{k-1}\right]$ is multivariate normal of $(k-1)$ dimensional with mean vector $\nu_{c,d;i,j}\theta I_{k-1}$ and dispersion matrix $\sum^* = \left(\left(\sigma^*_{lm}\right)\right)$, where*

$$\nu_{c,d;i,j} = \binom{c}{i} i \binom{d}{j} j \left[\int_{-\infty}^{\infty} (F(y))^{i+j-2}(1-F(y))^{c+d-i-j} f^2(y)dy \right.$$
$$\left. + \int_{-\infty}^{\infty} (F(y))^{c+d-i-j}(1-F(y))^{i+j-2} f^2(y)dy \right],$$

and $I_{k-1} = [1]_{(k-1)\times 1}$ and $\left(\left(\sigma^*_{l,m}\right)\right)$ is given in (3).

Now, we deal with finding the optimal weights, $a_l^\prime s$, as given by Rao [23].

Theorem 4 *Under the sequence of local alternatives $\left\{H_N^\prime(x)\right\}$, defined in (4), the efficacy of the test statistics $W_{c,d;i,j}$ is maximized for the weights:*

$$a_l^* = \frac{l(k-l)}{2k},\ l = 1, \ldots, k-1.$$

The efficacy of the test statistic $W_{c,d;i,j}$ is:

$$e\left(W_{c,d;i,j}\right) = \left[\frac{d}{d\theta} E_{H_N^\prime}\left[W_{c,d;i,j}\right]\big|_{\theta=0}\right]^2 \Big/ \left(\underline{a}^\prime \sum \underline{a}\right)$$
$$= \nu^2_{c,d;i,j}\left[\left(I^\prime_{k-1}\underline{a}\right)^2 \Big/ \left(\underline{a}^\prime \sum \underline{a}\right)\right].$$

Using the optimum choice of weights as given in Theorem 4, the efficacy of the test statistics $W_{c,d;i,j}$ becomes (see Rao [24] p. 60):

$$\underbrace{Sup}_{\underline{a}}\, e\left(W_{c,d;i,j}\right) = \frac{\left(k^2-1\right)\nu^2_{c,d;i,j}}{12c^2\rho_{c,d;i,j}}, \tag{5}$$

where expressions for $\rho_{c,d;i,j}$ and $\nu_{c,d;i,j}$ are given in Theorems 1 and 3, respectively.

5 Comparison of the Proposed Test

In this section, we study the Pitman asymptotic relative efficiencies (AREs) to compare the $W_{c,d;i,j}$ tests with respect to (w.r.t.) some existing tests for multi-sample location problem, namely Jonckheere [1] J test and some members of Kumar et al. [7] K_m test, Gaur [9] $T_{r,s}$ test and Gaur [11] G_c test.

After submitting the expression of $\rho_{c,d;i,j}$ and $\nu_{c,d;i,j}$ in the expression of efficacy of $W_{c,d;i,j}$ as given in (5), it can be seen that the efficacy of $W_{c,d;i,j}$ depends upon $(c, d; i, j)$ but through their sum only, i.e. $(c + d; i + j)$.

Remark 1 Due to symmetric nature of the test statistics, the asymptotic efficacy of $W_{c,d;i,j}$ test remains same by taking $(c + d; i + j)$ as $(c + d; c + d - i - j + 2)$. Moreover, if $2(i + j) = c + d + 2$, the asymptotic efficacy of $W_{c,d;i,j}$ test remains same by taking $(c + d; i + j)$ as $(c + d + 1; i + j)$.

The AREs of $W_{c,d;i,j}$ test w.r.t. J and some members of K_m, $T_{r,s}$ and G_c tests for different underlying distribution are given in Tables 1, 2, 3, 4, 5 and 6, for different combinations of $(c + d; i + j)$.

From the tables of AREs, we observe the following:

1. Performance of $W_{c,d;i,j}$ tests depends upon the tail behaviour of the underlying distribution.
2. For light-tailed distributions, to have some gain in efficiency w.r.t. competing tests, choose $(c, d; i, j)$ in such a way that $(c + d)$ as maximum as possible with $(i + j) = 2$.
3. For medium-tailed distributions, to have some gain in efficiency w.r.t. competing tests, choose $(c, d; i, j)$ in such a way that $(c + d) = 5$ with $(i + j) = 2$. In case of logistic distribution, J test has maximum efficiency in comparison with all other tests, which is well-known result of optimality test.
4. For heavy-tailed distributions, to have some gain in efficiency w.r.t. competing tests, choose $(c, d; i, j)$ in such a way that $(c + d)$ as maximum as possible with $(i + j) = (c + d + 1)/2$; if $(c + d)$ is odd, otherwise $(i + j) = (c + d + 2)/2$; if $(c + d)$ is even.

6 An Illustrative Example

To see the implementation of the test, we consider the study of Shirley [25], in which the time (in seconds) taken by mice to stimuli to their tails was measured at different levels of dose. In this experiment, four levels of dose (0–3) have been considered with ten mice in each group. Now, we wish to test whether increasing level of dose impacts the reaction time by mice, i.e. to test H_0:

$$H_0 : \theta_1 = \theta_2 = \theta_3 = \theta_4$$

against the ordered alternative

$$H_1 : \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4,$$

where θ_i's are appropriate parameters corresponding to level of dose (0–3).

Table 1 AREs of $U_{c,d;i,j}$ for uniform distribution w.r.t. different tests

Test $(c+d; i+j)$	J	K_1	K_2	K_3	$T_{2,1}$	$T_{2,2}$ or $T_{3,1}$	$T_{3,2}$	$T_{3,3}$	G_2	G_3
(2; 2)	1.000	1.701	1.969	2.119	1.000	0.814	0.657	0.543	1.457	1.701
(3; 2)	1.000	1.701	1.969	2.119	1.000	0.814	0.657	0.543	1.457	1.701
(4; 2)	1.228	2.089	2.418	2.602	1.228	1.000	0.807	0.667	1.789	2.089
(5; 2)	1.522	2.589	2.996	3.225	1.522	1.239	1.000	0.827	2.217	2.589
(6; 2)	1.841	3.131	3.624	3.901	1.841	1.499	1.210	1.000	2.682	3.131
(7; 2)	2.169	3.690	4.270	4.597	2.169	1.766	1.425	1.178	3.161	3.690
(8; 2)	2.501	4.254	4.924	5.299	2.501	2.036	1.643	1.359	3.644	4.254
(3; 3)	1.000	1.701	1.969	2.119	1.000	0.814	0.657	0.543	1.457	1.701
(4; 3)	0.686	1.168	1.351	1.454	0.686	0.559	0.451	0.373	1.000	1.168
(5; 3)	0.686	1.168	1.351	1.454	0.686	0.559	0.451	0.373	1.000	1.168
(6; 3)	0.762	1.297	1.501	1.616	0.762	0.621	0.501	0.414	1.111	1.297
(7; 3)	0.870	1.480	1.713	1.844	0.870	0.708	0.572	0.473	1.268	1.480
(8; 3)	0.992	1.688	1.954	2.103	0.992	0.808	0.652	0.539	1.446	1.688
(4; 4)	1.228	2.089	2.418	2.602	1.228	1.000	0.807	0.667	1.789	2.089
(5; 4)	0.686	1.168	1.351	1.454	0.686	0.559	0.451	0.373	1.000	1.168
(6; 4)	0.588	1.000	1.157	1.246	0.588	0.479	0.386	0.319	0.856	1.000
(7; 4)	0.588	1.000	1.157	1.246	0.588	0.479	0.386	0.319	0.856	1.000
(8; 4)	0.628	1.068	1.236	1.331	0.628	0.511	0.413	0.341	0.915	1.068
(8; 5)	0.538	0.916	1.060	1.141	0.538	0.438	0.354	0.292	0.784	0.916

Table 2 AREs of $U_{c,d;i,j}$ for exponential distribution w.r.t. different tests

Test $(c+d; i+j)$	J	K_1	K_2	K_3	$T_{2,1}$	$T_{2,2}$ or $T_{3,1}$	$T_{3,2}$	$T_{3,3}$	G_2	G_3
(2; 2)	1.000	1.701	1.969	2.119	1.000	0.814	0.657	0.543	1.457	1.701
(3; 2)	1.000	1.701	1.969	2.119	1.000	0.814	0.657	0.543	1.457	1.701
(4; 2)	1.228	2.089	2.418	2.602	1.228	1.000	0.807	0.667	1.789	2.089
(5; 2)	1.522	2.589	2.996	3.225	1.522	1.239	1.000	0.827	2.217	2.589
(6; 2)	1.841	3.131	3.624	3.901	1.841	1.499	1.210	1.000	2.682	3.131
(7; 2)	2.169	3.690	4.270	4.597	2.169	1.766	1.425	1.178	3.161	3.690
(8; 2)	2.501	4.254	4.924	5.299	2.501	2.036	1.643	1.359	3.644	4.254
(3; 3)	1.000	1.701	1.969	2.119	1.000	0.814	0.657	0.543	1.457	1.701
(4; 3)	0.686	1.168	1.351	1.454	0.686	0.559	0.451	0.373	1.000	1.168
(5; 3)	0.686	1.168	1.351	1.454	0.686	0.559	0.451	0.373	1.000	1.168
(6; 3)	0.762	1.297	1.501	1.616	0.762	0.621	0.501	0.414	1.111	1.297
(7; 3)	0.870	1.480	1.713	1.844	0.870	0.708	0.572	0.473	1.268	1.480
(8; 3)	0.992	1.688	1.954	2.103	0.992	0.808	0.652	0.539	1.446	1.688
(4; 4)	1.228	2.089	2.418	2.602	1.228	1.000	0.807	0.667	1.789	2.089
(5; 4)	0.686	1.168	1.351	1.454	0.686	0.559	0.451	0.373	1.000	1.168
(6; 4)	0.588	1.000	1.157	1.246	0.588	0.479	0.386	0.319	0.856	1.000
(7; 4)	0.588	1.000	1.157	1.246	0.588	0.479	0.386	0.319	0.856	1.000
(8; 4)	0.628	1.068	1.236	1.331	0.628	0.511	0.413	0.341	0.915	1.068
(8; 5)	0.538	0.916	1.060	1.141	0.538	0.438	0.354	0.292	0.784	0.916

Table 3 AREs of $U_{c,d;i,j}$ for normal distribution w.r.t. different tests

Test $(c+d; i+j)$	J	K_1	K_2	K_3	$T_{2,1}$	$T_{2,2}$ or $T_{3,1}$	$T_{3,2}$	$T_{3,3}$	G_2	G_3
(2; 2)	1.000	1.094	1.148	1.183	1.000	0.978	0.967	0.969	1.055	1.094
(3; 2)	1.000	1.094	1.148	1.183	1.000	0.978	0.967	0.969	1.055	1.094
(4; 2)	1.022	1.118	1.173	1.209	1.022	1.000	0.988	0.991	1.078	1.118
(5; 2)	1.035	1.132	1.187	1.224	1.035	1.012	1.000	1.003	1.091	1.132
(6; 2)	1.032	1.129	1.184	1.221	1.032	1.010	0.997	1.000	1.088	1.129
(7; 2)	1.017	1.113	1.167	1.203	1.017	0.995	0.983	0.986	1.073	1.113
(8; 2)	0.995	1.089	1.142	1.178	0.995	0.974	0.962	0.964	1.049	1.089
(3; 3)	1.000	1.094	1.148	1.183	1.000	0.978	0.967	0.969	1.055	1.094
(4; 3)	0.948	1.038	1.088	1.122	0.948	0.928	0.917	0.919	1.000	1.038
(5; 3)	0.948	1.038	1.088	1.122	0.948	0.928	0.917	0.919	1.000	1.038
(6; 3)	0.970	1.061	1.113	1.148	0.970	0.949	0.937	0.940	1.023	1.061
(7; 3)	0.993	1.087	1.140	1.175	0.993	0.972	0.960	0.962	1.047	1.087
(8; 3)	1.009	1.104	1.158	1.194	1.009	0.987	0.976	0.978	1.064	1.104
(4; 4)	1.022	1.118	1.173	1.209	1.022	1.000	0.988	0.991	1.078	1.118
(5; 4)	0.948	1.038	1.088	1.122	0.948	0.928	0.917	0.919	1.000	1.038
(6; 4)	0.914	1.000	1.049	1.081	0.914	0.894	0.883	0.886	0.964	1.000
(7; 4)	0.914	1.000	1.049	1.081	0.914	0.894	0.883	0.886	0.964	1.000
(8; 4)	0.931	1.019	1.069	1.102	0.931	0.911	0.900	0.903	0.982	1.019
(8; 5)	0.890	0.973	1.021	1.053	0.890	0.870	0.860	0.862	0.938	0.973

Table 4 AREs of $U_{c,d;i,j}$ for logistic distribution w.r.t. different tests

Test $(c+d;i+j)$	J	K_1	K_2	K_3	$T_{2,1}$	$T_{2,2}$ or $T_{3,1}$	$T_{3,2}$	$T_{3,3}$	G_2	G_3
(2; 2)	1.000	1.029	1.059	1.081	1.000	1.005	1.027	1.065	1.012	1.029
(3; 2)	1.000	1.029	1.059	1.081	1.000	1.005	1.027	1.065	1.012	1.029
(4; 2)	0.995	1.024	1.053	1.076	0.995	1.000	1.021	1.059	1.007	1.024
(5; 2)	0.974	1.002	1.031	1.053	0.974	0.979	1.000	1.037	0.986	1.002
(6; 2)	0.939	0.967	0.994	1.015	0.939	0.944	0.964	1.000	0.950	0.967
(7; 2)	0.896	0.923	0.949	0.969	0.896	0.901	0.920	0.955	0.907	0.923
(8; 2)	0.851	0.876	0.901	0.920	0.851	0.855	0.874	0.906	0.861	0.876
(3; 3)	1.000	1.029	1.059	1.081	1.000	1.005	1.027	1.065	1.012	1.029
(4; 3)	0.988	1.017	1.046	1.068	0.988	0.993	1.015	1.052	1.000	1.017
(5; 3)	0.988	1.017	1.046	1.068	0.988	0.993	1.015	1.052	1.000	1.017
(6; 3)	0.996	1.025	1.054	1.077	0.996	1.001	1.022	1.060	1.008	1.025
(7; 3)	0.999	1.028	1.058	1.080	0.999	1.004	1.026	1.064	1.011	1.028
(8; 3)	0.992	1.021	1.051	1.073	0.992	0.998	1.019	1.057	1.004	1.021
(4; 4)	0.995	1.024	1.053	1.076	0.995	1.000	1.021	1.059	1.007	1.024
(5; 4)	0.988	1.017	1.046	1.068	0.988	0.993	1.015	1.052	1.000	1.017
(6; 4)	0.972	1.000	1.029	1.051	0.972	0.977	0.998	1.035	0.983	1.000
(7; 4)	0.972	1.000	1.029	1.051	0.972	0.977	0.998	1.035	0.983	1.000
(8; 4)	0.981	1.010	1.039	1.061	0.981	0.986	1.007	1.045	0.993	1.010
(8; 5)	0.957	0.985	1.013	1.035	0.957	0.962	0.983	1.019	0.968	0.985

Table 5 AREs of $U_{c,d;i,j}$ for Laplace distribution w.r.t. different tests

Test $(c+d; i+j)$	J	K_1	K_2	K_3	$T_{2,1}$	$T_{2,2}$ or $T_{3,1}$	$T_{3,2}$	$T_{3,3}$	G_2	G_3
(2; 2)	1.000	0.900	0.866	0.848	1.000	1.064	1.168	1.303	0.933	0.900
(3; 2)	1.000	0.900	0.866	0.848	1.000	1.064	1.168	1.303	0.933	0.900
(4; 2)	0.940	0.846	0.814	0.797	0.940	1.000	1.098	1.225	0.877	0.846
(5; 2)	0.856	0.770	0.741	0.726	0.856	0.910	1.000	1.115	0.798	0.770
(6; 2)	0.768	0.691	0.665	0.651	0.768	0.817	0.897	1.000	0.716	0.691
(7; 2)	0.686	0.618	0.594	0.582	0.686	0.730	0.802	0.894	0.640	0.618
(8; 2)	0.615	0.554	0.533	0.522	0.615	0.655	0.719	0.802	0.574	0.554
(3; 3)	1.000	0.900	0.866	0.848	1.000	1.064	1.168	1.303	0.933	0.900
(4; 3)	1.072	0.965	0.929	0.909	1.072	1.140	1.253	1.397	1.000	0.965
(5; 3)	1.072	0.965	0.929	0.909	1.072	1.140	1.253	1.397	1.000	0.965
(6; 3)	1.038	0.934	0.899	0.880	1.038	1.104	1.212	1.352	0.968	0.934
(7; 3)	0.982	0.884	0.851	0.833	0.982	1.045	1.147	1.279	0.916	0.884
(8; 3)	0.916	0.825	0.794	0.777	0.916	0.975	1.071	1.194	0.855	0.825
(4; 4)	0.940	0.846	0.814	0.797	0.940	1.000	1.098	1.225	0.877	0.846
(5; 4)	1.072	0.965	0.929	0.909	1.072	1.140	1.253	1.397	1.000	0.965
(6; 4)	1.111	1.000	0.962	0.942	1.111	1.182	1.298	1.447	1.036	1.000
(7; 4)	1.111	1.000	0.962	0.942	1.111	1.182	1.298	1.447	1.036	1.000
(8; 4)	1.088	0.979	0.942	0.922	1.088	1.157	1.271	1.417	1.015	0.979
(8; 5)	1.137	1.023	0.984	0.964	1.137	1.209	1.328	1.480	1.060	1.023

Table 6 AREs of $U_{c,d;i,j}$ for Cauchy distribution w.r.t. different tests

Test $(c+d; i+j)$	J	K_1	K_2	K_3	$T_{2,1}$	$T_{2,2}$ or $T_{3,1}$	$T_{3,2}$	$T_{3,3}$	G_2	G_3
(2; 2)	1.000	0.796	0.743	0.721	1.000	1.132	1.356	1.670	0.857	0.796
(3; 2)	1.000	0.796	0.743	0.721	1.000	1.132	1.356	1.670	0.857	0.796
(4; 2)	0.883	0.703	0.656	0.636	0.883	1.000	1.198	1.475	0.757	0.703
(5; 2)	0.737	0.587	0.548	0.531	0.737	0.835	1.000	1.231	0.632	0.587
(6; 2)	0.599	0.477	0.445	0.432	0.599	0.678	0.812	1.000	0.513	0.477
(7; 2)	0.481	0.383	0.358	0.347	0.481	0.545	0.653	0.804	0.412	0.383
(8; 2)	0.387	0.308	0.287	0.279	0.387	0.438	0.524	0.646	0.331	0.308
(3; 3)	1.000	0.796	0.743	0.721	1.000	1.132	1.356	1.670	0.857	0.796
(4; 3)	1.167	0.929	0.867	0.841	1.167	1.321	1.583	1.949	1.000	0.929
(5; 3)	1.167	0.929	0.867	0.841	1.167	1.321	1.583	1.949	1.000	0.929
(6; 3)	1.095	0.872	0.814	0.789	1.095	1.240	1.486	1.829	0.939	0.872
(7; 3)	0.990	0.788	0.736	0.714	0.990	1.122	1.343	1.654	0.849	0.788
(8; 3)	0.874	0.696	0.649	0.630	0.874	0.990	1.185	1.459	0.749	0.696
(4; 4)	0.883	0.703	0.656	0.636	0.883	1.000	1.198	1.475	0.757	0.703
(5; 4)	1.167	0.929	0.867	0.841	1.167	1.321	1.583	1.949	1.000	0.929
(6; 4)	1.256	1.000	0.933	0.905	1.256	1.423	1.704	2.098	1.077	1.000
(7; 4)	1.256	1.000	0.933	0.905	1.256	1.423	1.704	2.098	1.077	1.000
(8; 4)	1.210	0.963	0.899	0.872	1.210	1.370	1.641	2.021	1.037	0.963
(8; 5)	1.310	1.043	0.974	0.944	1.310	1.484	1.778	2.188	1.123	1.043

Table 7 Computed values of $W_{c,d;i,j}$ and corresponding p values

$(c+d; i+j)$	(4; 2)	(4; 3) or (5; 3)	(6; 3)	(6; 4) or (7; 4)	(8; 4)	(8; 5)
$W_{c,d;i,j}$	1.74586	1.79333	1.75040	2.01063	2.06919	2.10501
p value	1.1×10^{-5}	1.9×10^{-6}	3.6×10^{-6}	2.9×10^{-7}	5.8×10^{-7}	2.7×10^{-7}

The computed value of $W_{c,d;i,j}$ test statistics and the corresponding p values for different combinations of $(c, d; i, j)$ through their sum $(c+d; i+j)$, are given in Table 7. For computing the values, we have used optimal choice of weights as given in Theorem 4.

From Table 7, test is to reject H_0 at 1% level of significance for all $W_{c,d;i,j}$ tests considered here. However, it may be noted that for the choice $(c+d; i+j) = (8, 5)$, $W_{c,d;i,j}$ test clarifies it more significantly. This is so, by Kolmogorov–Smirnov test, the data set follows Laplace distribution at 1% level of significance and by using the observation 4, of AREs, $W_{c,d;i,j}$ test with $(c+d; i+j) = (8, 5)$ has maximum efficiency as compared to another $W_{c,d;i,j}$ tests considered here. As test is to reject H_0, it concluded that increasing level of dose impacts in increasing the time taken by mice to stimuli to their tails.

7 Simulation Study

To see the performance of the proposed test, Monte Carlo simulation study is carried out. Here, we found power of $W_{c,d;i,j}$ test by considering $k = 3$, i.e. for three populations with sample size $n = 10\,(10)\,30$. Data is generated from three well-known distributions, namely (i) normal, (ii) Laplace and (iii) Cauchy data with shift parameter (θ) considered is $\theta = \theta_{i+1} - \theta_i = 0.2\,(0.2)\,0.6;\ i = 1, 2$, and level of significance is fixed at 0.05. The computation of power is based on 10,000 repetitions and is given in Tables 8, 9 and 10.

Based on the power computations, we have the following observations:

1. For normal distribution, shift of order 0.6 is detected at reasonable power for sample size 20, for $(c+d) = 5$ with $(i+j) = 2$ and required more sample size for other combinations of $(c, d; i, j)$ to detect the shift of same order.
2. For Laplace and Cauchy distribution, shift of order 0.6 is detected at reasonable power for sample size 20 and 30, respectively, for $(c+d) = 7$ with $(i+j) = 4$ and required more sample size for other combinations of $(c, d; i, j)$ to detect the shift of same order. This authenticates the computations of AREs as well.

Table 8 Estimated power of $W_{c,d;i,j}$ for normal distribution

n	θ	$(c+d;i+j)$							
		(2; 2)	(3; 2)	(4; 2)	(5; 2)	(5; 3)	(6; 3)	(7; 3)	(7; 4)
10	0.2	0.2209	0.2204	0.2327	0.2351	0.2105	0.2131	0.2185	0.1992
	0.4	0.5139	0.5227	0.5381	0.5396	0.5043	0.5065	0.5109	0.4904
	0.6	0.8103	0.8102	0.8189	0.8198	0.7847	0.7968	0.8019	0.7654
20	0.2	0.3377	0.3369	0.3502	0.3595	0.3310	0.3342	0.3361	0.3215
	0.4	0.7788	0.7813	0.7925	0.7942	0.7703	0.7727	0.7768	0.7586
	0.6	0.9459	0.9464	0.9485	0.9541	0.9405	0.9428	0.9443	0.9318
30	0.2	0.4510	0.4487	0.4608	0.4687	0.4454	0.4479	0.4498	0.4366
	0.4	0.9071	0.9083	0.9174	0.9243	0.9002	0.9034	0.9052	0.8914
	0.6	0.9880	0.9878	0.9895	0.9907	0.9811	0.9840	0.9861	0.9720

Table 9 Estimated power of $W_{c,d;i,j}$ for Laplace distribution

n	θ	$(c+d; i+j)$							
		(2; 2)	(3; 2)	(4; 2)	(5; 2)	(5; 3)	(6; 3)	(7; 3)	(7; 4)
10	0.2	0.1903	0.1902	0.1891	0.1879	0.2092	0.2061	0.1996	0.2146
	0.4	0.4437	0.4449	0.4280	0.4047	0.4641	0.4513	0.4401	0.4712
	0.6	0.6958	0.6963	0.6876	0.6484	0.7189	0.7006	0.6898	0.7297
20	0.2	0.2884	0.2909	0.2787	0.2639	0.2980	0.2908	0.2813	0.3052
	0.4	0.6796	0.6785	0.6655	0.6265	0.7054	0.6912	0.6694	0.7133
	0.6	0.9222	0.9240	0.9132	0.8922	0.9349	0.9274	0.9188	0.9504
30	0.2	0.3760	0.3787	0.3590	0.3461	0.3893	0.3806	0.3662	0.4053
	0.4	0.8261	0.8257	0.8129	0.7768	0.8455	0.8320	0.8180	0.8667
	0.6	0.9838	0.9844	0.9817	0.9696	0.9946	0.9875	0.9834	0.9972

Table 10 Estimated power of $W_{c,d;i,j}$ for Cauchy distribution

n	θ	$(c+d; i+j)$							
		(2; 2)	(3; 2)	(4; 2)	(5; 2)	(5; 3)	(6; 3)	(7; 3)	(7; 4)
10	0.2	0.1295	0.1227	0.1198	0.1155	0.1477	0.1451	0.1199	0.2101
	0.4	0.2496	0.2443	0.2369	0.2249	0.2882	0.2779	0.2402	0.3594
	0.6	0.4158	0.4077	0.3845	0.3487	0.4715	0.4417	0.3961	0.5453
20	0.2	0.1738	0.1709	0.1691	0.1550	0.1997	0.1829	0.1697	0.2682
	0.4	0.3897	0.3833	0.3715	0.3207	0.4279	0.4151	0.3784	0.4896
	0.6	0.6895	0.6605	0.6190	0.5878	0.7548	0.7356	0.6560	0.8233
30	0.2	0.2255	0.2229	0.1987	0.1832	0.2631	0.2508	0.2109	0.3348
	0.4	0.5191	0.5177	0.4726	0.4204	0.5484	0.5312	0.5014	0.6220
	0.6	0.8445	0.8239	0.7861	0.7452	0.9195	0.8924	0.8123	0.9507

8 Conclusion

In this paper, we proposed a general class of distribution-free tests for multi-sample location problem. The asymptotic distribution of the proposed class of tests is derived, and optimal choice of weights is obtained. The Pitman asymptotic relative efficiencies and simulation study of the proposed class of tests $W_{c,d;i,j}$ suggest that one should choose $(c, d; i, j)$ in such a way that (i) $(c + d)$ as maximum as possible with $(i + j) = 2$, for light-tailed distributions, (ii) $(c + d) = 5$ with $(i + j) = 2$, for medium-tailed distributions and (iii) $(c + d)$ as maximum as possible with $(i + j) = (c + d + 1)\big/2$; if $(c + d)$ is odd, otherwise $(i + j) = (c + d + 2)\big/2$; if $(c + d)$ is even, for heavy-tailed distributions.

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Modelling of Male Age at Marriage: Evidences from Western Region of Uttar Pradesh (India)

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1 Introduction

The act of wedding young ladies at a youthful age is most normal in sub-Saharan Africa, South Asia and particularly India (UNICEF [32]). Early marriage has universally been associated with early childbearing, particularly in South Asia, where marriage is very important social events, and childbearing outside of marriage does not receive any societal approval [1, 2]. Traditions encompassing marriage, including the age and the mechanism in which a mate is chosen, rely on upon a general public's perspective of the family, i.e. its structure and quality of life, and the individual and group responsibilities of its individuals.

Marriage is early and almost widespread in India. In spite of the fact that the lawful periods of marriage for males and females are 21 years and 18 years, respectively, a vast extent of female marriage still occurs before the legitimate age. According to Census [3], mean age at marriage for females is 21.2 years. There is a difference of two year between rural and urban population. However, there has been a tremendous decline in the percentage of marriages held below legal age from 35.3% in 1991 to 3.7% in 2011 [3].

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India has a long convention of marriage, even before completing the schooling, and consequently parenthood at younger age. The impact of marriage on fertility and the demographic move is very much recorded. Early marriage does not just increase fertility [22]; it can, if joined by early childbearing, affect the health of females and their offsprings and, therefore, cause the high child and maternal mortality [13]. Moreover, early marriage may affect the chance of any females of receiving any professional training for their livelihood and just deprive them of contributing to the social and economic development of the nation [15, 25].

Age at first marriage is a result of different social, economic and demographic factors. Albeit social and other social practices may support and keep up a young age at marriage, differentials of age at first marriage by financial characteristics are very much reported. Age at first marriage is decidedly involved with financial improvement. Other than economic, religion has been found as a vital element, influencing the planning of family formation and passage into parenthood [31, 33]. Marriage practices change from society to society inside a nation and different for a different nation.

The family formation process comprises of a progression of stages where ladies move progressively from marriage to first birth, from first to second, etc., until they reach their desired family size [27]. Besides, the marriage itself might be viewed as a point of starting childbearing within the familial and socially controlled atmosphere [4]. Hence, age at marriage in the selected groups receives incredible significance in the study of population, demography, family and society. This study endeavours to promote our comprehension of the economic and social components, influencing age at first marriage of the study region in India.

In India, marriages are not only compulsory but happen at early ages. In the huge majority of the cases, guardians wedded off their little girls once they achieve menarche because any further delay after that will result in the burden of more dowries. Also, in the absence of any infrastructure for vocational training (sewing, knitting, pottery and handicrafts) to make them self-reliant, marriage is considered to be the only viable and socially accepted solution. Though there are certain enactments and penalties to check the act of early marriage in India, part of marriages do take place beneath the eligible ages. Along these lines, the issue of early age at marriage in India is extremely intricate in nature.

In social orders where childbearing preceding the marriage is not socially accepted, delay in marriage contributes fundamentally towards diminishment in the levels of fertility by shortening the childbearing span of a female. This thus decreases the number of kids and has backwards effect on the populace growth rate of the nation [21]. The age at marriage changes by religion, caste, geographical area, residence (rural/urban), type of family and salary level. Therefore, age at marriage is an intriguing zone for the demographers, who are occupied with the anticipating fertility rates and population growth rate.

Despite being such an important event, the reliable and quality data on female age at marriage have not been available in India. Historically, India has been a patriarchal society. The birth of a male baby is considered to be a boon and celebrated in different manners in different societies. On the contrary, the birth of a daughter is believed to be a curse, particularly in a rural community. Consequently, the boy receives more attention in everything ranging from good education to proper nutrition [20]. Due to these problems, there has always been an ambiguity in the data related to female age at marriage. More importantly, husband plays a dominant role in taking all important decisions—either it is family size, use of contraceptives or any other thing. Therefore, to overcome this problem, we decided to use the male age at marriage for the present work and we have found a correlation coefficient of 0.95 between the age of husband and wife.

Several studies on differentials and determinants of age at marriage have been undertaken by various demographers. In this direction, [5] got the first breakthrough to approximate the age at marriage. They considered the marriage a combination of various events and then combined them together to get a closed-form mathematical expression. This model is considered a standard tool in demography for determining the age at marriage. Several other models have been proposed after that, but with more mathematical calculations. Coale and McNeil model was further refined and simplified by various researchers to make it practically more viable [8, 9, 17–19, 26] recently tried to improve this model. Kaneko [18] managed to get it in generalised log-gamma function, which still requires a lot of computations for practical purpose. Matthews et al. [23] proposed the PICRATE model for the marriage. In the Indian context, [24] used type I extreme-value model to study the age at marriage pattern. Later, [14] applied the same model to the data taken from Bangladesh. Recently, Singh et al. [29, 30] proposed the negative binomial distribution to approximate the age at marriage after discretising the data.

The problem with the entire above-discussed model is that they require complicated mathematical calculations and made for female age at marriage, but in this study we have used male age at marriage to understand their pattern through probability model. The reason behind taking male age at marriage is that this is a proxy of female age at marriage. Also, it is worthwhile to mention here that generally male partner is found more educated than their female partner; thus, it is expected that they provide better information about their age at marriage than females. The objective of this work is to propose a new model which is simpler in its approach and easier to apply.

2 Description of Data

The data utilised as a part of the present investigation have been gathered from the provincial region of locale Meerut, Uttar Pradesh (India). There has been a fast spurt of modern advancement in and around the city amid the previous decades, and it has turned into a noteworthy centre point for different economic, cultural, educational and developmental activities over the years. The study of about 3300 family units was attempted to get the solid and important information relating to the problem under examination. Singh [28] suggested a methodology to collect the data from three types of villages through a stratified clustered sampling method. These three sorts of the village have been identified as semi-urban, remote and growth centres. We classified the whole of Meerut rural into three regions:

(i) Semi-urban areas, (ii) remote areas and (iii) growth centres

The villages of Meerut district were classified into two groups according to the distance from Meerut City (boundary of Meerut Nagar Nigam) to form two strata. The villages at a distance of fewer than 8 kilometres formed the first stratum of semi-urban villages, while rest constitute the second stratum called as remote villages. For the selection of growth centres, villages, which were located nearby industrial areas and sugar mills, were taken into consideration. Random selection of 8 and 6 villages was done from these two strata, respectively, to get approximately 1100 households of each type.

All semi-urban villages were put in increasing order of their population size and then divided into three almost equal groups. These groups consist of

(a) Large-size village (first group)
(b) Mid-size (second group)
(c) Small village (third group).

From the first group, with the help of random number table, we selected a two-digit random number, and the village corresponding to this number was selected for the study. The remote village was also put in order of their size and divided in the same way as a semi-urban village.

The frequency distribution of selected subjects with respect to their socioeconomic and demographic characteristics has been shown by Table 1. In the study region, 23% males are very low educated (up to primary), 48.5% attained a secondary level, and rest went for higher education. Around 80% males of the area are engaged in labour and agriculture work, and only 20% makes their living from other sources, indicating the dominance of the agrarian economy. In the study area, only 14% males belonged to a small family (up to four members) and rest were from mid-size or large families. This is due to the prevalence of joint family culture in rural areas of the district. The majority of selected males came from the Hindu community. The semi-urban and remote village have almost equal participation, while 20.8% males belonged to growth centres.

Table 1 Frequency distribution of males according to the various background characteristics

Explanatory variable	Number of males	Percentages
Education		
Primary	165	23.0
Secondary	347	48.5
Higher	204	28.5
Occupation		
Labour	279	39.0
Agriculture	293	40.9
Others	144	20.1
Family size		
Small	102	14.2
Mid	323	45.1
Large	291	40.6
Religion		
Hindu	594	83.0
Muslim	122	17.0
Type of village		
Semi-urban	282	39.4
Remote	285	39.8
Growth centres	149	20.8

Table 2 demonstrates the mean age at marriage and their standard deviation (SD) according to different background characteristics. It is evident from the table that lower educated males have a lower mean age at marriage. Age at marriage keeps on increasing with the level of education. Similarly, the males engaged in labour and agriculture work have a lower mean age at marriage as compared to males working in other areas. Family size has a very important contribution in determining the age at marriage. It is because in small families more emphasis is placed on education and enhancing the quality of life and males tend to get married late relative to their large family counterparts. Also, the mean age at marriage of Hindu males is found higher than Muslim males. Type of village has very little effect on age at marriage with the exception of males from growth centres.

Table 2 Some statistical measures of age at marriage according to various background characteristics

Explanatory variable	Mean	S.D.
Education		
Primary	23.59	2.16
Secondary	24.12	2.31
Higher	25.26	2.47
Occupation		
Labour	24.05	2.48
Agriculture	24.09	2.26
Others	25.32	2.29
Family size		
Small	26.02	2.62
Mid	24.46	2.13
Large	23.57	2.28
Religion		
Hindu	24.56	2.37
Muslim	23.19	2.22
Type of village		
Semi-urban	24.30	2.11
Remote	24.08	2.68
Growth centres	24.85	2.29

3 Model

Assumptions (a) All the individuals of a cohort are in the marriageable state.

(b) Let X be the age at marriage. There are two groups of individuals with probability α and $1 - \alpha$, respectively, on the basis of their average age at marriage. The age at marriage in both the groups follows the logistic distribution; thus, the resulting distribution becomes a mixture model, i.e.

$$f(x) = \alpha f_1(x, \mu_1, s_1) + (1 - \alpha) f_2(x, \mu_2, s_2) \tag{1}$$

where μ_1 and s_1 are the location and scale parameter of the first group, while μ_2 and s_2 are location and scale parameter of the second group, respectively. f_1 and f_2 are the probability density function of first and second groups, respectively. It is worthwhile to mention here the probability density function of logistic distribution as follows

$$\frac{e^{-(x-\mu)/s}}{s\left(1 + e^{-(x-\mu)/s}\right)^2}, \quad s > 0, -\infty < X < \infty \tag{2}$$

4 Parameter Estimation

The expectation–maximisation (EM) algorithm has been used for estimating the parameters. It is an iterative method for approximating the maximum of a likelihood function. This method was first explained by Dempster et al. in [7]. There were some convergence issues in his work which was later corrected by Jeff Wu in 1983 [16]. The theory and application of EM algorithm were nicely discussed by [12] in his book. This algorithm is used when we have unobserved latent variables, like in mixture distributions. In such cases, we have a set of distributions and each observation comes from one of these distributions with some probability. But we do not know the distribution it came from. The EM algorithm bounces back and forth between two steps;

(i) Given the current parameters and the observed data, estimate the latent variables.

(ii) Given the observed data and the latent variables, estimate the parameters.

And this process goes on repeating until the convergence is established. With the application of EM algorithm, we estimated the following values of parameters (Fig. 1 and Table 3);

$$\mu_1 = 23.86,\ \ \mu_2 = 28.47,\ \ s_1 = 1.078,\ \ s_2 = 1.2279,\ \ \alpha = 0.90$$

$$\chi^2 = 2.80,\ \ \text{p}-\text{value} = 0.73$$

Determinants of Age at Marriage Through a Proportional Hazard Model: Cox proportional hazard model was used to investigate the study variable (age at marriage), in addition to the descriptive methods. The Cox model [6] has the advantage of both life tables and multivariate regression approach. The expression for the Cox model is

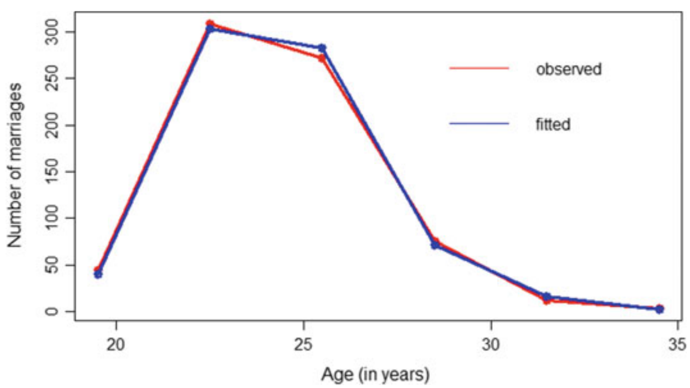

Fig. 1 Graphic representation of observed and fitted frequencies

Table 3 Distribution of observed and expected frequencies

Age at marriage	Observed number of males	Expected number of males
18–21	44	40
21–24	309	303
24–27	272	283
27–30	76	71
30–33	12	16
33–36	3	2
Total	716	716

$$h(t, z) = h_0(t) \cdot \mathrm{Exp}(\beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 + \cdots + \beta_i z_i)$$

In the above equation, time variable t denotes the age at marriage. The outcome variable h(t) denotes the hazard rate, i.e. the rate at which marriage takes place or the risk of getting married at time t. The term h_0 is the baseline hazard function that varies only with t. The terms β_1, β_2, β_3, …, β_i are the regression-like coefficients, showing the effect of covariates on the outcome variable.

For additional analysis, hazard regression model was used to estimate the effect of all the covariates on age at marriage. It is observed from Table 4 that males, who have attained only primary- or secondary-level education, are at higher risk of entering marital union early than the highly educated males. They have around 57 and 32% more chances of getting married early. The young man engaged in labour and agriculture work has a higher probability of starting their marital life early than the males involved in other professions. A very important finding can be observed from the family size. In small and mid-size families, males have the tendency of getting married late than their large family counterparts and their odds are highly significant. This result validates and recommends all-round effort in promoting the adoption of small family norms. Traditionally, Hindu males go for marriage late as relative to the Muslim males due to their different social and religious exposure. This fact has also been observed in Table 4. Surprisingly, the type of village has nothing to do with the age at marriage. The one reason behind these phenomena might be given that due to industrial, agriculture and infrastructural development, now, the importance of distance of villages from cities and other developed villages has become very less important.

Table 4 Risk of getting marriage in various categories through proportional hazard model

Model-I				Model-II		
Variable	Exp(β)	S.E.	p value	Exp(β)	S.E.	p value
Education						
Primary	–			1.570	0.115	0.000
Secondary	–			1.320	0.093	0.003
Higher	–			1.000		
Occupation						
Labour	1.339	0.108	0.007	1.288	0.108	0.019
Agriculture	1.388	0.106	0.002	1.270	0.108	0.027
Others	1.000			1.000		
Family size						
Small	0.547	0.123	0.000	0.551	0.124	0.000
Mid	0.840	0.087	0.045	0.829	0.087	0.032
Large	1.000			1.000		
Religion						
Hindu	0.717	0.114	0.004	0.772	0.118	0.029
Muslim	1.000			1.000		
Type of villages						
Semi-urban	1.099	0.105	0.370	1.130	0.108	0.260
Remote	1.003	0.106	0.976	1.024	0.107	0.830
Growth centres	1.000			1.000		
−2log likelihood	7673.034			7656.226		

5 Conclusion

From the above analysis and discussion, it can be concluded that the mixture of logistic distribution fully justifies the age at marriage pattern of the study region. It is evident from the above model that the population selected consists of two groups of males. One group prefers early marriage, while the second group has some other priorities before settling in life. It is apparent from the study that educated males tend to get married late as compared to the lower educated males. The same is the case with males engaged in labour and agriculture activities. Our findings are in agreement with that of some previous researchers [10]. Astonishingly, a new variable, family size, has been found to play a crucial role in determining the age at marriage. The reason behind this fact is that in the joint family system an individual has a very little choice in taking a decision on their personal matters. All the decisions regarding marriage and all other important events are taken by the head or elders of the household, and one has to abide by those decisions. Historically, religion has a very important role in marriage-related decisions [11]. Hazard model analysis (Table 4) also shows the importance of all these factors. Due to increasing transportation, communication and outreach of media to the distant and remote areas has diminished the role of distance from urban centres. Therefore, our policy makers should pay attention towards all these factors. First and foremost, promotion and awareness of education with full thrust may alone tackle the problem of early marriage, which has been the root cause of high population growth, unemployment, epidemics, health and nutrition, rising crimes and overloaded infrastructure.

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Copula Functions and Applications in Engineering

Pranesh Kumar

1 Introduction

A copula function provides an easy way to connect distribution functions of two or more random variables to their marginal distribution functions. Precisely, a copula is a multivariate distribution function expressed in terms of marginally uniform random variables on the unit interval. Thus, a copula separates the joint distribution into two factors, the marginal distributions of the individual variables and their mutual dependency. Copulas are also known as *dependence functions* or *uniform representations* [55]. In mathematical sense, copula word was first time used by Sklar [63]; however, this copula appeared earlier in papers of Hoeffding [26, 27] and Fréchet [13] also. Over the four decades, copulas have been applied in areas of statistics, insurance, finance, economics, survival analysis, image processing, and engineering [41–49]. In the following sections, we present the copula functions, their properties, relationships with dependence measures, simulations, and some examples of copula applications. To learn more in details about copulas, some valuable resources are monographs by Hutchinson and Lai [29], Joe [30] and Nelsen [56], and conference proceedings edited by Beněs and Štěpán [2], Cuadras et al. [4], Dall'Aglio [6], Dall'Aglio et al. [7], and Rüschendorf et al. [58].

2 Copula Functions and Properties

For simplicity, we will consider a bivariate copula function of two continuous random variables denoted by X and Y with respective (probability) distribution functions $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ and joint (probability) distribution func-

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tion $H(x, y) = P(X \leq x, Y \leq y)$. By convention, we will denote by $f(x)$ and $g(y)$, respectively, the (probability) density functions and by $f(x, y)$ the joint (probability) density function of X and Y. A two-dimensional copula is defined [55].

2.1 Copula Function Definition

For every (x, y) in $[-\infty, \infty]^2$, and the point in $\mathbf{I}^3(\mathbf{I} = [0, 1)]$ with coordinates $(F(x), G(y), H(x, y))$, a two-dimensional (bivariate) copula is a function $C : \mathbf{I}^3 \to \mathbf{I}$ such that

(C1) $C(0, x) = C(x, 0) = 0$ and $C(1, x) = C(x, 1) = x$ for $x \in \mathbf{I}$;
(C2) function $V_C([a, b] \times [c, d]) = C(b, d) - C(a, d) - C(b, c) + C(a, c) \leq 0$.

The function C is called the C—*volume* of the rectangle $[a, b] \times [c, d]$. A copula C is the restriction to $\mathbf{I}^2$ of a bivariate distribution function with uniform margins on $\mathbf{I}$. A copula is a probability measure on $\mathbf{I}^2$ via $V_C([0, u] \times [0, v]) = C(u, v)$. Thus, it may be noted that the function $\Pi(u, v) = uv$ which satisfies conditions (C1) and (C2) is a copula and is known as the product copula.

The following theorem by Sklar [63] is an important contribution to copulas applications in practice.

2.2 Sklar's Theorem

Let H be a two-dimensional distribution function with marginal distribution functions F and G. Then, for any random variables X and Y with marginal distribution functions $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ and joint distribution function $H(x, y) = P(X \leq x, Y \leq y)$ [10], there exists a copula C such that $H(x, y) = C(F(x), G(y))$.

Conversely, for any distribution functions F and G and any copula C, the function H defined above is a two-dimensional distribution function with marginal F and G. If F and G are continuous, C is unique.

We can thus express joint distribution H in terms of marginal F and G and copula C. From modeling perspective, Sklar's theorem allows to separate modeling of the marginal distributions from the dependence structure which is expressed in C [60, 63].

Given joint distribution function H with continuous marginals F and G, the corresponding copula is $C(u, v) = H\big(F^{(-1)}(u), G^{(-1)}(v)\big)$, where $F^{(-1)}(.)$ is the cadlag inverse function given by $F^{(-1)}(u) = sup\{x|F(x) \leq u\}$ (and similarly for $G^{(-1)}\big)$. Note that if X and Y are continuous random variables, then C is the joint distribution function for the random variables $U = F(X)$ and $V = G(Y)$.

2.3 Copula Density Function

If $H(x, y)$ and $C(F(x), G(y))$ are differentiable, then $H(x, y) = C(F(x), G(y))$ implies that the joint density function satisfies $c(F(x), G(y)) = f(x, y)/f(x)g(y)$, where $c(.)$ is the copula density function $c(u, v) = \partial^2 C(u, v)/\partial u \partial v$. Thus, copula density is the ratio of the joint density function to what it would have been under independence. That means, copula provides an adjustment required to convert the independent density function into the joint density function.

2.4 Fréchet–Hoeffding Bounds

The following Fréchet–Hoeffding bounds inequalities hold:

$$\max\{F(x) + G(y) - 1, 0\} \leq H(x, y) \leq \min\{F(x), G(y)\}, \tag{1}$$

$$W(u, v) = \max\{u + v - 1, 0\} \leq C(u, v) \leq \min\{u, v\} = M(u, v). \tag{2}$$

The functions W and M are known as the Fréchet–Hoeffding *lower* and *upper* bounds, respectively. Note that M and W are themselves copulas.

For the continuous random variables X and Y, copulas M, W and Π possess the following properties:

(1) Copula $C(u, v)$ is $M(u, v)$ if and only if each of X and Y is almost surely an increasing function of the other.
(2) Copula $C(u, v)$ is $W(u, v)$ if and only if each of X and Y is almost surely a decreasing function of the other.
(3) Copula $C(u, v)$ is $\Pi(u, v) = uv$ if and only if X and Y are independent. That means, X and Y are independent if and only if the copula of X and Y satisfies $C(u, v) = \Pi(u, v)$.

3 Copula Families

Multivariate distributions having any marginal distributions can be constructed using Sklar's theorem and copulas. Several examples of copulas are found in [11, 15–20, 29, 30, 56]. Some commonly used parametric copula families are described below.

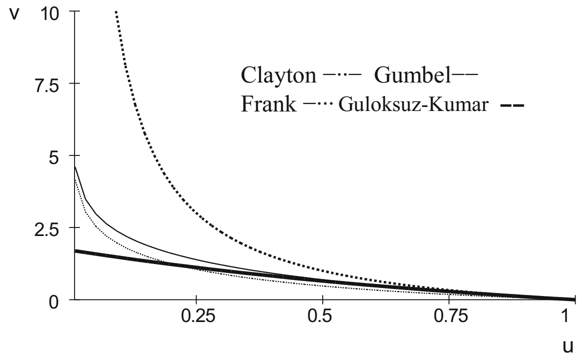

Fig. 1 Archimedean copula generator functions, $\varphi(t)$

3.1 Archimedean Copulas

Archimedean copulas can be generated using a function called generator function φ which is defined as follows (adapted from Nelson [56]):

Archimedean copula definition: Let Φ be the class of functions φ such that the function φ satisfies conditions (i) $\varphi(1) = 0$, (ii) $\varphi(0) = \infty$, (iii) $\varphi'(t) < 0$ and (iv) $\varphi''(t) > 0$, for $0 < t < 1$. Thus, the function φ is continuous, strictly decreasing and convex function and always has an inverse, φ^{-1}. The copula that can be expressed using generator function φ in form $C(x, y) = \varphi^{-1}[\varphi(F(x)) + \varphi(G(y))]$ is called the *Archimedean* copula.

Further, generator functions φ need not satisfy the condition (ii). The generator function when $\varphi(0) < \infty$ can also generate the *Archimedean* copula. The generator function is *non-strict* if $\varphi(0) < \infty$, and strict if $\varphi(0) = \infty$. For *non-strict* generators, the *pseudo*-inverse exists and is defined as:

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 < t < 1, \\ 0, & \varphi(0) < \infty. \end{cases} \tag{3}$$

If *Archimedean* copula is generated by a *non-strict* generator function, i.e., $\varphi^{[-1]}(t) = \varphi^{-1}(t)$, then it takes the form $C(u, v) = \max(C(u, v), 0)$.

Further advantage of using *Archimedean* copulas is that an *Archimedean* copula can be uniquely determined by a univariate function known as Kendall distribution function defined as: $K(t) = t - \frac{\varphi(t)}{\varphi'(t)}$.

Archimedean copulas have one or more parameters. However for simplicity, one-parameter (θ) Archimedean copulas families are popular in practice. Some examples of one-parameter Archimedean copulas are presented in Table 1 and shown in Fig. 1.

Table 1 Archimedean copulas, generator functions and Kendall's τ

Family	Copula $C(u, v)$	Parameter θ	Generator $\varphi(t)$	Kendall's τ
Clayton	$\left[u^{-\theta} + v^{-\theta} - 1\right]^{-1/\theta}$	$[-1, \infty)$	$t^{-\theta} - 1$	$\frac{\theta}{\theta+2}$
Gumbel	$\exp\left[-\left\{(-\ln u)^{\theta} + (-\ln v)^{\theta}\right\}\right]^{1/\theta}$	$[1, \infty)$	$[-\ln t]^{\theta}$	$\frac{\theta-1}{\theta}$
Frank	$\left(\frac{-1}{\theta}\right)\ln\left[\frac{(1-e^{-\theta})-(1-e^{-u\theta})(1-e^{-v\theta})}{(1-e^{-\theta})}\right]$	$[-\infty, \infty)$	$-\ln\frac{e^{-t\theta}-1}{e^{-\theta}-1}$	$1 - \frac{4}{\theta}[1 - D_1(\theta)]$
Guloksuz–Kumar	$1 - \frac{\ln\left(e^{\theta(1-u)}+e^{\theta(1-v)}-1\right)}{\theta}$	$(0, \infty)$	$e^{\theta(1-t)} - 1$	$1 + 4\left(\frac{1-\theta-e^{-\theta}}{\theta^2}\right)$

3.2 Gaussian Copulas

For a bivariate normal distribution of the random variables X and Y, denoted by $N_\rho(x, y)$, with correlation coefficient ρ, the Gaussian copula is defined as $C(u, v) = N_\rho\big(\Phi^{-1}(u), \Phi^{-1}(v)\big)$, where Φ denotes the standard normal distribution. The Gaussian copula cannot be expressed in a closed form. If the location or scale of the distribution is changed, the copula does not change. Gaussian copula can be calculated in order to construct bivariate distribution functions with the same dependence structure as the standard bivariate normal distribution function but with non-normal marginals.

3.3 Survival Copulas

When the random variable of interest X refers to the lifetime or survival time, the probability of survival above time x is given by the survival function $\bar{F}(x) = P(X > x) = 1 - F(x)$. In case of two random variables X and Y, joint survival function is $\bar{H}(x, y) = P(X > x, Y > y)$. The function which couples the joint survival function $\bar{H}(x, y)$ to its marginal survival functions $\bar{F}\big(x\big)$ and $\bar{G}\big(y\big)$ is called the survival copula $\hat{C}(\bar{F}\big(x\big), \bar{G}(y)\big) = \bar{H}(x, y)$. The survival copula $\hat{C}$, in terms of copula C, can be expressed $\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ [28].

4 Copula and Dependence Measures

There are several dependence measures available which are used to study association between random variables [50, 54, 57, 59, 61]. Copulas are used to measure nonparametric, distribution-free, or scale-invariant nature of the association between random variables. There exist relationships between copulas and scale-invariant measures of association like Kendall's τ and Spearman's ρ which are the measures of dependence known as concordance. The Kendall's τ using sample data can be calculated as [38]:

In a random sample of size n with observations $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ from a continuous bivariate distribution, there are $\binom{n}{2}$ distinct pairs of observations and each pair is either *concordant* or *discordant*. Kendall's τ = [(number of concordant pairs)–(number of discordant pairs)]/total number of pairs.

Kendall's τ, Spearman's ρ, and Gini's index γ, respectively, in terms of copula C are given

$$\tau = 4 \int\int_{\mathrm{I}^2} C(u, v) dC(u, v) - 1, \tag{4}$$

$$\rho = 12 \int\int_{\mathrm{I}^2} uv\,dC(u,v) - 3, \tag{5}$$

$$\gamma = 2 \int\int_{\mathrm{I}^2} (|u+v-1| - |u-v|)\,dC(u,v). \tag{6}$$

The Kendall's τ in terms of Archimedean copula generator function $\varphi(t)$ is: $\tau = 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt + 1$.

5 Tail Dependence Measures

Tail dependence is important to be considered in studying behavior of the lower or upper extreme values of the distribution [35]. For example, in risk management, it is useful to understand the tail distribution of losses. Often, large losses in a portfolio are caused by simultaneous large moves in several components. Tail dependence measure calculates the probability that one variable is extreme given that other is extreme and describes the amount of dependence in the upper right tail or lower left tail of the distribution. That means, it measures the proportion that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold. Joe [30] defines the tail dependence:

Tail dependence definition: If a bivariate copula $C(u, v)$ is such that $\lambda_U = \lim_{u\to 1} P[F(x) > u|G(y) < u] = \lim_{u\to 1}[\{1 - 2u + C(u, u)\}/(1 - u)]$ exists, then $C(u, v)$ has upper tail dependence for $\lambda_U \in (0, 1]$ and no upper tail dependence for $\lambda_U = 0$. Similarly, lower tail dependence in terms of copula is $\lambda_L = \lim_{u\to 0} \mathrm{P}[F(x) < u|G(y) < u] = \lim_{u\to 0}[C(u, u)/u]$. Copula has lower tail dependence for $\lambda_L \in (0, 1]$ and no lower tail dependence for $\lambda_L = 0$.

Thus, the tail dependence is symmetric. The Gaussian copula has no tail dependence that means its left and right tail measure values are zero.

6 Fitting Copula

To identify which copula may be the best fit, several copulas may be estimated and then most appropriate (as per criterion) is chosen. The estimation procedure consists of mainly two steps: (i) Estimate the marginal distributions, and (ii) specify the copula function. Marginals can be estimated by empirical or parametric methods.

6.1 Empirical Copula

Let $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ be a random sample from a bivariate distribution with marginal distribution functions $F(x)$ and $G(y)$. To fit the empirical copula, Genest and Rivest Embrechts et al. [9] suggested:

(1) Estimate the copula parameter θ.
(2) Obtain the empirical estimate of copula function, say $k_n(t)$ by defining pseudo-observations $T_i = \sum_{i=1}^{n} \frac{I\left(X_j \leq X_i \text{ and } Y_j \leq Y_i\right)}{n+1}, i = 1, \ldots, n$, and then calculate $K_n(t) = \sum_{i=1}^{n} \frac{I(T_i \leq t)}{n+1}, i = 1, \ldots, n$.
(3) Construct parametric estimate of $K_\varphi(t) = t - \frac{\varphi(t)}{\varphi'(t)}$.
(4) Compare the distance between $K_n(t)$ and $K_\varphi(t)$ by $MD = \int \left[K_n(t) - K_\varphi(t)\right]^2 dK_n(t)$ [14].

6.2 Simulating Data

There are several ways to simulate data (x, y) from a bivariate (X, Y) distribution having joint distribution $H(x, y)$ or copula function $C(u, v)$ [31]. One such method is the conditional distribution method [56]:

(1) Generate two independent random numbers between (0, 1), say u and t.
(2) Let $v = c_u^{(-1)}(t)$, where $c_u^{(-1)}$ is a quasi-inverse of the conditional distribution for V given $U = u$, i.e., $c_u(v) = \partial C(u, v)/\partial u$.
(3) The copula simulated data is (u, v), or the pair of values $\left(x = F^{-1}(u), y = G^{-1}(v)\right)$, in original variables (X, Y).

7 Copulas and Information Measures

Since the Shannon's [62] seminal work on the mathematical foundation of information theory in the context of communication theory, there have been much interest with regard to the applications of information-theoretic framework in physical, engineering, biological, and social sciences and more so in the fields of information technology, nonlinear systems, and molecular biology. Shannon's probabilistic measure of uncertainty, known as entropy, quantifies and analyzes uncertainty present in the probabilistic systems. Akaike [1] provided information theoretic criterion using the maximum likelihood principle.

For the continuous random variable X with density function $f(x)$, entropy is defined as $h(X) = -\int_X f(x) \log f(x) dx$. Entropy $h(X) = 0$ implies that variable X is deterministic (not random) and the value of $h(X)$ closer to zero indicates that the uncertainty in X is lesser. $h(X)$ is a monotonic increasing function of total number of random outcomes, n. The joint entropy $h(X, Y)$ of pair of variables X

and Y, having joint density function $f(x, y)$ and marginals as $f(x)$ and $f(y)$, is defined: $h(X, Y) = -\int_{X,Y} f(x, y) \log f(x, y) dxdy$. Assuming X and Y as input and output, respectively, of a stochastic system, entropy $h(X)$ represents the uncertainty of input X before output Y is observed while $h(X|Y)$ is the uncertainty of input X after output Y is realized. Conditional entropy $h(X|Y)$ is defined as $h(X|Y) = -\int_{X,Y} f(x, y) \log f(x|y) dxdy$, where $f(x|y)$ is the conditional density function of X given $Y = y$. The difference $I(X, Y) = h(X) + h(Y) - h(X, Y) = h(X, Y) - h(X|Y) - h(Y|X)$ is known as the mutual information and can be interpreted as the distance from independence between two variable X and Y [32]. Alternatively, mutual information in terms of the Kullback–Liebler divergence between joint distribution $F(x, y)$ and the two marginal distributions $F(x)$ and $G(y)$ is defined as [39]:

$$I(X, Y) = \int\limits_{X,Y} f(x, y) \log \frac{f(x, y)}{f(x) f(y)} dxdy. \tag{7}$$

$I(X, Y) \geq 0$ with equality iff X and Y are independent.

Different entropy measures in terms of copula joint density function $c(u, v)$ [37]:

$$\text{Joint entropy} : h(X, Y) = -\int\limits_{U,V} c(u, v) \log c(u, v) dudv. \tag{8}$$

$$\text{Conditional entropy} : h(X|Y) = -\int\limits_{U,V} c(u, v) \log c(u|v) dudv. \tag{9}$$

$$\text{Mutual information} : I(X, Y) = -\int\limits_{U,V} c(u, v) \log \frac{c(u, v)}{c(u)c(v)} dudv. \tag{10}$$

For the Marshall–Olkin copula [8, 52, 53], copula and entropy measures [45]:

$$\text{Copula} : C(u, v) = \min\left(u^{1-\theta} v, uv^{1-\theta}\right), \quad u, v, \theta \in (0, 1], \tag{11}$$

$$\text{Copula parameter} : \theta = 2\tau/(1 + \tau), \tag{12}$$

$$\text{Density function} : c(u, v) = \begin{cases} (1-\theta)u^{-\theta}, & \text{if } u > v, \\ (1-\theta)v^{-\theta}, & \text{if } u < v, \\ 0, & \text{if } u = v. \end{cases} \tag{13}$$

$$\text{Mutual information} : I(X, Y) = -2\frac{1-\theta}{2-\theta}\left[\log(1-\theta) + \frac{\theta}{2-\theta}\right]. \tag{14}$$

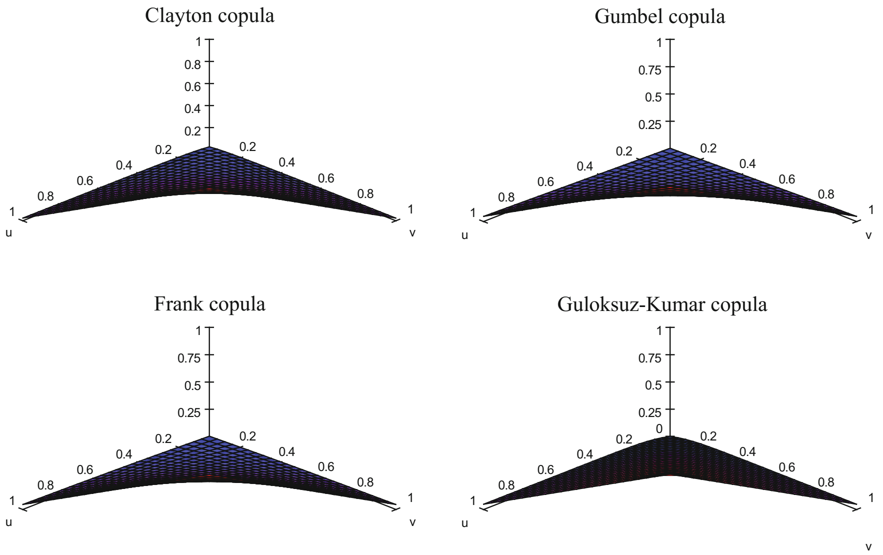

Fig. 2 Archimedean copulas

8 Copula Applications: Some Examples

8.1 Clinical Chemistry

Guloksuz and Kumar [21] developed a new Archimedean copula (refer to Table 1 and Figs. 1 and 2) and described its application in a clinical chemistry study which was conducted to analyze amylase in saliva for a particular purpose. They considered data set consisting of measurements of amylase levels that are taken from saliva at 12 a.m and 12 p.m on Thursday. The dependence structure of amylase levels at two time points is investigated. Kendall's tau is estimated as 0.64835. Copulas considered are Gumbel [22], Clayton [3], and Frank [12] copulas and the new bivariate G-K copula. First, copula parameters are estimated using Kendall's tau, and then empirical distributions of all copulas are compared to theoretic distributions by using the minimum distance measure. Their study indicated that the new bivariate Archimedean copula (G-K) gives the best-fitted model followed by Frank, Gumbel, and Clayton copulas.

8.2 Aerial Image Registration

Kashanchi and Kumar [33] have investigated information-theoretic and stochastic approach for the concept of image alignment to develop a more robust and general method for image registration. Image registration is the problem of finding the best possible geometrical transformation in order to align the moving or test image based on the fixed or reference image. They considered copula functions to find the best possible geometrical transformation for the tested images. Aerial images were used as the main test cases (input to the copula-based image registration approach). They narrowed down the image registration algorithm and monitored the performance of the metric function where all other parameters were equal. In the algorithm, copula-based divergence and mutual information-based image registration were assembled in which two copula functions, namely Gaussian and Frank copulas, were used. Their performance was compared using the peak signal-to-noise ratio (PSNR) with the joint histogram mutual information-based image registration which is one of the popular methods in area-based image registration. These algorithms were tested on seven 2-D gray-scale agricultural aerial images. In this comparison, Frank, Gaussian, and joint histogram performed very similar, but Gaussian copula was faster than other methods.

8.3 Titanium Welds

In commercial uses, pure titanium is strengthened by a mixture of oxygen, nitrogen, and carbon. It is noted that hydrogen which is typically less than 50 ppm is present as an impurity. In a study by Harwig, Ittiwattana, and Castner [5], they investigated Grade 2 titanium which is preferred in welding applications requiring good strength, ductility, and corrosion resistance. They evaluated the effects of higher cooling rates and iron and developed a preferred oxygen equivalent relationship that could be used to predict the properties of titanium welds. To develop oxygen equivalent relationship model between oxygen content and ultimate testing strength of titanium welds, Kumar [42] considered the Farlie–Gumbel–Morgenstern (FGM) Copula : $C(u, v) = uv[1 + \theta(1 - u)(1 - v)]$, and constructed confidence bands and prediction bands for testing weld strength.

8.4 Pricing Spread Options

Spread options are options on the price difference between two or more traded assets. Crack term is used in refining which refers to the cracking of crude oil into other products (e.g., heating oil or gasoline). A crack spread (also known as a refinery spread) is the simultaneous purchase or sale of crude against the purchase or sale of refined petroleum products. Crack spreads, introduced on the New York Mercantile Exchange (NYMEX), are used for managing risk in the oil industry. Commonly used models for pricing spread options are the bivariate binomial model and the approximate analytical formula known as the Kirk model [36]. Herath et al. [24] examined various Archimedean copulas for pricing crack spread options and determined which copula function provides the best model based on the goodness of fit criterion. They developed two copula-based Monte Carlo simulation algorithms and compared the copula models with settlement prices and the currently used standard models. Their results indicated that the Gumbel copula and standard models (binomial, and Kirk and Aron [36]) may mis-price a crack spread option and that the Clayton model is more appropriate. More copula applications in option price modeling are carried out in [23, 25].

8.5 Crude Oil Prices and Export Model

The open-source software WinBUGS is a very useful Windows-based computer program for the Bayesian analysis of complex statistical models using Markov Chain Monte Carlo numerical methods. Kumar and Kashanchi [49] proposed a novel method for performing the regression modeling using the WinBUGS and copulas. WinBUGS and copula-based methodology are illustrated by modeling the relationship of Iran's light and heavy crude oil prices with Iran's crude oil export [34]. These estimated models can be used to predict the future oil prices based on oil export or vice versa, future oil export/production for the given crude oil prices.

8.6 Benford's Law

Kumar [45] considered the probability distribution known as the Benford's law and the Marshall–Olkin copula [52] to calculate the information-theoretic uncertainty measures. Benford's law became basis for many algorithms to detect frauds, embezzlement, tax evasions, and computer bugs. He noted in many instances that the pages of logarithmic table starting with number 1 were dirtier and much worn out than other pages and thought that it was unlikely that users have special preference for logarithms starting with number 1. In all cases, the number 1 was found to be the first digit about 30 percent of the time, more often than any other number. He derived a probability distribution to explain this phenomenon. The leading digit $d \in [1, b-1], (d = 1, 2, \ldots 9,$ in base $b = 10)$ occurs with probability $p(d) = \log_{\mathrm{b}}\left(1 + \frac{1}{d}\right)$. For this probability distribution, entropy value is $H(d) = -\sum p(d)\log_{10}\left(1 + \frac{1}{d}\right) = 0.87$ dits/digit and the maximum entropy value is $H_{\max}(d) = -\log_{10} n = -\log_{10} 9 = 0.95$ dits/digit. It means that the uncertainty in the probability distribution of digits is less than the maximum possible uncertainty which is case of no preference (randomness). Thus, this reduction in uncertainty is because all digits in the table do not occur equally or in the same proportion.

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A Preservation Technology Model for Deteriorating Items with Advertisement-Dependent Demand and Trade Credit

Himanshu Rathore

1 Introduction

In present global world, handling of organization–customer relation seems quite typical which becomes more critical in case of deteriorating inventory. For high-level customer satisfaction, they have to manage all the parameters simultaneously like order amount, demand, security. To attract the customers, suppliers usually use many types of tools; for example, they showcase their items through advertisement which definitely affects the demand rate. The trade credit is also another tool to attract the customers.

In the literature, researchers have developed a single-period inventory model in [6], to maintain coordination between advertising and quantity decisions. In article [15], authors have studied a coordination system for pricing and advertising strategy in a manufacturer-retailer channel of supply chain system. In [14], researchers have explored a supply chain coordination model with two production model and random demand which is directly related to advertisement and selling price. The study [3] has described a coordinating policy for two-level supply chain model. The model is dealing with two types of policies: first is ordering policy, and the second one is advertising policy for a single-period commodity.

In the work [7], authors have explored an adjustment system for a supply chain dealing with the demand which is directly proportional to price and advertisement frequency. In article [10], authors have described an EOQ model for deteriorating item with deterioration rate which is non-instantaneous in nature. They have considered the demand rate is taken in the form of price and advertisement frequency. Partial backlogging is considered under inflation and finite time horizon. In article [1], researchers have settled a two-storage inventory study for deteriorating items

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inculcating time-, price-, and advertisement-related demand rate. The partially backlogged shortages are put into focus.

In article [16], authors have established an optimal dynamic advertising model for deteriorating items. In the study [8], researchers have developed an economic production inventory model with the concept of imperfection in which the demand rate is directly related to frequency of the advertisement. The work [4] has also focused on advertisement in a study of EOQ model with partial backlogging for an imperfect lot. In article [11], authors have generated a research work for deteriorating item with time- and advertisement-related demand. They have calculated the amount of economic order quantity in an inflationary environment.

In [2], authors have studied a fuzzy inventory model in which they have taken demand rate as direct function of advertisement and price. They have optimized the fuzzy model. In article [5], researchers have established an EOQ study for deteriorating item with deterioration rate which is non-instantaneous by nature- and price-dependent demand rate. To attract the customer, supplier offers a credit period with two-storage facilities. The study [9] has finite time horizon inventory model for two-level supply chain system. The items are taken deteriorating in nature with time-, price-, and production cost-dependent demand rate.

The preservation technology is another tool to improve the performance of the inventory system. In the literature work [12], authors have explored an optimal strategy for deteriorating items in which they have used the concept of preservation to reduce the losses due to deterioration. The complete work is carried under the effect of trade credit and inflation. In [13], researchers have explored a two-storage inventory model for the items which are deteriorating in nature with preservation technology investment and partially backlogged shortages. In similar manner, we can find the literature related to preservation technologies in different aspects.

In this scenario, we have put a theory to deal with an inventory system for deteriorating items with preservation technology investment. In this model, the behavior of the demand rate is linear with time, price, and advertising frequency. The supplier offers the credit period to enhance the demand rate in an inflationary environment. There are different sections in this paper; in the continued subheading, we have described the notations and assumptions those are applied in the creation of mathematical model.

2 Assumptions and Notations

These are the assumptions and notations which we have incorporated in mathematical model formulation.

2.1 Assumptions

- The rate of demand is taken as direct function of time, prices, and directly with frequency of advertisement. The demand rate is defined as $D(N, p, t) = N^a(\alpha - \beta p + \gamma t)$ where $a, \alpha, \beta, \gamma > 0$;
- The shortages are not permitted;
- Instantaneous replenishment and lead time is zero;
- The time horizon is taken of infinite length;
- The items are deteriorating in nature with constant rate of deterioration;
- The preservation technologies are considered to preserve the items;
- The inflation rate is constant;
- The supplier offers a trade credit period (M) for the purchaser. During credit period, the purchaser invests sales revenue in an interest-bearing account. After credit period, the purchaser has to pay a high interest charged by the supplier for unsold amount of the stock. So there are two choices for the purchaser; they can pay whole amount before commencement of the credit period or after the time period of credit period.

2.2 Notation

- C: The order cost per order at t = 0;
- p: The unit selling price at t = 0;
- Cp: The purchase cost per unit at t = 0;
- h: The holding cost per unit per time unit without any interest charges;
- N^{α}: The frequency of advertisement where $a > 0$;
- G: The advertisement cost per advertisement;
- θ: The deterioration rate, 0 < θ < 1;
- m(ξ): The preservation technology function with $m(\xi) = \theta\left(1 - e^{-\mu\xi}\right)$ where $\mu > 0$;
- ξ: The preservation technology cost per unit per time unit;
- τ_θ: The reduced deterioration rate, $\tau_\theta = (\theta - m(\xi))$;
- r: The inflation rate, where $0 \leq r < 1$;
- I_e: The interest earned per unit per year;
- I_p: The interest paid by purchaser for unsold stock per year;
- M: The credit period;
- T: The total cycle length;
- $I(t)$: The instantaneous inventory at any time t;
- Q: The maximum limit of inventory level during [0, T];
- $TC_1(T, N, \xi)$: The present worth of total cost per time unit, when T $\leq$ M where T, N, ξ are the decision variables;
- $TC_2(T, N, \xi)$: The present worth of total cost per time unit, when M < T where T, N, ξ are the decision variables;

- The total cost includes the cost functions which are mentioned below:
  - OC: The order cost;
  - PC: The purchase cost;
  - HC: The present worth of holding cost;
  - IP: The present worth of interest paid;
  - NG: The present worth of advertisement;
  - IE_1 and IE_2: The present worth of interest earned from sales revenue while credit period in both the cases when T $\leq$ M and M $<$ T, respectively;

Note: The notations with superscript * represents the optimal value of that particular parameters.

3 Mathematical Model Formulation

At the time $t = 0$ the inventory level is Q as the time precedes the stock level depletes continuously just because of total effect of deterioration and demand. The complete functioning of inventory is elaborated by following differential equations.

$$\frac{dI(t)}{dt} + \tau_\theta I(t) = -N^a(\alpha - \beta p + \gamma t); \quad 0 \leq t \leq T \tag{1}$$

By applying the limits of stock amount like $I(t = 0) = Q$ and $I(t = T) = 0$. Solving (1), we obtain the solution for stock level at time t which is given below.

$$I(t) = -N^a e^{-\tau_\theta t}\left[(\alpha - \beta p)t + \frac{t^2}{2}(\tau_\theta(\alpha - \beta p) + \gamma) + \left(\frac{\gamma \tau_\theta t^3}{3}\right)\right] + Q \tag{2}$$

where

$$Q = N^a e^{-\tau_\theta T}\left[(\alpha - \beta p)T + \frac{T^2}{2}(\tau_\theta(\alpha - \beta p) + \gamma) + \left(\frac{\gamma \tau_\theta T^3}{3}\right)\right] \tag{3}$$

The cost functions are as follows:

- The order cost $= C$
- The purchase cost $= QC_p$
- The advertisement cost $= NG$
- The holding cost in the effect of inflation $= hC_p \int_0^T I(t)\, e^{-rt} dt$

$$\begin{aligned} HC = hC_p\Bigg[&-N^a\left(\frac{T^2}{2} + \frac{T^3}{6}(\tau_\theta(\alpha - \beta p) + \gamma) + \tau_\theta \gamma \frac{T^4}{12}\right) \\ &+ (\tau_\theta + r)N^a\left(\left(\frac{T^3}{3} + \frac{T^4}{8}(\tau_\theta(\alpha - \beta p) + \gamma) + \tau_\theta \gamma \frac{T^5}{15}\right)\right) + Q\left(T - \frac{rT^2}{2}\right)\Bigg] \end{aligned} \tag{4}$$

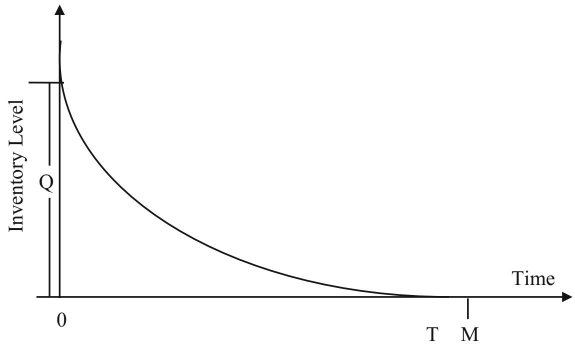

Fig. 1 (Case 1: T ≤ M) Inventory system with trade credit

Now we will calculate the total paid and earned interest by the purchaser. There are two cases according to time period of credit limit as described by: case 1: $T \leq M$ case 2: $M < T$.

Case 1: $\boldsymbol{T \leq M}$ In this case, as the credit time limit is more than the total cycle period, there is no paid interest for the purchaser. So the total interest earned is given by as follows. The diagrammatic representation of present case is given in Fig. 1

$$IE_1 = I_e p \left[\int_0^T e^{-rt} t N^a (\alpha - \beta p + \gamma t)\, dt + T \int_T^M e^{-rt} N^a (\alpha - \beta p + \gamma t)\, dt \right]$$

$$IE_1 = I_e p N^a \Bigg[(\alpha - \beta p) \frac{T^2}{2} + (\gamma - r(\alpha - \beta p)) \frac{T^3}{3} - \gamma r \frac{T^4}{4} + T \left\{ (\alpha - \beta p)(M - T) + (\gamma - r(\alpha - \beta p)) \left(\frac{M^2 - T^2}{2} \right) - \gamma r \left(\frac{M^3 - T^3}{3} \right) \right\} \Bigg] \tag{5}$$

Hence, here is the total cost function:

$$TC_1(T, N, \xi) = OC + PC + NG + HC - IE_1 \tag{6}$$

Case 2: $\boldsymbol{M} < T$ In second case, the credit time limit is less than the total cycle length. In such case, purchaser has to pay interest for the unsold amount which is charged by the supplier when credit time limit gets over. Hence, interest paid by purchaser is given as below. The pictorial representation of this case is given in Fig. 2.

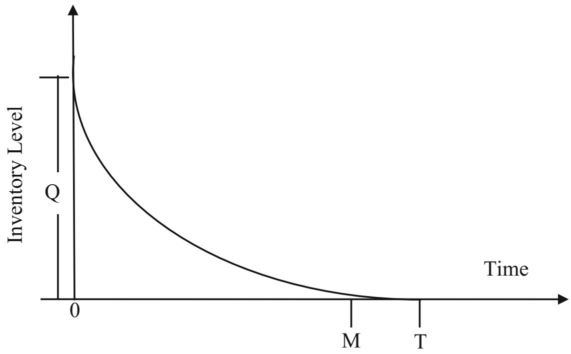

Fig. 2 (Case 2: M<T) Inventory system with trade credit

$$IP = I_p C_p \int_M^T I(t) e^{-rt}\, dt$$

$$IP = I_p C_p N^a \Bigg[-\left\{ \left(\frac{T^2 - M^2}{2}\right) + (\tau_\theta(\alpha - \beta p) + \gamma)\left(\frac{T^3 - M^3}{6}\right) + \tau_\theta \gamma \left(\frac{T^4 - M^4}{12}\right) \right\}$$
$$+ (\tau_\theta + r)\left\{ \left(\frac{T^3 - M^3}{3}\right) + (\tau_\theta(\alpha - \beta p) + \gamma)\left(\frac{T^4 - M^4}{8}\right) + \tau_\theta \gamma \left(\frac{T^5 - M^5}{15}\right) \right\} + Q\left(T - M - r\left(\frac{T^2 - M^2}{2}\right)\right)\Bigg] \tag{7}$$

In this case, the interest which is earned by the purchaser is as follows:

$$IE_2 = I_e \left[p \int_0^T e^{-rt} t N^a (\alpha - \beta p + \gamma t) dt + (Mp - C_p T) \int_M^T e^{-rt} N^a (\alpha - \beta p + \gamma t)\, dt \right]$$

$$IE_2 = I_e N^a \Bigg[p\left((\alpha - \beta p)\frac{T^2}{2} + (\gamma - r(\alpha - \beta p))\frac{T^3}{3} - \gamma r \frac{T^4}{4} \right) + (Mp - C_p T)$$
$$\left\{ (\alpha - \beta p)(T - M) + (\gamma - r(\alpha - \beta p))\left(\frac{T^2 - M^2}{2}\right) - \gamma r \left(\frac{T^3 - M^3}{3}\right) \right\}\Bigg] \tag{8}$$

Hence, the total cost function is as presented below:

$$TC_2(T, N, \xi) = OC + PC + NG + HC + IP - IE_2 \tag{9}$$

The core objective of the system in both the cases is to find the optimum value of the total cost function. For the optimization, the necessary and sufficient conditions for $i = 1, 2$ are as follows:

$$\frac{\partial TC_i(T, N, \xi)}{\partial T} = 0, \frac{\partial TC_i(T, N, \xi)}{\partial N} = 0, \frac{\partial TC_i(T, K, \xi)}{\partial \xi} = 0;$$

Provided DET(H1) > 0, DET (H2) > 0 where H1, H2 are the principal minor of the Hessian matrix. Here is the Hessian matrix of the total cost function for $i = 1, 2$:

$$\begin{bmatrix} \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial T^2} & \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial T \partial N} & \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial T \partial \xi} \\ \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial N \partial T} & \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial N^2} & \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial N \partial \xi} \\ \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial \xi \partial T} & \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial \xi \partial N} & \frac{\partial^2 TC_i(\mathrm{T}^*,N^*,\xi^*)}{\partial \xi^2} \end{bmatrix}$$

4 Numerical Illustration

We have verified our model by giving this numerical example. The values of different inventory parameters are described as below:

$C = 150\$$, $h = 0.15\$$, $C_p = 20\$$, $G = 50\$$, $p = 30\$$, $I_e = 0.06\$$, $I_p = 0.09\$$,
$\theta = 0.03$, $\alpha = 150$, $\beta = 3.5$, $\gamma = 0.2$, $a = 3$, $M = 1$, $\mu = 0.2$

The optimal values of decision variables are as follows:

$$T^* = 0.940245,\ \xi^* = 12.5797,\ N^* = 0.673576,\ TC^* = 69.7404$$

5 Sensitivity Analysis

We have done a sensitivity analysis to sensitize the study. We have explored the numerical calculations by varying values of some parameters to study the respective change in the optimal values of decision variables. The outputs are mentioned below in Table 1.

According to the study of data given in table, it is quite clear that this model is very sensitive as by changing the parameters the optimal values are also changed. So, we can say that this model is physically verified model. The convex curve of the total cost function is given as below in Figs. 3, 4, and 5.

6 Conclusion

In this chapter, we have explored the system in which we can easily calculate the optimal cost function for the inventory system dealing with deteriorating items under preservation technology effect. The demand rate is taken as a linear function of advertising frequency, price, and time. This study also presents the optimal values of advertising frequency and preservation cost. The present work is numerically verified, and an investigation of sensitivity is also executed. The whole calculative

Table 1 Sensitivity analysis

Parameter	Change in parameter	Preservation cost (ξ^*)	Advertising frequency (N^*)	Total cycle length (T^*)	Total cost (TC^*)
θ	0.02	8.24846	0.673576	0.940245	65.7404
	0.05	9.54785	0.673576	0.940245	66.5468
	0.07	16.3996	0.673576	0.940245	73.3696
α	140	11.76	0.756941	0.94601	72.6668
	145	12.1889	0.711316	0.94295	71.0693
	155	12.9378	0.641691	0.937818	59.822
β	3.0	13.5717	0.590426	0.933615	66.7842
	3.3	13.0058	0.635888	0.937361	68.4057
	3.6	12.3495	0.695406	0.941831	70.51
γ	0.1	12.4205	0.674125	0.940767	69.7853
	0.5	13.052	0.671891	0.938857	69.6043
	0.6	13.2076	0.671318	0.938453	69.5585
p	25	11.254	0.62582	0.936343	68.3464
	35	13.0645	0.791681	0.947856	73.6266
	46	13.0025	0.833175	0.950085	75.0267
μ	0.1	25.1595	0.673576	0.940245	69.7404
	0.3	8.3865	0.673576	0.940245	69.7404
	0.4	6.28987	0.673576	0.940245	69.7404
r	0.02	12.63	0.67167	0.939835	69.6551
	0.04	12.5292	0.675505	0.940655	69.8264
	0.05	12.4783	0.677457	0.941066	69.9131
M	1.2	13.9985	0.585241	1.09498	65.3327
	1.3	14.2808	0.554803	1.16898	63.4875
	1.4	14.2977	0.530838	1.24152	61.7923

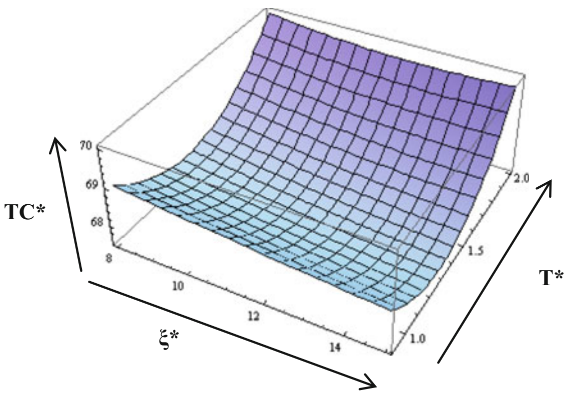

Fig. 3 Convex curve of total cost TC* in regard to preservation cost ξ* and total cycle length T*

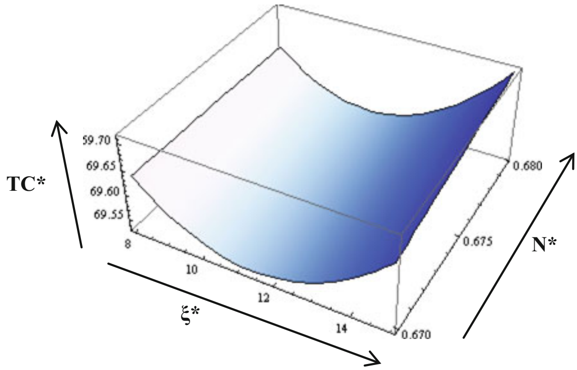

Fig. 4 Convex curve of total cost TC* in regard to preservation cost ξ* and advertising frequency N*

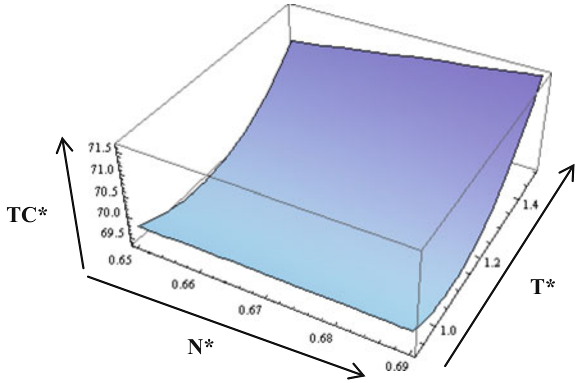

Fig. 5 Convexity of total cost TC* in regard to advertising frequency N* and total cycle length T*

part is performed by implementing the software MATHEMATICA11. The present work further can be modified by inculcating other constraints of the system dealing with stock of materials.

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Time Series Model for Stock Price Forecasting in India

A. Mohamed Ashik and K. Senthamarai Kannan

1 Introduction

Predicting the trends in stock market trading prices is an extremely challenging task due to the uncertainties factors involved that influence the market value on a day such as economic conditions, investor's sentiment, political events. A stock is a share in the pretension of a company. The stock represents a claim on the company assets and incomes. Most shares are traded on exchanges, which are locations where buyers and sellers will meet and make a decision on a price. The scope of a stock market is to make possible the trade of securities between buyers and sellers and reduce investment risks. The National Stock Exchange (NSE) is a large stock market traded in Republic of India, which is set up in November 1992. NSE is being the first exchange in the country to provide a modern and fully automated screen-based electronic trading structure which accessible easy and quick trading facilities to the investors over the country.

Similarly, the Nifty is an indicator of the several sectors such as automobiles, bank, real estate, media, pharmaceuticals, information technology, energy. If the Nifty 50 sectors go increasing, it means that the Nifty 50 stocks in India went up during the period and the Nifty 50 sectors go downstairs, that the Nifty 50 stocks on down. Stock market price is most concerned about the stock open, low, high, close, and volume. In nature, a trading day closing price is not only associated with the previous trading day closing price. In this paper, consider about only seven sectors of Nifty closing stock price (in Rs.) data was used, i.e., automobiles, bank, energy, information technology, media, metal, pharmaceuticals. The Nifty sectors data was

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initially examined and then Box–Jenkins method is used to well the data. The data time period is between April 2016 and March 2017, with 248 observations. Data is obtained from the part of nseindia.com, and the computations are done by using SPSS software package.

2 Review of Related Study

Box and Jenkins (1970) proposed the autoregressive integrated moving average model using stationary concept for the forecast purpose [1]. Naylor et al. (1972) observed that the accuracy of ARMA models of Box–Jenkins methodology was significantly better than the accuracy of the econometric model [2]. Lesseps and Morrell (1977) and their studies establish that the stock price follows a long-term trend with short-term fluctuation and forecast the exchange rate [3]. Pankratz (2008) studied and is helpful to the proper construction of univariate cases for Box–Jenkins forecasting method in various fields [4]. Suresh and Krishna Priya (2011) developed ARIMA model to forecast values for sugarcane area production for subsequent years [5]. Bagherifard et al. (2012) compare study to ANN and ARIMA models in predicting stock prices with higher accuracy [6]. Jawahar Farook and Senthamarai Kannan (2014) developed Box–Jenkins method for predicting the global atmospheric carbon dioxide emissions data [7]. Rotela Junior et al. (2014) described ARIMA model to obtain a short-term forecast for the next month in order to minimize prediction errors for the Bovespa Stock Index [8]. Renhao Jin et al. (2015) used ARIMA model for predicting the closing stock prices of Shanghai Composite Stock Price Index [9]. Mohamed Ashik and Senthamarai Kannan (2017) applied ARIMA model to forecast the upcoming daily closing stock price of Nifty 50 [10].

3 Methodology

3.1 Stationary Process

A stationary process is a stochastic process whose statistical properties (joint distribution) do not change with time. For a strict-sense stationary process, its joint probability distribution is constant. In a wide-sense stationary process, its first- and second-order moments are constant.

$$\left(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right) = \left(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_{n+h}}\right)'$$

R-Squared. R-squared is a statistical measure of how close the data is to the integrated regression line. It is referred to the coefficient of determination. In general, the higher (%) of R-squared is better the model fits in data.

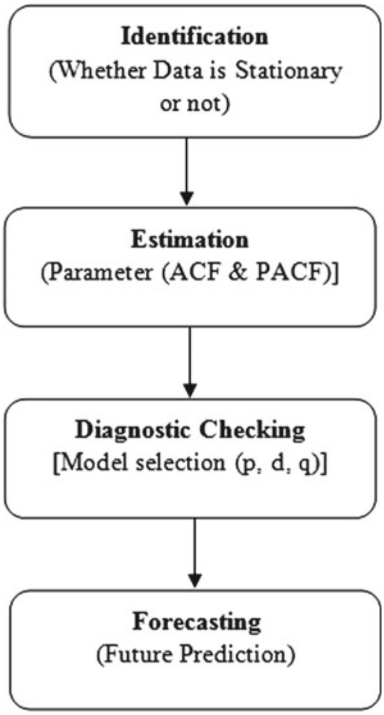
Error Rates. Error rates are used to measure of forecast precision of a forecasting technique in statistics. Forecast error is a measure of how accurate our forecast was in a given time period. The performances of different approaches have been evaluated on the basis of root mean square error (RMSE), mean absolute error (MAE), and mean absolute percentage error (MAPE) which are given [$E_t = Y_t - F_t$]; Y_t—actual value; F_t—forecast value.

$$RMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(Y_t - F_t)^2} \quad MAE = \frac{1}{n}\sum_{i=1}^{n}|E_t| \quad MAPE = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{Y_t - F_t}{Y_t}\right| \times 100$$

Box–Jenkins (ARIMA) Method. In time series study, an ARIMA model is a mixture of autoregressive and moving average with a difference. These models are fitted to time series data either to better identify with the data or to predict upcoming points in the series. It is also known as Box–Jenkins method. In case, the data is found to be non-stationary and is achieved by differencing technique and it is reduced. Generally, Box–Jenkins method are marked ARIMA (p, d, q) where parameters are p, d, q and refer to autoregressive (φ), difference (∇), and moving average (θ) also can be expressed as

$$y_t = c + \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \cdots + \varphi_p y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \quad (1)$$

φ—Autoregressive model; θ—Moving Average model; ∇(Difference)—$Y_t - Y_{t-1}$; c—constant.
Box–Jenkins methods for modeling and analyzing time series are characterized by four steps:



Identification. The identification stage finding the time series data is stationary or not and compare the estimated autocorrelation function (ACF) and partial autocorrelation function (PACF) to find a match.
Estimation. The parameters are estimated by modified least squares and maximum likelihood estimation techniques appropriate (ACF and PACF) to time series data.
Diagnostic Checking. The third stage of Box–Jenkins method (diagnostics check) is necessary to test the appropriateness of the selected model. Model selection can be made based on the minimum values of Bayesian information criteria (BIC) given by

$$BIC = -2\log(L) + k\ \log(n) \tag{2}$$

If the model selection is done, it is requisite to verify the satisfactoriness of the estimated model. The estimated residuals can be computed as $\hat{e} = Y_t - \hat{Y}_t$ where $\hat{Y}_t$ is the estimated observation at time t. The values of serial correlation can be studied to verify that the series of correlation residuals is white noise. After the tentative model has been fitted to the data, it is important to perform the diagnostic check to test the suitability of the model. It has been found that it is effective to measure the chosen model by examining as Ljung–Box statistic follows chi-square and computed as follows:

$$Q = n(n+2)\sum_{p=1}^{h}(n-k)^{-1}r_p^2 \tag{3}$$

The Ljung–Box statistic value is compared to critical values from chi-square distribution. While the diagnostic checking stage is fulfilled effectively and the model is found sufficient, the suitable model can be used for next level (forecast) purpose.
Forecasting. The final step of ARIMA model (forecast) is an essential application of time series analysis. It is the prediction values based on identified past values of that variable or other related variables. In analysis part, the suitable model is found satisfactory and the fitted model can be used for future prediction purpose.

4 Results

4.1 Descriptive Statistics

During the period of selected sectors in Nifty closing stock price, there are no outliers identified (Fig. 1). In Table 1, the summary statistics results for minimum, maximum, mean, standard deviation, skewness, and kurtosis are computed in the selected sectors closing stock price of Nifty 50.

Stationary Sequence. A time plot diagram is firstly used of NSE Nifty sectors data based on closing price. As shown in Fig. 2, a clear non-stationary fluctuations trend can be found, which is corresponding to the Indian Economics. This fluctuation trend

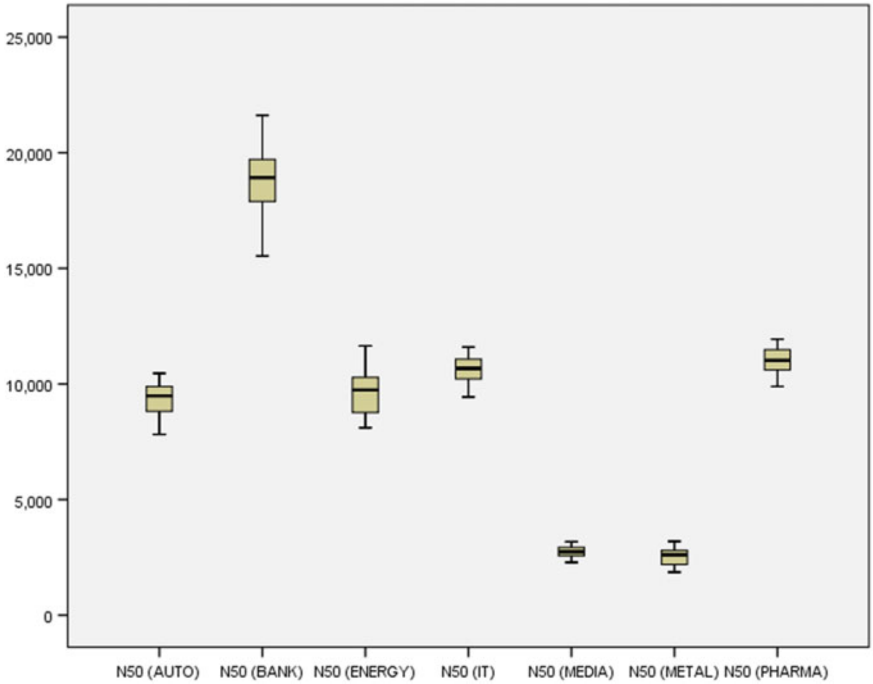


Fig. 1 Whisker–Box diagram of selected sectors of Nifty closing stock price

Table 1 Descriptive statistics of selected sectors of Nifty closing stock price

Sector	N	Min	Max	Mean	Std. Dev	Skew	Kurt
Auto	248	7821.3	10458.65	9321.98	672.95	−0.414	−0.946
Bank	248	15530.75	21620.7	18769.12	1367.67	−0.248	−0.492
Energy	248	8105.55	11648.7	9690.23	954.52	0.159	−0.865
IT	248	9434.6	11597.25	10617.58	517.19	−0.119	−0.837
Media	248	2281.65	3174.25	2743.5	220.44	−0.04	−0.99
Metal	248	1861.65	3189.4	2561.06	385.38	−0.182	−1.038
Pharma	248	9889.3	11936.65	11020.46	486.4	−0.026	−1.177

breaks the hypotheses of weaker stationary. In many application cases, the weaker stationary is used instead of strongly stationary.

In Box–Jenkins method, a first-order differencing is computed for the data after that time plot of the differencing data is shown in Fig. 3. The differencing data shows a stationary model and thus the value of d (I) was 1. In Fig. 4, the ACFs (MA) and PACFs (AR) are done based on the differentiating data, which display a short-term serial correlation and confirm the difference data is stationary. To construct a precise inference of the data, serial correlation is a check for white noise.

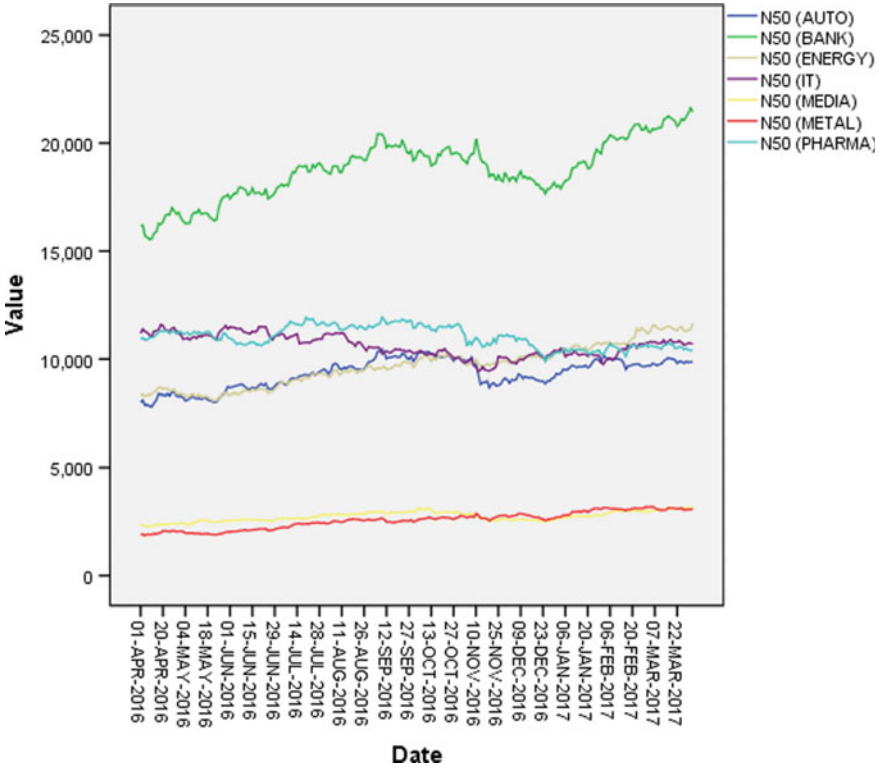


Fig. 2 Time plot of selected sectors of Nifty closing stock price

ARIMA Modeling. Modern statistical methods have been able to aid companies to predict the future in the certain process. According to the identification rules on time series, the corresponding model can be established. If PACF and ACF of the stationarity sequence are truncated and tailed, it can be concluded the sequences for AR(p) model. If PACF and ACF of the stationarity sequence are tailed and truncated, it can be strong that the MA(q) model can be fitted for the sequence. If PACF of a stationarity sequence and ACF is a tail, ARMA (p, q) model is appropriate for the sequence. Based on the results from an ARIMA model can be fitted to the selected sectors of Nifty 50, also the parameters in ARIMA (p, 1, q) need to be determined. From Fig. 5, the autocorrelation and partial autocorrelation are safe to the concluded.

In addition, using SPSS package for different values of p and q (0, 1 and 2), various ARIMA models were monitored from Eq. 2. The appropriate model was chosen to correspond minimum value of BIC. In this approach, ARIMA (0, 1, 1) model was established to be the most suitable model for all sectors (Table 2). Then the model authentication is described with checking of residuals of the chosen ARIMA model. This can be done through examine autocorrelations and partial autocorrelations of

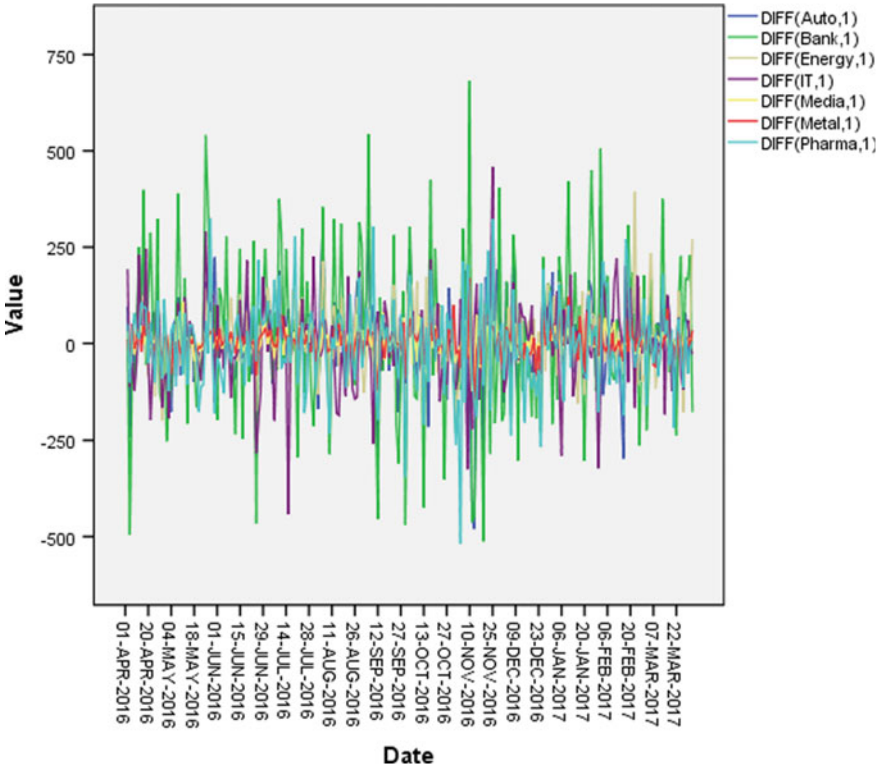


Fig. 3 Time plot of the first order differencing on selected sectors of Nifty closing stock price

the residuals of diverse orders. In Fig. 5, the residuals of ACF and PACF of ARIMA (0, 1, 1) model are suitable for the original data.

From Table 2, the energy sector is highest R-squared values and lowest MAPE. Therefore in (Eq. 1), the fitted ARIMA (0, 1, 1) model for energy sector closing stock price is

$$\hat{Y}_t = C + Y_{t-1} - \theta_1 \varepsilon_{t-1} + \varepsilon_t$$
$$\hat{Y}_t = 13.325 + Y_{t-1} - 0.037\varepsilon_{t-1} + \varepsilon_t$$

As Table 3, model parameters of energy sector concluded with difference (1), moving average (lag 1). These measures show that the future prediction factual error is low and the forecasting graph of Nifty energy sector closing stock price is given in Fig. 6. The volatility of Nifty sectors stock prices are can be caused by various factors, such as India financial, RBI policy, international events, and policies.

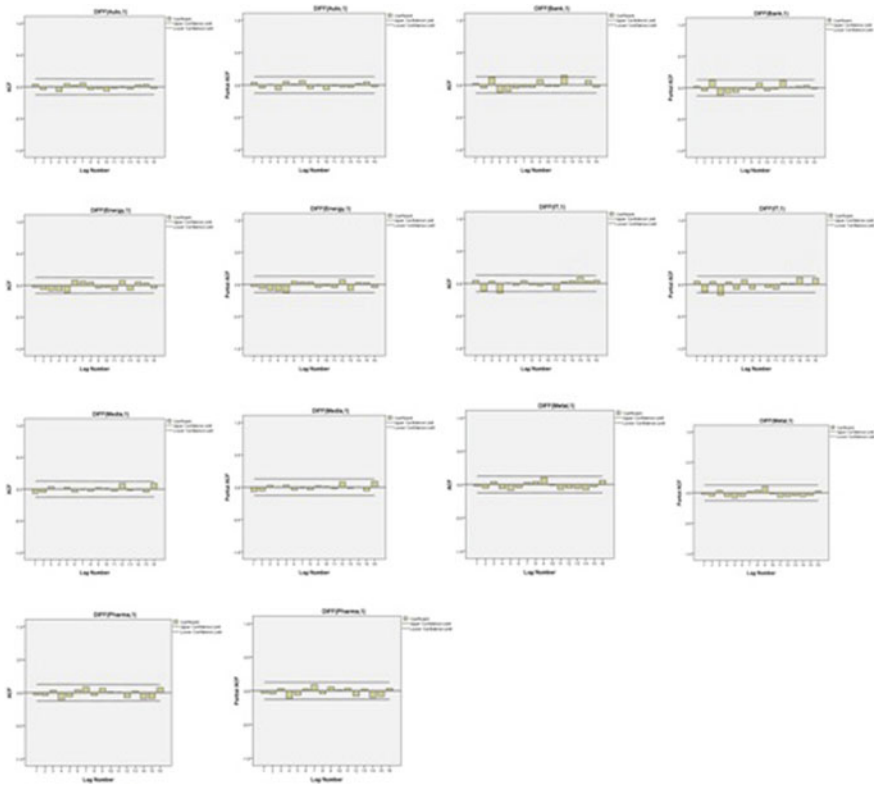


Fig. 4 ACF and PACF of the first order differencing on selected sectors of Nifty closing stock price

5 Discussion

In this study, Box–Jenkins method offers an excellent technique for forecasting the importance of any variables. In the function of model construction, the originally mentioned sector's data of Nifty is found to be non-stationary, but the first-order differentiating of all sectors data is stationarity. Then the monitoring of BIC values for various tentative ARIMA models, we obtained the lowest BIC values for the fitted model of selected sectors. After that, ARIMA (0, 1, 1) model is developed for analyzing and forecasting of selected sectors closing stock prices. From the empirical results, it can be observed that the energy sector of influence R-squared value is (99%) high and MAPE is (0.745) very small for other sectors of Nifty closing stock price. Thus, the future prediction accuracy is more fitting for the energy sector. Hence, the energy sector was a minimum risk of investors and increasing fluctuations trend for upcoming trading days.

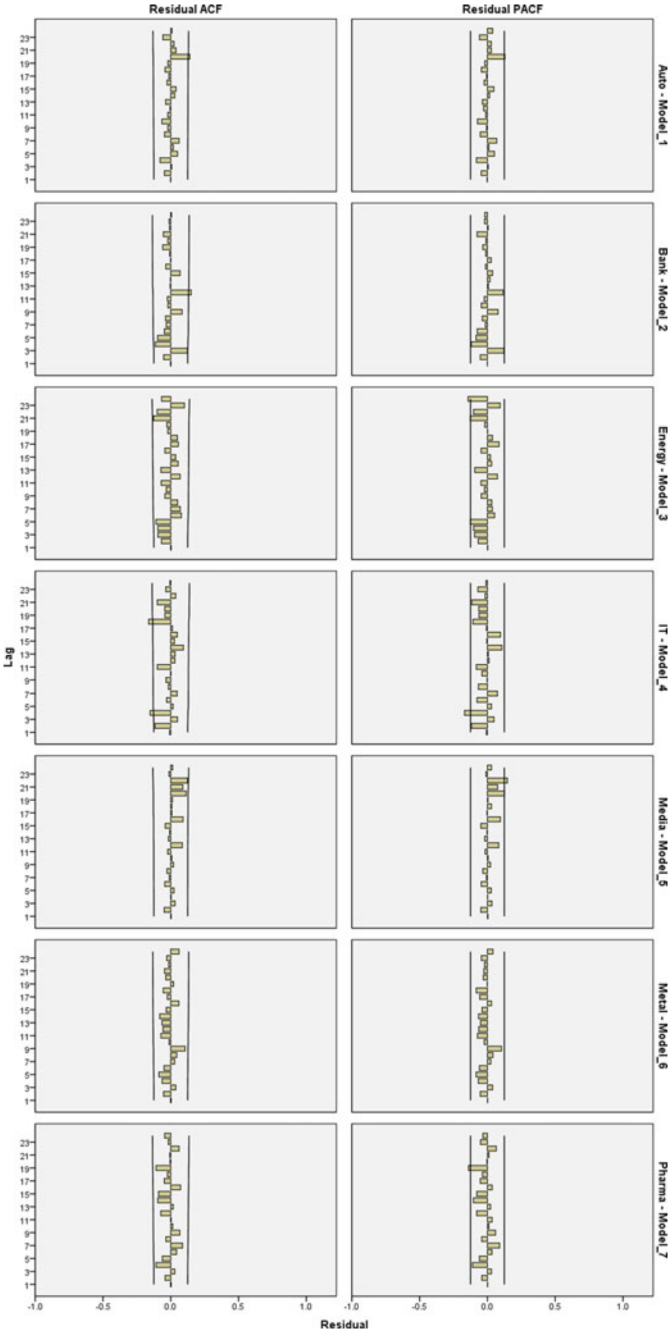


Fig. 5 Residual ACF and PACF diagram of selected Nifty sectors

Table 2 Model statistics of selected sectors of Nifty closing stock price

Sector	Model	R^2	RMSE	MAE	MAPE	maxAPE	Ljung–Box Q (18)		
	(ARIMA)						Statistics	DF	Sig
Auto	(0, 1, 1)	0.971	114.61	85.83	0.924	5.29	7.481	17	0.976
Bank	(0, 1, 1)	0.979	195.86	144.88	0.776	3.29	20.901	17	0.231
Energy	(0, 1, 1)	**0.991**	81.91	72.58	**0.745**	3.26	19.997	17	0.274
IT	(0, 1, 1)	0.955	109.56	79.34	0.75	4.49	24.533	17	0.106
Media	(0, 1, 1)	0.973	86.01	75.88	0.946	4.99	6.656	17	0.988
Metal	(0, 1, 1)	0.989	89.55	79.21	1.14	5.7	15.637	17	0.55
Pharma	(0, 1, 1)	0.946	113.38	85.49	0.778	4.92	16.773	17	0.47

Table 3 Model parameters of Nifty energy sector closing stock price

	Estimate	SE	T	Sig
Constant	13.325	5.635	2.365	0.019
CL Price No transformation difference	1			
MA Lag 1	0.037	0.065	−0.262	0.572

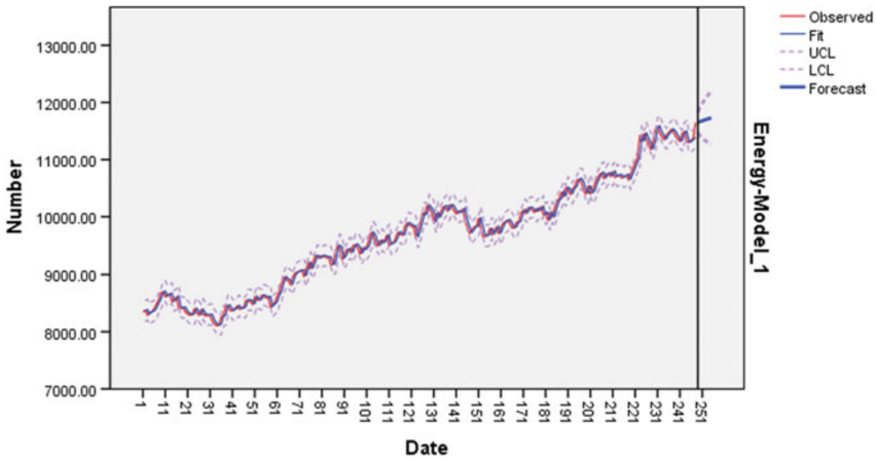


Fig. 6 Forecast graph on the energy sector of Nifty data

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Asset Pricing Through Capital Market Curve

Dipankar Mondal and N. Selvaraju

1 Introduction

Based on Markowitz's [12] mean–variance portfolio selection model, Sharpe [17] introduced capital asset pricing model (CAPM) to explain the equilibrium price of risky assets. Using different techniques, Lintner [11] and Mossin [14] also derived the same asset pricing model. These pioneering works of Sharpe, Lintner, and Mossin are widely accepted as a practical tool for asset pricing. Apart from all its glory, the model is highly debated because of its measurement of systematic risk, the beta, which mainly depends on variance and treats the downside and upside deviations of returns equally. Such symmetrical treatment on both sides of mean makes this model unrealistic and undesirable to the investors. To overcome these deficiencies, downside beta was introduced as an alternative measure of systematic risk. The sole purpose of downside beta is to identify the downside deviation of returns. The concept of downside risk has emerged mainly as a replacement for traditional risk, measured through variance. It was first introduced by Markowitz [13] in terms of semivariance. Rather than measuring both sides deviations of returns, it only measures dispersion below the expected return of a random variable. Thus, it is called one-sided variance or semivariance. Since a rational investor worries about only underperformance rather than overperformance, semivariance is realistic and an appropriate measure of risk. In fact, Markowitz [13] states that semivariance tend to produce better portfolios than those based on variance. Some good articles on downside portfolio selection problems are [3, 6, 7, 10, 16].

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Following on the steps of Sharpe [17], Hogan and Warren [9] developed another capital market equilibrium model based on semivariance. Traditional CAPM is a special version of this model. Nantell and Price [15] showed that the mean–semivariance CAPM reduces to mean–variance CAPM if return follows multivariate normal distribution [15]. Later, Bawa [1] introduced lower partial moment (LPM) as a measure of downside risk. Using the LPM, Bawa and Linderberg [2] also derived downside asset pricing model when the target is the risk-free rate of return. In 1989, Harlow and Rao [8] developed a generalized version of mean-downside equilibrium model for arbitrary targets. All the previous downside asset pricing models are special cases within this generalized framework and that includes the classical mean–variance CAPM.

All of the aforementioned downside asset pricing models have been derived using the techniques described by Sharpe [17]. Moreover, their methods depend on certain distributional assumptions on returns. In this paper, we present an alternative technique to derive the downside and classical equilibrium models. We mainly use the geometry of capital market curve. Our method does not have any distribution restrictions and hence works for all cases.

The paper is organized as follows. In Sect. 2, we describe the theoretical background of lower partial moment as a downside risk measure. In Sect. 3, we define risk and return of a portfolio and formulate a mean-LPM portfolio optimization model. We talk about some well-known results and their deficiencies. Then, by including one risk-free asset to the opportunity set, we reformulate the portfolio selection model and present the concept of an investment curve. We prove that regardless of the target, the curves are always convex in the mean-LPM space. In Sect. 4, we derive the downside asset pricing model with the help of the optimization model and the geometry of investment curves. We conclude this paper by summarizing, in the final section, the important results along with future directions of research.

2 Lower Partial Moments

Lower partial moments of order α with target τ is defined as

$$\mathrm{LPM}_{\alpha}(\tau; r) = \int_{-\infty}^{\tau} (\tau - r)^{\alpha} dP(\omega) = E\Big[(\tau - r)^{\alpha}\mathbb{1}_{\tau > r}\Big].$$

Because of the threshold or target return, investors are now free to choose their own level of risk. LPM is similar to semivariance if the target is the expected return and $\alpha = 2$. Similarly, for $\alpha = 1$, it is the same as the lower semideviation. Higher values of α imply higher-order partial moments of return. Now, for various values of α and τ, one can generate many risk measures according to one's choice. The parameter α reflects the risk-taking attitude of the investors. Fishburn [5] shows that investors are risk-neutral for $\alpha = 1$, but $\alpha > 1$ and $\alpha < 1$ indicates risk-averse and risk-seeker attitudes, respectively. Thus, the higher the values of α, the more the investors are risk-averse.

3 Portfolio Selection Model with LPM_α

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and r_i, for $i = 1, 2, \ldots, n$, be random return of asset i defined on the space, with $E(r_i) = \mu_i$. Let $x_1, x_2, \ldots, x_n$ be proportion of initial wealth invested in these n assets. Random and expected return of the portfolio can be expressed as $r_p = \sum_{i=1}^{n} x_i r_i$ and $\mu_p := E(r_p) = \sum_{i=1}^{n} x_i \mu_i$, respectively. Now, the risk of the portfolio can be defined in terms of the lower partial moment as $\theta_p := \mathrm{LPM}_\alpha(\tau; r_p) = E\Big[(\tau - r_p)^\alpha \mathbb{1}_{\tau > r_p}\Big]$.

Thus, the investor's optimization problem in the mean-LPM space is to find the investment weights in order to minimize the risk subject to fixed level of expected return μ^* and is given as

$$\begin{cases} \underset{x}{\text{Minimize}} & \mathrm{LPM}_\alpha(\tau; r_p) = E\Big[(\tau - r_p)^\alpha \mathbb{1}_{\tau > r_p}\Big] \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} x_i \mu_i = \mu^* \\ & x \in X \end{cases}$$

Here X, the opportunity set or feasible set of portfolios, is defined as $X = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1, x_i \geq 0\}$ without short sales or $X = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1\}$ with short sales. Bawa [1] has proved that LPM_α is a convex function of $\boldsymbol{x}$. Since LPM is convex, the above optimization problem becomes a convex optimization problem and hence the minimum LPM frontier will be convex in the mean-LPM space. All these derivations are based on distributional assumptions on returns. In this note, we exhibit an alternative proof that does not depend on the distribution of returns.

Let us form a portfolio with one risk-free asset with return r_f and the n risky assets. Hence, the investor's portfolio now contains $n + 1$ assets, i.e., $r_p = \sum_{i=1}^{n} x_i r_i + x_f r_f$. In order to find the optimal weights for efficient portfolios, the following optimization problem can be formulated

$$\begin{cases} \underset{x, x_f}{\text{Minimize}} & \mathrm{LPM}_\alpha(\tau; r_p) \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} x_i \mu_i + x_f r_f = \mu^* \\ & x \in X_f \end{cases}$$

where x_f is weight for risk-free asset and the new opportunity set $X_f = \{(x_1, x_2, \ldots, x_n, x_f) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i + x_f = 1, x_i \geq 0\}$ when short sales are restricted or $X_f = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i + x_f = 1\}$ when short sales are allowed.

An investment curve or line is a convex combination of the risk-free asset and a risky portfolio in mean-risk space. Sharpe [17] described that every investment curves is linear in mean–standard deviation space and the capital market line is the efficient investment frontier. Brogan and Stidham [4] have showed that the curves are also linear in mean-LPM$_\alpha^{1/\alpha}$ space when the target is risk-free rate or the expected return of the portfolio whose risk is being measured. But they are not always linear in the mean-LPM plane. In our context, the investment curves are not only nonlinear but also they exhibit some special characteristics. In the following theorem, we show that the curves are always convex in mean-LPM$_\alpha$ plane regardless of any target.

Theorem 1 *All investment curves are convex in mean-LPM$_\alpha$ plane.*

Proof Let $r_p = x_f r_f + (1 - x_f) r_q$ be the random return of the convex combination of a risk-free asset r_f and a risky portfolio r_q. Hence the expected return is $\mu_p = x_f r_f + (1 - x_f)\mu_q$. Since $\tau - r_p = \tau - x_f r_f - (1 - x_f) r_q = \tau + x_f (r_q - r_f) - r_q$, LPM can be written as $LPM_\alpha(\tau; r_p) = E\big[(\tau - r_q + x_f (r_q - r_f))^\alpha \mathbb{1}_{\tau > r_q - x_f (r_q - r_f)}\big]$.

In order to show the convexity of investment curve, we need to prove that $\frac{\partial^2 \mathrm{LPM}_\alpha(\tau; r_p)}{\partial \mu_p^2} \geq 0$. Since

$$\frac{\partial \mathrm{LPM}_\alpha(\tau; r_p)}{\partial \mu_p} = \frac{\partial \mathrm{LPM}_\alpha(\tau; r_p)}{\partial x_f} \frac{\partial x_f}{\partial \mu_p}$$

and

$$\begin{aligned} \frac{\partial^2 \mathrm{LPM}_\alpha(\tau; r_p)}{\partial \mu_p^2} &= \left[\frac{\partial \mathrm{LPM}_\alpha(\tau; r_p)}{\partial \mu_p \partial x_f} \frac{\partial x_f}{\partial \mu_p} + \frac{\partial \mathrm{LPM}_\alpha(\tau; r_p)}{\partial \mu_p \partial x_f} \frac{\partial^2 x_f}{\partial \mu_p^2}\right] \\ &= \frac{\partial^2 \mathrm{LPM}_\alpha(\tau; r_p)}{\partial x_f^2} \left(\frac{\partial x_f}{\partial \mu_p}\right)^2 \qquad [\because \frac{\partial^2 x_f}{\partial \mu_p^2} = 0], \end{aligned}$$

we only have to show that

$$\frac{\partial^2 \mathrm{LPM}_\alpha(\tau; r_p)}{\partial x_f^2} \geq 0.$$

Using differentiation under integration sign, we get

$$\begin{aligned} \frac{\partial^2 \mathrm{LPM}_\alpha(\tau; r_p)}{\partial x_f^2} &= E\left[\frac{\partial^2}{\partial x_f^2}\{(\tau - r_p)^\alpha \mathbb{1}_{\tau > r_p}\}\right] \\ &= \alpha(\alpha - 1) E\left[(\tau - r_p)^{\alpha - 2} \mathbb{1}_{\tau > r_p} (r_q - r_f)^2\right]. \end{aligned}$$

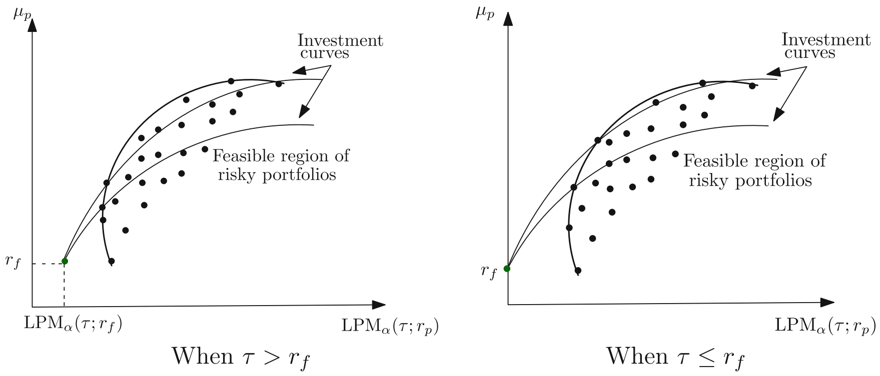


Fig. 1 Investment curves

As the expression inside the bracket is nonnegative, the expectation is also nonnegative. Hence, for all $\alpha \geq 1$, $\frac{\partial^2 \mathrm{LPM}_\alpha(\tau; r_p)}{\partial x_f^2} \geq 0$ or the investment curves are convex. □

Every investment curve in mean-LPM plane originates from a fixed point either $\left(\mathrm{LPM}_\alpha(\tau; r_f), r_f\right)$ (when $\tau > r_f$) or $\left(0, r_f\right)$ (when $\tau \leq r_f$) and passes through different risky portfolios. The risk-free asset is not always actually risk-free under this framework, particularly when $r_f < \tau$(see Fig. 1).

4 Equilibrium Model with Lower Partial Moment

Using the preceding results and exploiting the geometry of investment curves, we derive an equilibrium pricing relationship when target is either the risk-free rate or the return of the portfolio whose risk is being measured. Standard assumptions of Sharpe–Lintner–Mossin's asset pricing model are also considered.

Let $r_p = x_f r_f + (1 - x_f) r_q$ be the random return of a portfolio containing a risk-free asset and risky portfolio. The expected return of the portfolio can be written as

$$\mu_p = x_f r_f + (1 - x_f)\mu_q. \tag{1}$$

Similarly, the risk is given by

$$
\begin{aligned}
\mathrm{LPM}_\alpha(\tau; r_p) &= E\Big[(\tau - r_p)^\alpha \mathbb{1}_{\tau > r_p}\Big] \\
&= E\Bigg[\Big\{\tau - x_f(r_f - r_q) - r_q\Big\}^\alpha \mathbb{1}_{\tau > x_f(r_f - r_q) + r_q}\Bigg] \\
&= E\Bigg[\Big\{\tau - \frac{\mu_p - \mu_q}{r_f - \mu_q}(r_f - r_q) - r_q\Big\}^\alpha \mathbb{1}_{\tau > \frac{\mu_p - \mu_q}{r_f - \mu_q}(r_f - r_q) + r_q}\Bigg] \\
&= g(\mu_p)
\end{aligned}
$$

Therefore, LPM_α is a nonlinear function of μ_p. We have shown (in Theorem 1) that all investment curves are convex in mean-LPM space regardless of the target. Moreover, for $\tau = \mu_p$ and $\tau = r_f$, we can get explicit equation of investment curve. For $\tau = \mu_p$,

$$
\mathrm{LPM}_\alpha(\mu_p; r_p) = (1 - x_f)^\alpha \mathrm{LPM}_\alpha(\mu_q; r_q). \tag{2}
$$

Using (1) and (2), we get

$$
(\mu_p - r_f)^\alpha = \frac{(\mu_q - r_f)^\alpha}{\mathrm{LPM}_\alpha(\mu_q; r_q)} \mathrm{LPM}_\alpha(\mu_p; r_p). \tag{3}
$$

Investment curve originates from $(0, r_f)$ and passes through the point (LPM_α $(\mu_q; r_q)$, μ_q) in $\mathrm{LPM}_\alpha(\mu_p; r_p)$-$\mu_p$ plane. The capital market curve (CMC) or the efficient frontier curve is the optimal curve in which any portfolio has maximum return in its risk class or minimum risk with the same level of return. Harlow and Rao [8] have described that the CMC is tangent to the efficient frontier of risky portfolios. The tangency point is called the market portfolio. This is the unique risky portfolio located on the CMC and the efficient frontier of risky feasible region. Every investor wants to reach the CMC by maximizing expected return and minimizing risk. Thus, in equilibrium, every optimal portfolio must lie on the CMC.

We define

$$
Q = \frac{(\mu_p - r_f)^\alpha}{\mathrm{LPM}_\alpha(\mu_p; r_p)}.
$$

Q is a function of μ_p and $\theta_p (= \mathrm{LPM}_\alpha(\mu_p; r_p))$ such that $\frac{\partial Q}{\partial \mu_p} > 0$ and $\frac{\partial Q}{\partial \theta_p} < 0$. Now, investors would like to choose his optimal portfolio by maximizing Q subject to their budget level because at the maximum values of Q all the optimal portfolios lie on the capital market curve or efficient frontier. Hence, the following optimization model can be set up.

$$
\begin{cases}
\underset{(x_1, x_2, \ldots, x_n, x_f)}{\text{Maximize}} & Q \\
\text{subject to} & \displaystyle\sum_{i=1}^{n} x_i + x_f = 1.
\end{cases}
$$

Substituting $x_f = 1 - \sum_{i=1}^{n} x_i$ in Q, we can make the model unrestricted as

$$\left\{ \underset{(x_1,x_2,\ldots,x_n)}{\text{Maximize}} \quad Q(\mu_p, \theta_p) \right. \tag{4}$$

Hence, μ_p and θ_p are functions of $(x_1, x_2, \ldots, x_n)$. Now we derive the capital asset pricing model in mean-LPM plane using the model defined in Eq. (4). Following proposition establishes the LPM version of CAPM.

Proposition 1 *The optimal solution of the optimization problem defined in Eq. (4) satisfies:*

$$\mu_i - r_f = [\mu_m - r_f]\beta_i^-$$

Proof Let us consider n risky assets with random return $r_1, r_2, \ldots, r_n$ and a risky-free asset with return r_f be available in a market. Since the market portfolio contains every marketable asset in a certain proportion x_{im} (assume), the random return of the market can be expressed as $r_m = \sum_{i=1}^{n} x_{im} r_i$ such that $\sum_{i=1}^{n} x_{im} = 1$.

Now in order to find the maximum, we need to find the derivative with respect to x_i for all $i = 1, 2, \ldots, n$.

$$\frac{\partial Q}{\partial x_i} = 0 \text{ for all } x_i \ (i = 1, 2, \ldots, n)$$

$$\frac{\partial Q}{\partial \mu_p}\frac{\partial \mu_p}{\partial x_i} + \frac{\partial Q}{\partial \theta_p}\frac{\partial \theta_p}{\partial x_i} = 0 \text{ (using chian rule)}$$

$$\mu_i - r_f = -\frac{\frac{\partial Q}{\partial \theta_p}}{\frac{\partial Q}{\partial \mu_p}}\frac{\partial \theta_p}{\partial x_i}$$

It is a condition for any arbitrary portfolio p to be located on the capital market curve AB (see Fig. 2). Since $\mu_p - r_p = \sum_{i=1}^{n} x_i(\mu_i - r_i)$, using differentiation under integral sign, we get

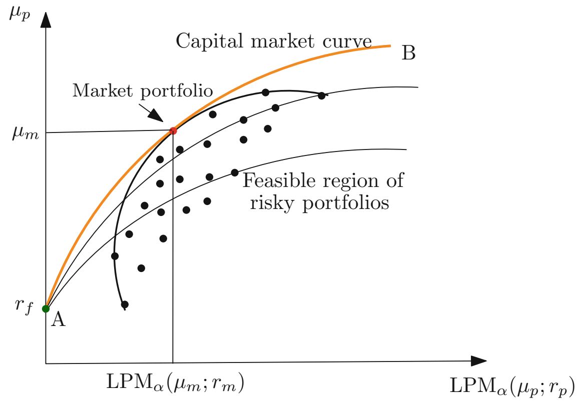
$$\frac{\partial \mathrm{LPM}_\alpha(\mu_p; r_p)}{\partial x_i} = E\left[(\mu_p - r_p)^{\alpha-1}\mathbb{1}_{\mu_p > r_p}(\mu_i - r_i)\right] \text{ for } i = 1, 2, \ldots, n.$$

As the market portfolio lies on the capital market curve,

$$\mu_i - r_f = -\frac{\left(\frac{\partial Q}{\partial \theta_p}\right)\Big|_{p=m}}{\left(\frac{\partial Q}{\partial \mu_p}\right)\Big|_{p=m}}\frac{\partial \theta_p}{\partial x_i}\Big|_{p=m}$$

for all $i = 1, 2, \ldots, n$. Thus,

Fig. 2 Capital market curve



$$\mu_i - r_f = k\frac{E\left[(\mu_m - r_m)^{\alpha-1}\mathbb{1}_{\mu_m > r_m}(\mu_i - r_i)\right]}{E\left[(\mu_m - r_m)^{\alpha}\mathbb{1}_{\mu_m > r_m}\right]} = k\beta_i^- \tag{5}$$

where $k = -\frac{\left(\frac{\partial Q}{\partial \theta_p}\right)\Big|_{p=m}}{\left(\frac{\partial Q}{\partial \mu_p}\right)\Big|_{p=m}} E\left[(\mu_m - r_m)^{\alpha}\mathbb{1}_{\mu_m > r_m}\right]$. Since Eq. (5) is true for all i,

$$\begin{aligned}\mu_m - r_f &= \sum_{i=1}^{n} x_{im}\mu_i - r_f \quad [\because r_m = \sum_{i=1}^{n} x_{im} r_{im}] \\ &= \sum_{i=1}^{n} x_{im}\left(\mu_i - r_f\right) \quad [\because \sum_{i=1}^{n} x_{im} = 1] \\ &= k\sum_{i=1}^{n} x_{im}\beta_i^- \quad \text{[using Eq. (5)]}\end{aligned}$$

Now

$$\sum_{i=1}^{n} x_{im}\beta_i^- = \frac{\sum_{i=1}^{n} x_{im} E\left[(\mu_m - r_m)^{\alpha-1}\mathbb{1}_{\mu_m > r_m}(\mu_i - r_i)\right]}{E\left[(\mu_m - r_m)^{\alpha}\mathbb{1}_{\mu_m > r_m}\right]}$$

$$= \frac{E\left[(\mu_m - r_m)^{\alpha-1}\mathbb{1}_{\mu_m > r_m}(\sum_{i=1}^n x_{im}\mu_i - \sum_{i=1}^n x_{im} r_i)\right]}{E\left[(\mu_m - r_m)^{\alpha}\mathbb{1}_{\mu_m > r_m}\right]}$$

$$= \frac{E\left[(\mu_m - r_m)^{\alpha-1}\mathbb{1}_{\mu_m > r_m}(\mu_m - r_m)\right]}{E\left[(\mu_m - r_m)^{\alpha}\mathbb{1}_{\mu_m > r_m}\right]} = \beta_m^- = 1,$$

$$\mu_m - r_f = k \tag{6}$$

Eliminating k from Eqs. (5) and (6) results

$$\mu_i - r_f = [\mu_m - r_f]\beta_i^-.$$

It is the mean-downside CAPM when the target is the expected return of the portfolio. □

Remark 1 Similarly, for $\tau = r_f$, we can also derive a slightly different downside CAPM given by

$$\mu_i - r_f = [\mu_m - r_f]\beta_{i;r_f}^-$$

where

$$\beta_{i;r_f} = \frac{E\left[(r_f - r_m)^{\alpha-1}\mathbb{1}_{r_f > r_m}(r_f - r_i)\right]}{E\left[(r_f - r_m)^{\alpha}\mathbb{1}_{r_f > r_m}\right]}.$$

This version of CAPM is first derived by Bawa and Linderberg [2]. Hence, this beta $\beta_{i;r_f}^-$ is known as BL beta.

Remark 2 If we choose variance instead of LPM as a measure of risk, then Q will be function of μ_p and σ_p^2. Then, using the similar techniques, we will get the classical asset pricing model.

5 Conclusion

In this paper, we have considered the lower partial moment as a portfolio risk measure and derived a capital asset pricing model in the mean-LPM plane. We proved the convexity of the investment curves in mean-LPM space. We also derived exact equation of an investment curve when the target is the risk-free rate or the expected return of the portfolio whose risk is being measured. We then formulated an optimization model to find the optimal portfolios located on CMC. With the help of these results and using the geometry of capital market curve, we showed that the optimal solution of the model satisfies downside capital asset pricing model. Our method of deriving the asset pricing model not only different from earlier ones but also does not have any distributional restrictions. Theoretical developments of this paper indeed enrich the portfolio selection theory under the mean-LPM framework and also form a strong platform for further research on asset pricing theory.

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TQM Indicators Implemented by Teachers of the Primary School

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1 Introduction

TQM is both a philosophy and a set of controlling principles that represent the equilibrium of a continuously improving organization. TQM is not a programme, it is a process which is continuous, involves everyone in an organization, and links business process. Further, to meet their customers' needs and expectations. TQM helps in cooperating to furnish products and services. TQM is the catalyst which has helped turn the company in terms of profitability and customer perceptions. TQM utilizes both quantitative methods and human resource practices to improve material and service inputs, intra- and inter-organizational processes, and to sharpen the focus on meeting customer's needs. TQM may be defined as a continuous quest for excellence by creating the right skills and attitudes in people to make prevention of defects possible and satisfy customers/users totally at all times. TQM is an activity in an association that has to reach every member within an association. Oakland [29] has defined TQM as follows: "TQM is an approach to improving the effectiveness and flexibility of business as a whole". It is necessarily a way of bringing together and involving the whole business with the support of every department, all activities performed, each person at every stage in the workplace.

A perspective on quality in education: the quest for zero defects was studied by Sayed [33] to cross-examine critically some of the prominent aspects of the present dissertation on quality in education. Dahlgaard et al. [10] have introduced five key principles of TQM by using a new management pyramid called the TQM pyramid. The key elements studied are leadership, focus on the customer and the employee, continuous improvement, everybody's participation, and focus on facts. Using quality management procedures in education for managing the learner-centred educational environment, research was carried out by Stukalina [35]. Zabadi [39] is in

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the opinion that crucial role is played by higher education in the economic and cultural reconstruction and development of the nations. Chizmar [6] has focused on quality teaching management and learning. The influence of a TQM teaching and learning model suggests hypothesis regarding teaching strategies that increases learning and its importance on the quality of product, orientation to students, encouragement of teamwork and a continuing desire to improve.

2 Literature Review

Quality in higher education with reference to various characteristics as employee satisfaction, leadership, managing teaching and learning performances, recruiting procedure, service quality and improving in teaching, and infrastructure are also studied. Zhang [40] has carried research in developing a TQM model at university. Kanji and Tambi [17] have carried out research on TQM and higher education in Malaysia. The critical success factors studied are measurement of resources, leadership, people management, internal and external customer satisfaction, continuous improvement and process improvement. Grygoryev and Karapetrovic [13] aimed to introduce an integrated system for measuring, modelling and managing teaching and learning performance in a university classroom environment. A new perspective for quality improvement in teaching and learning processes was done by Yeap [38]. He emphasizes that any continuous improvement effort will help to identify, analyse and redesign the teaching and learning process. Feedback data is essential and important in the evaluation procedure of teaching and learning with the output clearly specified and measured. Farhadi [11], overall objective of his work is to highlight the principles of TQM intricate and to point out how this line has been and can be used to improve the quality of an academic institution. Their work has been specified for higher education and enclosed the entire institution; their objective was conducted to an evaluation and assessment of the quality work at university.

Some exemplary models of managing school, parental involvement, teaching and learning methods, evaluation of school performance and relation among the principal and teachers are also discussed. Rampa [30] believes that integrated TQM framework would be flexible enough to accommodate differences in schools with regard to contexts, requirements, strengths and weaknesses. Further, educators' views on total quality management in secondary schools in Eshowe circuit were studied by Magwaza [21]. Ncube [26] analysed how the management of the quality of education of Rural Day Secondary Schools has been affected by the internal efficiency of the school system. The legal framework governing parental involvement with education in Zimbabwe was studied by Ngwenya and Pretorius [27]. Teacher perceptions of classroom management practices in public elementary schools were studied by Wilson [37]. Past few years only awareness of TQM in service and educational sector has been developed. Unfortunately, the scarcity of research with empirical proof in favour of TQM issues and its implementations in school education system is mostly unpredictable, surprisingly scant and prevented the conception towards the universal road map which can be further used by other schools to accomplish their quality-oriented targets.

2.1 Theoretical Research Framework

Irrespective of any discipline, theoretical research framework is essential for conducting research at the initial phase only. It is a conceptual exemplary of how a researcher uses the concept to theories or relates logical aspects of the relationships among various indicators that have been identified as important to the research problem. Further, it explores how these indicators are created, spread, approved and adapted over space and time [18]. Based on this philosophy, various theoretical models that describe the indicators of TQM which are relevant to this study would explain the conceptual model in context of school. TQM has the potential to not only enhance internal performance but also improve bottom line result. This research indicates the relationship between education system in primary schools and TQM indicators.

2.2 Indicators of TQM

2.2.1 Customer Focus

Quality begins with customers and ends with customers [32]. The objective of quality is to recognize and describe the quality targets which give customer satisfaction at various processes in the workplace [34]. Customer requirements are moving targets, which have to be constantly examined, identified and satisfied. Developing a successful customer service system can be one of the most rewarding goals you achieve for your company. Customer focus means expressing not only what the customers want, but also manipulative internal system and processes within the organization where alarm for customer and customers' needs get rooted in the organization's processes and related activities. Education largely belongs to a human service enterprise which is more or less narrowed to a customer base. The broadest definition of a customer as defined by Killedar [19] is the person, department or organizational unit that next receives the value-added product, service or client.

2.2.2 Top Management Support

The term top management has gained currency in the last few years and has often been synonymously used with improvement, upliftment and contribution. It is rightly said that, as the driver of any vehicle has to be perfect and good, similarly is the importance of top management who are the leading drivers of TQM activities [9]. Zabadi [39] argued that top management has an ability to create vision and promote change which is at the heart of successful implementation of TQM. The principal reasons for the failure of quality improvement in an organization are due to indifference and lack of involvement and commitment of the top management support. According to Joseph and Blanton [16] to demonstrate commitment to quality, the management should establish a quality council which would coordinate the company's various activities regarding quality. The duty of top management support is to inspect how the principal guides the school, determines strategies and action plans, sets school values, performance expectations and how the school addresses its responsibilities to the students, parents and other stakeholders. It addresses distribution of plans for better process and how activities are measured and sustained, Manaf and Seng [23].

2.2.3 Relation

The relationship an employee has with his or her supervisor is a central element to the employee's affiliation to the organization, and it has been argued that many employee behaviours are largely a function of the way they are managed by their supervisors [31]. Further, the major source of employee turnover is related to management issues which alone speak to the multiple repercussions the employee/management relationship has on an organization. Developing operative communication practices and respecting employees' work and opinions lead to better relationships between managers and their staff [4]. Further, the human resource indicated, that includes employee training and employee relation which were positively related to quality improvement. Thus, maintaining the relationship between student and teacher is a must to attain a quality school. Employers of a school can build a bridge between administrator, student, parents, teachers and principal by imparting regularly training and involving them in strategy meetings and activities.

2.2.4 Empowerment

Empowerment has gained much importance in last few years. Very largely the success of TQM depends on the efficiency and effectiveness of its employees in the organization for performing the process for improvement [24]. By Sarkar [32] "Empowerment means sharing with non-managerial employees the power and authority to make and implement decisions". Empowerment is the process of allowing or permitting an individual to think, behave and take action and control work and decision-making in a self-directed way. It is the state of feeling self-empowered to take control of one's own destiny [36]. Marks and Louis [25] defined teacher empowerment as a function of the readiness of building-level administrators to share their autonomy with those whose commitment is necessary to make the educational programme function at the highest degree of efficiency.

2.2.5 Continuous Improvement

Continuous improvement of the entire organization constitutes the basics of TQM, which represents process of systematically defining and separating root causes of performance deficiency delaying the improvement, refining and improving the products, services and organizational systems to produce gradual improvement towards total quality and value to the customer [32]. Continuous improvement should be considered as something normal process in any organization to realize total quality. The commitment to total quality schools to continuous improvement refers to both incremental as well as breakthrough improvement [24]. The focus of continuous improvement concept is to find the shortfalls in the organization and the sources of variations in administrative, manufacturing, teaching and other service processes which can diminish from a quality output, thus improving the process to reduce undesirable outputs and generate customer satisfaction [21]. In this context, the author confirms on using continuous improvement as a need in the educational area particularly for attaining quality in school. Thus, educational professionals must be firmly tied with the process to assess continuously to prevent problems from occurring, and they must correct process problems as they develop and make improvements.

2.2.6 Feedback

Feedback is a process which makes self-regulation possible by managing itself towards its goals while interacting with environmental disturbance and describes “circular casual processes” [8]. To support this, Yeap [38] says that the feedback policy in an organization monitors and determines the corrective steps required for the next improvement stage. Consistently, by implementing the feedback policy, the employee and employer both can assist in the process of refining, designing and redesigning which leads to continuous improvement effort in the organization which has often been practiced at the end of every semester or year, Alani et al. [2]. Further, according to Ah-Teck [1], feedback tools are a must for successful conflict resolution. The analysis of the feedback helps for continuous improvement process as it provides the background for evaluating objectives, evaluate outcomes and improve the teaching methodology and strategies that are critical for attaining and exceeding school performance which are the goals of school to make a quality school.

2.2.7 Process Management

Process is defined as “a systematic series of actions directed to the achievement of goals” [3]. Process refers to business, product design and production activities which are the building blocks of all organization. Process management is the means for assurance, control of energy, planning, administering the activities necessary to achieve a high level of performance and transforming inputs like materials, money, information and data into valuable outputs as information, products, service and data for which the customer will pay [34]. TQM is process-oriented. It is important to understand that process management can also be applied in academic institution along with manufacturing units. Quality planning has to begin from the top management support towards every person within the school, involving principals, teachers, administrators, students and parents and creating better ways to integrate with them to establish good relationships as to provide higher quality teaching to students.

2.3 Research Gap

The existing literature study is focused on the initial advancement of the perception of TQM, historical analysis and diffusion of TQM awareness and implementation in different sectors in abroad and in India. Most of the studies in education area have done intensive study on analysing data on indicators such as attendance problems of student, dropout ratio in schools, level of appreciation by school principal and teachers, parental involvement in education, student recruitment in schools. Other authors have studied the quality of education by type of school management, office management and teachers' perceptions of classroom management in public school and school management environment based on infrastructure and socio-economic status of students. Lastly, the studies based on TQM principle in university, higher education and secondary schools are done to some extent but the literature studies show that research in primary school using TQM indicators for improving quality is rare. Thus, to overcome the limitations mentioned, the current research targets to go beyond an analysis of those indicators. However, this research is trying to explore the relevance of TQM indicators in primary education. The present research focuses teachers' attitude towards implementing TQM indicator in the school. Hence, the objective of this research is to identify critical factors of TQM in the primary schools' context from the teachers' perspectives. Thus, to accomplish the purpose of the study, top management support, customer focus, continuous improvement, feedback, relation, empowerment and process management are studied.

3 Research Methodology

This research is an empirical study conducted on school teachers of primary school from fifth to eighth of Ahmedabad district. Convenience sampling method was employed to select respondents for the study. There were a total of 20 schools from which 311 questionnaires were administered to teaching staff willing to participate in the study after brief informative address on the objective of the study. The questionnaire administered was collected, sorted and processed using SPSS version 22. Out of 346 questionnaires administered, 311 were returned giving a response rate of 90%.

Each teacher evaluated the degree of quality factors in his/her school by rating each question on five-point Likert scale ranging from 1 (strongly disagree) to 5 (strongly agree). The structured questionnaires consisted of 41 questions on TQM construct and six questions on the demographic profile which were filled by primary school teachers from fifth to eighth standard. For TQM dimensions, the measurements based on the seven dimensions of TQM were developed in the study by the author. At first instance, the pilot study was performed by 20 teachers in the field of TQM and education, and some of the questions were revised based on the feedback from the pilot study. The data collected was validated and analysed. To identify the dimensions of TQM practices in the primary schools as perceived by the school teachers, factor analysis was performed on all 41 questions using SPSS version 22.

4 Data Analysis

To identify how many and what factors were identified as TQM practices in schools as perceived by teachers, factor analysis is used in this study. Factor analysis is a statistical method in which variables are not classified as independent or dependent. Rather, it can be applied to examine interrelationships among a large number of variables, and also to explain these variables in terms of their underlying factors [22].

To summarize the structure of a set of variables, factor analysis is used as an exploratory technique. To assess both convergent and discriminant construct validity, exploratory factor analysis (EFA) is executed. In general practice, EFA is used to explore the data and provide evidence about the number of factors required to best represent the data [14, 20]. Further, the Bartlett's test of sphericity and measure of sampling adequacy (MSA) test were to be executed before using EFA to make sure that the variable is adequately inter-correlated to produce descriptive factors [20]. To assess the factorability of the correlation matrix, Bartlett's test of sphericity is assumed to be statistically significant (i.e. $p < 0.05$). To check the appropriateness of factor analysis and to measure the degree of inter-correlations among the variable, measure sample adequacy (MSA) test is used [14]. On the other hand, the value of Kaiser–Meyer–Olkin (KMO) has to be referred before the analysis of EFA [15]. According to George [12], the MSA is measured by the value of KMO. An overall MSA (KMO) value of above 0.50 is accepted always before proceeding with the factor analysis [14]. Further, the varimax rotation assumes that the factors are uncorrelated and independent of each other when rotated. Hence, in this study the varimax orthogonal rotation is chosen for analysing EFA. Another important value to be checked when performing EFA is the factor loading. Factor loading indicates the strength of the relationships between the factors (latent variable) and the indicators. A coefficient value of factor greater than 0.40 validates an adequate loading [15]. Eigenvalue represents that amount of variance accounted for by a factor. Factors having eigenvalue greater than one are considered significant and usually accepted in common practice [14]. Thus, in order to determine factors from the variables such as factors responsible for TQM practices, factor analysis was performed as discussed above.

4.1 Exploratory Factor Analysis for TQM Factors

The reliability analysis of the teachers' data was found to be 0.893 having 41 variables which are above the threshold value of 0.6 which ensures that the variables are statistically reliable. Hair et al. [14] suggested that alpha value of 0.7 is sufficient to run exploratory analysis. The minimum size of the samples required to run factor analysis as recommended by Nunnally and Bernstein [28] ranges from 100 to 300. In this study, a sample size of 311 is considered sufficient to run factor analysis.

Initially, EFA was performed on 41 items of TQM factors. Factor analysis was carried out after removing the items with cross-loadings, deleting items having poor factor loadings less than 0.4 and also those which are not significantly loaded on any other factors. An eigenvalue greater than one is considered significant and is retained in this study [14]. Finally, after the preliminary analysis based on the above criteria, a clear simple structure of TQM factors is attained after the seventh iteration. The KMO measure of sampling adequacy was found to be 0.811 interpreted as meritorious, and hence, data sets are considered to be appropriate for EFA to produce reliable outcomes. Further, Bartlett's test of sphericity was also found to be significant having chi-square value 3136.646, ($p = 0.000$). Thus, the absence of multicollinearity is seen which confirms the data was appropriate for factor analysis.

The result of factor extraction using principal component analysis (PCA) with varimax rotation as seen in Table 1 suggested only 24 items should be used out of original 41 items. Further investigation shows that all these 24 items are loaded into seven factors with eigenvalues of more than one and accounted for 66.247% of total variation, which is above the cut-off point 60% [14]. Thus, all the seven factors are sufficient enough to explain the concept of TQM practices of teachers. Seven factors interpreted were continuous improvement, process management, customer focus, empowerment, relation, feedback and top management support. Most of the extracted factors were in line with some of the factors that were developed by Ciptono [7].

These seven factors are compared with the original theoretical constructs as discussed earlier. As expected, mostly items were grouped according to its original construct. The dominant factor contains the first factor in consistent with the continuous improvement (CI) dimensions with an eigenvalue of 5.993 which accounted for 23.972% of variance and has five items. Meanwhile, second factor is well coordinated with the dimensions of process management (PM) with an eigenvalue 2.794 and accounted for 11.175% of variance and has three items. The third factor contains a blend of three items of relations, one item of continuous improvement and one item of top management support and is relabelled into the factor relations based on the meaning suggested within the context of relationship in education field. Thus, the third factor consists of items relation (RE) with an eigenvalue of 2.159 which accounted for 8.638% of variance and has five items. The fourth factor consists of items from customer focus (CF) and empowerment (EM). Since items extracted were more focused on customer focus, it was finally renamed as customer focus (CF) with an eigenvalue 1.738 which accounted for 6.950% of variance and has three items.

Table 1 Factor analysis results and Cronbach's Alpha of seven factors

Factor name	Factor loadings	Cronbach's Alpha
Continuous improvement	0.845	0.8932
	0.819	
	0.808	
	0.790	
	0.747	
Process management	0.901	0.858
	0.893	
	0.776	
Customer focus	0.826	0.701
	0.699	
	0.601	
Empowerment	0.821	0.702
	0.719	
	0.656	
Feedback	0.854	0.769
	0.821	
	0.648	
Relation	0.702	0.706
	0.670	
	0.657	
	0.605	
	0.573	
Top management support	0.858	0.702
	0.809	

The fifth factor can be classified as empowerment (EM) with an eigenvalue of 1.564 which accounted for 6.256% of variance and has three items. Meanwhile, sixth factor is well matched with the dimensions of feedback (FB) having eigenvalue 1.265 and variance 5.060%. Lastly, seventh factor was named top management support (TMS) with an eigenvalue of 1.049 and accounted for 4.196% of variance and has two items.

The seven factors clearly specify the importance of total quality in primary schools and provide evidence to support the construct validity as all the factor loadings were greater than 0.4, as recommended cut-off limit for validity. Thus, overall variance of 66.247% was explained by these seven factors. Followed by this, the details of the factor analysis including the reliability of the factors are shown in Table 1.

4.2 Reliability

Reliability is the property by which consistent results are achieved when we repeat the measurement of something. Reliability refers to the consistency, stability or dependability with which an instrument measures a set of dimensions. Reliability is related not to what ought to be measured, but rather to how it is measured, Hair et al. [14]. Reliability is the degree to which a measurement instrument is free from error and therefore yields consistent results. Hence, the results of reliability analysis were kept an eye on each of the scales adopted in this study [5].

Now, the overall Cronbach's alpha value after factor analysis of 24 items which extracted seven factors is 0.854 which is again very good. In summary, the seven new factors from the results of EFA are compared with the original theoretical factors as being discussed in this study earlier. Finally, the internal consistency of the measures was evaluated using Cronbach's alpha as seen in Table 1, and all values were found to be greater than the recommended threshold of 0.60. (Continuous improvement = 0.892; process management = 0.858; relation = 0.706; customer focus = 0.701; empowerment = 0.702; feedback = 0.762; and top management support = 0.702.)

Thus, based on the result of factor analysis and the reliability test, the final dimensions that can identify the concepts of TQM in primary schools are top management support, customer focus, continuous improvement, feedback, relation, empowerment and process management.

5 Conclusion, Limitation and Recommendations

The study provides an initial point to promote the quality aspects in the primary schools. Further, the study can also supplement other data related to implementation of TQM practices in the schools and university. Through the results of the data under study, the identified TQM dimensions will promote the use of TQM in primary school education. Further, the study has significant implication for the development, in the context of implementation of TQM in the overall educational field.

The findings of the present study are useful for both educational institutes and researchers. The top management can practise this model to:

1. Evaluate the scope of TQM practice within their school.
2. Identify those regions of TQM where some improvements could be focused.
3. Recognize those areas of TQM where excellence exists currently and could be planned for its sustainability.

The schools, which are the hub of the grass root of education, should provide quality education to the students. Quality ought to be the heart of the educational framework so that different fields will enable, progress and get each kind of help from the educational organizations. The result offers a set of seven TQM indicators and recommends that

1. The school should invest in training programmes on quality management for the principals, teachers and even at the management level.
2. To compete with the global standards of quality in education, the government should provide additional support and fund to improve educational facilities, renewal of curriculum, infrastructure and an adequate number of trained teachers.
3. Further, similar studies in specific educational industries with a reasonably large size should be carried on in order to authenticate and extend the result of the current study.

The present study is limited to primary schools in Ahmedabad locale. The current study utilized a cross-sectional configuration, and likewise conclusions with respect to level of generalization are restricted. Further, based on the limitation of the cross-sectional study, the further study can be extended to other schools situated in other parts of the states and also to establish interconnection, longitudinal research.

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